Illinois Journal of Mathematics
Volume 56, Number 1, Spring 2012, Pages 251–263
S 0019-2082

FUNDAMENTAL SOLUTIONS AND COMPLEX COTANGENT LINE FIELDS

SIDNEY M. WEBSTER

Abstract. We consider a fundamental solution for the $\overline{\partial}$-operator on a complex n-manifold, which is given by an $(n, n-1)$-form of the Cauchy–Leray type $\Theta = \theta \wedge (\overline{\partial}\theta)^{n-1}$, where θ is a suitable $(1,0)$-form. On the open submanifold M^n where θ is smooth and nonzero, its multiples generate a complex line sub-bundle $E \subset T^*_{(1,0)}M$, which we assume to satisfy a certain integrability condition. To such an E we attach a global holomorphic invariant, in the form of a complex Godbillon–Vey ∂-cohomology class, provided a certain primary obstruction class vanishes. If θ is also Levi nondegenerate, in that $\Theta \neq 0$, then it determines an invariant connection on the hyperplane bundle given by $\theta = 0$. This provides θ formally with a complete system of local holomorphic invariants.

0. Introduction

The fundamental solution for the Cauchy–Riemann operator $\overline{\partial}$ in one complex variable z is provided by the Cauchy kernel, $\theta = dz/(2\pi i z)$, which is a (1,0)-form on $\mathbf{C} - \{0\}$. Via Green's theorem (or the Goursat lemma) it gives the Cauchy integral formula for holomorphic functions. On the Riemann sphere θ has poles at 0 and ∞, with residues $+1$ and -1, respectively. For an arbitrary Riemann surface, one may consider an elementary Abelian differential of the third kind with poles at points $p_\pm$, with residues ± 1, and suitably normalized periods. It turns out that we should regard p_+ as a point, but p_- as a hypersurface, as we seek to generalize this to higher dimensions.

On a complex n-dimensional manifold M, $n > 1$, the situation is more complicated, and the possibilities more varied. A general theory of fundamental solutions on $\mathbf{C}^n$ has been formulated in [4] by Harvey and Polking.

Received November 21, 2011; received in final form May 23, 2012.
2010 *Mathematics Subject Classification.* Primary 32V40. Secondary 32N05.

In the present work, we take a more limited view and consider a fundamental solution with a fixed pole. It may be defined by a suitable $(n,n-1)$-form Θ on M, with singularities. Stokes' theorem will then lead to a formula of Bochner–Martinelli type.

In the first section, we consider such a form Θ of the Cauchy–Fantappiè–Leray type. Thus, it can be constructed from a $(1,0)$-form θ, which is ∂-integrable and Levi nondegenerate, that is,

$$\theta \wedge \partial\theta = 0, \qquad \Theta \equiv \theta \wedge (\overline{\partial}\theta)^{n-1} \neq 0, \tag{0.1}$$

respectively, away from its zeros and singularities. These conditions, as well as the basic nature of a singularity, are preserved if θ is multiplied by a smooth nonzero factor. It follows that they are really properties of the complex line bundle E spanned by θ. The integrability condition holds if locally $\theta = f\,\partial g$, for smooth functions f, g. The nondegeneracy condition will be used mainly in the last section.

More generally, we consider a complex vector sub-bundle $E \subseteq T^*_{(1,0)}M$, of rank $r \geq 1$, and the exterior ideal $\mathcal{I}(E)$ that it generates. We say that E is ∂-integrable if $\partial\mathcal{I}(E) \subseteq \mathcal{I}(E)$. We let $\mathcal{I}_k(E)$ denote the kth power of the ideal $\mathcal{I}(E)$, and $\mathcal{I}_k(E)^{(p,q)}$ the (p,q)-forms in it. Then we have differential complexes $(\mathcal{I}_k(E), \partial)$, and the associated cohomology groups, $H^{(*,0)}_{\partial}(M, \mathcal{I}_k(E))$. For $k = 0$, we get the usual (anti-)Dolbeault cohomology of M, whereas $\mathcal{I}_k(E) = 0$ for $k > r$.

We let $F \subset T_{(1,0)}M$ denote the sub-bundle annihilated by E. It is closed under Lie brackets, $[F,F] \subseteq F$, and so is a complex analogue of the tangent bundle to a real foliation. Classical foliation theory, [1], [5], [7] which we refer to as "real theory," provides us with a useful guide, although there are some significant departures.

For example in Section 3, we derive a complex analogue of the Godbillon–Vey (G–V) invariant [3], [7], in the case when the complex line bundle $\Lambda^r E$ is trivial. This is a "secondary" ∂-cohomology class on M, which may be given by a ∂-closed $(2r+1,0)$-form Γ. Γ lies in $\mathcal{I}_r(E)$, and its ∂-cohomology class $[\Gamma]$ satisfies the following.

PROPOSITION 0.1. *Let $E \subset T^*_{(1,0)}M$ be a ∂-integrable sub-bundle of rank r, $1 \leq r < n$, for which $\Lambda^r E$ is a trivial line bundle. Then we have a well defined complex G–V cohomology class $[\Gamma] \in H^{(2r+1,0)}_{\partial}(M, \mathcal{I}_{r-1}(E))$. If M is simply connected, we have $[\Gamma] \in H^{(2r+1,0)}_{\partial}(M, \mathcal{I}_r(E))$.*

In case the line bundle $\Lambda^r E$ is nontrivial, so its (first) Chern class is nonzero, we derive, in Section 4, a "primary obstruction" sheaf cohomology class $[\xi] \in H^2_{\delta}(M, \hat{\mathcal{I}}_r(E)^{(2r,0)})$, with coefficients in the sheaf of germs of ∂-closed sections of $\mathcal{I}_r(E)$ (δ is the Cech coboundary operator, and "hat" will generally mean ∂-closed). In Section 4 we prove the following.

THEOREM 0.2. *Suppose that the sheaf cohomology class* $[\xi] \in H^2_\delta(M, \hat{\mathcal{I}}_r(E)^{(2r,0)})$ *vanishes. Then there exists a global* ∂*-closed complex G–V* $(2r+1,0)$*-form* Γ, *which belongs to the ideal* $\mathcal{I}_r(E)$.

The class $[\xi]$ may be considered with the less refined $(\hat{\mathcal{I}}_0(E) \equiv \hat{\mathcal{A}}(M))$-coefficients ($\mathcal{A}$ generally denotes C^∞ coefficients). Then we may apply the (anti-)Dolbeault isomorphism to it in two directions, to get (1) a ∂-closed $(2r+2,0)$-form representing a class in $H^{(2r+2,0)}_\partial(M,\mathcal{A})$; or (2) a sheaf coholomology class in the group $H^{2r+2}_\delta(M,\overline{\mathcal{O}})$, the coefficients being anti-holomorphic. The vanishing of this class $[\xi]$ guarantees the existence of a global smooth G–V $(2r+1,0)$-form Γ, which, however, may not lie in any $\mathcal{I}_k(E)$, $k>0$. The class $[\Gamma]$ is then an obstruction to a never vanishing global section of E of the form $\theta = f\,\partial g$.

One major difference from the real theory is the probable failure of the ∂-Frobenius theorem, in any particular case. This is related to the probable failure of the corresponding Poincaré lemma in the differential complex $\{\mathcal{I}(E),\partial\}$, as discussed in Section 2. For this reason we proceed without making use of local integrals, which are safely assumed in the real case. We note that Kamber and Tondeur considered holomorphic sub-bundles in Section 8 of [5].

Yet another major difference from the real theory is the existence of local differential invariants. This is developed in Section 5.

THEOREM 0.3. *Let* $\theta \neq 0$ *be a fixed Levi nondegenerate*, ∂*-integrable* $(1,0)$*-form on a complex* n*-manifold* M, $n \geq 2$. *Then there exist* two *intrinsic connections on the hyperplane sub-bundle* F *annihilated by* θ.

The result is motivated by the corresponding invariant theory of a strictly pseudo-convex real hypersurface, and we follow the formalism of [2], [10]. As compared to [10], the derivation is complicated somewhat by the appearance of additional terms, roughly speaking. However, these terms seem to be the more important ones. The theorem is a little out of the ordinary, in that the two normalizing procedures producing the two connections seem to be equally natural. The curvature, torsion, and covariant differentiation of either connection lead to a complete system of differential invariants, at least formally. The more comprehensive and difficult pseudoconformal theory is developed in [11].

1. Fundamental solutions

Let p be a point in the complex n-manifold M, $n \geq 2$, and Θ a differential form of type $(n,n-1)$ which is smooth away from p. Then Θ is a *parametrix* for the Cauchy–Riemann operator $\overline{\partial}$ on functions f, if

$$d\Theta \equiv \overline{\partial}\Theta = (\delta_p + K)\,dV, \tag{1.1}$$

where δ_p is the delta function, and K is a smooth, or at least absolutely integrable, function on M. If M has smooth boundary ∂M, Stokes' theorem gives a generalized Cauchy formula,

$$\int_{\partial M} f\Theta - f(p) = \int_M (\overline{\partial} f \wedge \Theta + fK\,dV). \tag{1.2}$$

We have a *fundamental solution*, if $K=0$.

We shall consider such Θ of the Cauchy–Fantappie–Leray type,

$$\Theta = \theta \wedge (\overline{\partial}\theta)^{n-1}, \quad \overline{\partial}\Theta = (\overline{\partial}\theta)^n. \tag{1.3}$$

Here $\theta \neq 0$ is a $(1,0)$-form smooth away from p. We say it is Levi nondegenerate if the $(1,1)$-form $\overline{\partial}\theta$ is nondegenerate on the hyperplane field $\theta = 0$, or equivalently if $\Theta \neq 0$.

For the special form $\theta = \partial \log r$, where $r(z) = \overline{r(z)}, z \in \mathbf{C}^n$, we have $\overline{\partial}\Theta = (\overline{\partial}\partial \log r)^n$, and (subscripts denoting $z-, \overline{z}-$ derivatives)

$$K = \det\left[(\log r)_{i\overline{j}}\right] = r^{-n-1} \det \begin{bmatrix} r & r_{\overline{j}} \\ r_i & r_{i\overline{j}} \end{bmatrix}. \tag{1.4}$$

Thus, we have a fundamental solution, if r satisfies a familiar complex Monge–Amperè equation, and yields the correct singularity. For $r = |z|^2 = \overline{z} \cdot z$, we have $K=0$,

$$\theta = \overline{z} \cdot dz / (\overline{z} \cdot z), \tag{1.5}$$

and Θ is the original Bochner–Martinelli form. To consider it on complex projective space $\mathbf{P}_n\mathbf{C}$, we use new nonhomogeneous coordinates w,

$$w = \left(w_1, w'\right), \qquad z_1 = w_1^{-1}, \qquad z' = w_1^{-1} w'; \tag{1.6}$$

$$\Theta = -w_1^{-1} dw_1 \wedge \chi^{n-1}, \qquad \chi = \overline{\partial}\partial \log\left(1 + \left|w'\right|^2\right). \tag{1.7}$$

For f compactly supported in the w-coordinate system, we get

$$-2\pi i \int_{H_\infty} f\chi^{n-1} = c_n f(0) + \int_{\mathbf{P}_n} \overline{\partial} f \wedge \Theta, \tag{1.8}$$

where $c_n \neq 0$, and H_∞ is the hyperplane at infinity. Note that $H_\infty \cong \mathbf{P}_{n-1}$, and χ is its Fubini–Study form. Formulae (1.8) is the type of result that we would hope for, on a more general compact complex manifold.

Somewhat more general than (1.5) is the following construction. Let θ_0 be a smooth and suitably nondegenerate $(1,0)$-form, and V be a holomorphic, or meromorphic, $(1,0)$-vectorfield, with $\theta_0(V)$ not identically zero. Put $\theta(\cdot) = \theta_0(\cdot)/\theta_0(V)$, so that $\iota_V\theta = 1$. Since $\iota_V(\overline{\partial}\theta) = \overline{\partial}(\iota_V\theta) = 0$, it follows that $(\overline{\partial}\theta)^n = 0$. The zeros (and poles) of V contribute to the singularities of Θ.

The foregoing serves mainly as motivation. For the most part of this work, we leave aside the equation $\overline{\partial}\Theta = 0$, and the singularities, and consider smooth $(1,0)$-forms $\theta \neq 0$, which satisfy the ∂-integrability condition in (0.1),

$$\theta \wedge \partial\theta = 0 \quad \Longleftrightarrow \quad \partial\theta = \theta \wedge \omega, \tag{1.9}$$

where ω is a $(1,0)$-form. The implication $\Rightarrow$ is clear locally using a coframe, then globally using a partion of unity. It is the formal integrability condition for achieving $\theta = f\,\partial g$. This is not a priori called for, but in retrospect, it restricts us to an interesting class of forms θ. In the last section, they are required to be Levi nondegenerate.

As mentioned above, Levi nondegeneracy and ∂-integrability are preserved under changes $\theta \mapsto v\theta$, $v \neq 0$, so they are really properties of the complex line bundle $E \subset T^*_{(1,0)}M$ of multiples of θ. Such nondegenerate, integrable line bundles E are a main subject of this work. One may envision first finding such an E, then constructing a suitable section θ.

For the dual point-of-view, let $F \subset T_{(1,0)}M$ be the complex hyperplane field annihilated by E. Then the sections of F are closed under Lie brackets. Symbolically,

$$[F,F] \subseteq F. \tag{1.10}$$

The (global) Levi form λ of F is the vector-valued form

$$\lambda : F \times F \longrightarrow (TM \otimes \mathbf{C})/(F \oplus \overline{F}), \tag{1.11}$$

where $\lambda_x(Z_x, W_x) = i[Z, \overline{W}]_x$, for Z, W sections of F extending Z_x, W_x. It is to be nondegenerate in the last section. It has no analogue in the real theory.

2. Complex Frobenius problems

To clarify the integrability condition (1.10) further, we momentarily consider an almost complex manifold (M, J), $J^2 = -I$ on TM. For an almost complex sub-bundle $F_0 = JF_0 \subset TM$, we have the decomposition

$$F_0 \otimes \mathbf{C} = F \oplus \overline{F}, \qquad F = (F_0 \otimes \mathbf{C}) \cap T_{(1,0)}M, \tag{2.1}$$

where $F, \overline{F}$ are the $(\pm i)$-eigenbundles of J acting on F_0.

We say, temporarily, that F_0 is "$(1,0)$-integrable," if (1.10) holds. For (real) vector fields X, Y in F_0, let

$$Z = [X - iJX, Y - iJY] = [X,Y] - [JX,JY] - i\big([JX,Y] + [X,JY]\big). \tag{2.2}$$

Then $[F,F] \subseteq F_0 \otimes \mathbf{C}$ is equivalent to

$$X, Y \quad \text{in } F_0 \quad \Longrightarrow \quad [X,Y] - [JX,JY] \quad \text{in } F_0. \tag{2.3}$$

We have $(J - iI)Z = (J - iI)N(X,Y)$, where

$$N(X,Y) = [X,Y] - [JX,JY] + J\big([JX,Y] + [X,JY]\big) \tag{2.4}$$

is the familiar (real) Nijenhuis vector field. So $Z \in F$ implies $N(X,Y) = 0$. Thus $[F,F] \subseteq F$ is equivalent to (2.3), and $N(X,Y) = 0$.

For M a complex manifold, which we assume henceforth, we have $N(X,Y) = 0$, for all X, Y in TM (and conversely by the Newlander–Nirenberg theorem [8]). Thus F_0 is $(1,0)$-integrable, if and only if (2.3) holds. The much stronger real Frobenius condition is that X, Y in $F_0 \Longrightarrow [X,Y]$ in F_0, which we do *not* assume. It would imply that F_0 is the tangent bundle to a smooth real foliation of M. Since $F_0 = JF_0$, each leaf would be an almost complex (hence complex since $N = 0$) submanifold.

Let $E \subseteq T^*_{(1,0)}M$ be the sub-bundle of $(1,0)$-forms annihilating $F \subseteq T_{(1,0)}M$; and let $\mathcal{I}(E) \subseteq \mathcal{A}(M)$ be the exterior ideal generated by E, as in the introduction. Then by Cartan's formula for d, the integrability condition $[F,F] \subseteq F$ is equivalent to

$$d\mathcal{I}\big(E \oplus T^*_{(0,1)}M\big) \subseteq \mathcal{I}\big(E \oplus T^*_{(0,1)}M\big), \tag{2.5}$$

which is easily equivalent to

$$\partial\mathcal{I}(E) \subseteq \mathcal{I}(E). \tag{2.6}$$

At this point we drop the expression "$(1,0)$-integrable" in favor of "∂-integrable."

For a complex function f on M, we clearly have

$$df \in \mathcal{I}\big(E \oplus T^*_{(0,1)}M\big) \iff \partial f \in \mathcal{I}(E). \tag{2.7}$$

Thus, on a complex manifold, the ∂-Frobenius problem for the sub-bundle E is equivalent to the (complex) d-Frobenius problem for $E \oplus T^*_{(0,1)}M$.

Unfortunately, complex d-Frobenius problems [8], [9], are usually much more difficult than the real ones, and they may even be unsolvable, as the CR-embedding problem shows [6], [8]. This (probable) failure of actual integrability is related to the (probable) failure of the Poincaré lemma in the complex $(\mathcal{I}(E), \partial)$. This is, in turn, related to H. Lewy unsolvability on hypersurfaces in $\mathbf{C}^n$ (see [8], [9]). Therefore, as mentioned in the introduction, we base most of our considerations on the (formal) integrability conditions (1.9), (1.10), and make no assumption on the existence of integrals.

3. A complex Godbillon–Vey invariant

We essentially adapt the arguments of Godbillon and Vey [3], [7], but with a few additional considerations. Let $E \subset T^*_{(1,0)}M^n$ be a ∂-integrable smooth complex *line* sub-bundle. In this section, we assume that E is (topologically) trivial, and spanned by a global smooth $(1,0)$-form $\theta \neq 0$, satisfying (1.9), $\partial\theta = \theta \wedge \omega$. Applying the ∂-operator to this gives

$$0 = \theta \wedge \partial\omega \implies \partial\omega = \theta \wedge \xi, \tag{3.1}$$

where ξ is a global $(1,0)$-form. The complex G–V $(3,0)$-form is

$$\Gamma = \omega \wedge \partial\omega = \omega \wedge \theta \wedge \xi, \qquad \partial\Gamma = (\partial\omega)^2 = (\theta \wedge \xi)^2 = 0. \tag{3.2}$$

Thus, $\Gamma \in \hat{\mathcal{I}}_1(E)^{(3,0)} \subseteq \hat{\mathcal{A}}(M)^{(3,0)}$, where again "hat" means ∂-closed, and $\mathcal{I}_1(E) \equiv \mathcal{I}(E)$ is the first power of the ideal $\mathcal{I}(E)$. A change in the form ω, $\omega' = \omega + b\theta$, gives

$$\Gamma' = \omega' \wedge \partial\omega' = \Gamma + \partial(\theta \wedge \partial b), \tag{3.3}$$

by (3.1). Note that $\Gamma, \Gamma' \in \hat{\mathcal{I}}_1(E)^{(3,0)}$, while $\Gamma' - \Gamma \in \partial\mathcal{I}_1(E)^{(2,0)}$.

Next, we change θ, $\tilde{\theta} = v\theta$, $v \neq 0$,

$$\partial\tilde{\theta} = \tilde{\theta} \wedge \tilde{\omega}, \qquad \tilde{\omega} = \omega - v^{-1}\,\partial v, \tag{3.4}$$

so that $\tilde{\Gamma} = \tilde{\omega} \wedge \partial\tilde{\omega} = (\omega - v^{-1}\,\partial v) \wedge \partial\omega$. Hence,

$$\tilde{\Gamma} - \Gamma = \partial\bigl(-\omega \wedge v^{-1}\,\partial v\bigr). \tag{3.5}$$

However, if we have a single-valued logarithm, for example, if M is simply connected, then by (3.1) we have

$$\tilde{\Gamma} - \Gamma = \partial\bigl((-\log v)\theta \wedge \xi\bigr). \tag{3.6}$$

Thus, $\tilde{\Gamma}, \Gamma \in \hat{\mathcal{I}}_1(E)^{(3,0)}$, and $\tilde{\Gamma} - \Gamma \in \partial\mathcal{A}(M)^{(2,0)}$, in general; while $\tilde{\Gamma} - \Gamma \in \partial\mathcal{I}_1(E)^{(2,0)}$, if M is simply connected. This proves Proposition 0.1 in the case $r = 1$.

Next, we consider a ∂-integrable sub-bundle $E \subseteq T^*_{(1,0)}M$ of rank $r, 1 \leq r \leq n-1$. We assume that the rth exterior power $\Lambda^r E$ is trivial, with global nonzero section Θ, *not* to be confused with the Θ of Section 1. Locally E is spanned by $(1,0)$-forms θ^α, $1 \leq \alpha \leq r$, satisfying $\partial\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta$, for $(1,0)$-forms ω^α_β; and $\Theta = f\theta^1 \wedge \cdots \wedge \theta^r$, $f \neq 0$. It follows that $\Theta \in \mathcal{I}_r(E)^{(r,0)}$, and

$$\partial\Theta = \Theta \wedge \omega, \tag{3.7}$$

for a global $(1,0)$-form ω. (At first this holds locally with $\omega = (-1)^r(f^{-1}\,\partial f - \omega^\alpha_\alpha)$, then globally via a partition of unity.)

Applying the operator ∂ to (3.7) gives

$$0 = \Theta \wedge \partial\omega \quad \Longrightarrow \quad \partial\omega \in \mathcal{I}_1(E)^{(2,0)}. \tag{3.8}$$

The complex G–V $(2r+1, 0)$-form is

$$\Gamma = \omega \wedge (\partial\omega)^r \in \hat{\mathcal{I}}_r(E)^{(2r+1,0)}, \qquad \partial\Gamma = (\partial\omega)^{r+1} \in \hat{\mathcal{I}}_{r+1}(E) = 0. \tag{3.9}$$

If we change ω, $\omega' = \omega + \sigma$, then $0 = \Theta \wedge \sigma$, so that $\sigma \in \mathcal{I}_1(E)^{(1,0)}$, and $\partial\sigma \in \mathcal{I}_1(E)^{(2,0)}$. Since $\partial\omega, \partial\omega' \in \mathcal{I}_1(E)^{(2,0)}$ as well, we have $\sigma \wedge (\partial\omega')^r \in \mathcal{I}_{r+1}(E) = 0$. Hence,

$$\Gamma' = \omega' \wedge \bigl(\partial\omega'\bigr)^r = \omega \wedge (\partial\omega + \partial\sigma)^r = \omega \wedge \bigl((\partial\omega)^r + \zeta \wedge \partial\sigma\bigr), \tag{3.10}$$

where, by binomial expansion, $\zeta \in \hat{\mathcal{I}}_{r-1}(E)^{(2r-2,0)}$. It follows that $\Gamma' - \Gamma = \partial(-\omega \wedge \zeta \wedge \sigma)$, since $\partial\omega \wedge \zeta \wedge \sigma \in \mathcal{I}_{r+1}(E) = 0$. Thus we have

$$\Gamma' - \Gamma \in \partial\mathcal{I}_r(E)^{(2r,0)}. \tag{3.11}$$

Next, we change Θ, $\tilde{\Theta} = v\Theta$, $v \neq 0$, giving $\partial\tilde{\Theta} = \tilde{\Theta} \wedge \tilde{\omega}$, $\tilde{\omega} = \omega + (-1)^r v^{-1}\,\partial v$. Hence,

$$\tilde{\Gamma} = \tilde{\omega} \wedge (\partial\tilde{\omega})^r = \bigl(\omega + (-1)^r v^{-1}\,\partial v\bigr) \wedge (\partial\omega)^r = \Gamma + (-1)^r v^{-1}\,\partial v \wedge (\partial\omega)^r; \tag{3.12}$$

so that

$$\tilde{\Gamma} - \Gamma = \partial\eta, \tag{3.13}$$

where $\eta = (-1)^r \omega \wedge (\partial\omega)^{r-1} \wedge v^{-1}\,\partial v \in \mathcal{I}_{r-1}(E)^{(2r,0)}$, in general. If we have a single-valued log, then $\eta = (-1)^r \log v (\partial\omega)^r \in \mathcal{I}_r(E)^{(2r,0)}$.

This finishes the proof of Proposition 0.1.

4. The primary obstruction

In the last section, we assumed that $\Lambda^r E$ was a trivial complex line bundle. Now we drop this condition and quantify the obstruction to defining a global complex G–V form. This obstruction will be in the form of a sheaf cohomology class with certain coefficients. In this section, $E \subseteq T^*_{(1,0)}M$ is an arbitrary ∂-integrable sub-bundle of rank $r \geq 1$.

We select a log-simple open covering $\mathcal{U} = \{U_\mu\}$ of M (i.e., all nonempty finite intersections of the U_μ are contractible), such that $\Lambda^r E$ is trivial on each U_μ, and spanned by a nonzero $(r,0)$-form $\Theta_\mu \in \mathcal{I}_r(E)^{(r,0)}$.

On U_μ we have, from the last section, $\partial\Theta_\mu = \Theta_\mu \wedge \omega_\mu$, where $\partial\omega_\mu \in \mathcal{I}_1(E)^{(2,0)}$. We have the indeterminacy $\omega_\mu \mapsto \omega'_\mu = \omega_\mu + \sigma_\mu$, with $\sigma_\mu \in \mathcal{I}_1(E)^{(1,0)}$. We define $\Gamma_\mu = \omega_\mu \wedge (\partial\omega_\mu)^r \in \hat{\mathcal{I}}_r(E)^{(2r+1,0)}$, with the indeterminacy $\Gamma'_\mu - \Gamma_\mu \in \partial\mathcal{I}_r(E)^{(2r,0)}$.

On $U_\mu \cap U_\nu$, we have $\Theta_\nu = v_{\nu\mu}\Theta_\mu$, where $v_{\nu\mu} = v^{-1}_{\mu\nu}$, $v_{\mu\lambda}v_{\lambda\nu} = v_{\mu\nu}$ are the transition functions defining $\Lambda^r E$ as an element of the Cech cohomology group $H^1(\mathcal{U}, \mathcal{A}^*)$. With δ the codifferential, $\delta^2 = 0$, we have $(\delta\Gamma)_{\mu\nu} \equiv \Gamma_\nu - \Gamma_\mu = \partial\eta_{\mu\nu}$, where $\eta_{\mu\nu} = -\eta_{\nu\mu} \in \mathcal{I}_r(E)^{(2r,0)}$. (Here $\eta_{\mu\nu}$ is $(-1)^r \log v_{\mu\nu}(\partial\omega_\mu)^r$, skew-symmetrized in μ,ν.) Thus,

$$\Gamma = \{\Gamma_\mu\} \in \mathcal{C}^0\big(\mathcal{U}, \hat{\mathcal{I}}_r(E)^{(2r+1,0)}\big), \qquad \eta = \{\eta_{\mu\nu}\} \in \mathcal{C}^1\big(\mathcal{U}, \mathcal{I}_r(E)^{(2r,0)}\big); \tag{4.1}$$

$$\delta\Gamma = \partial\eta, \tag{4.2}$$

as in the classical work of A. Weil [12].

We define $\xi \equiv \delta\eta$.

Then $\delta\xi = 0$, and $\partial\xi = \partial\delta\eta = \delta\,\partial\eta = \delta^2\Gamma = 0$. Thus, $\xi \in Z^2(\mathcal{U}, \hat{\mathcal{I}}_r(E)^{(2r,0)})$. Let $\mathcal{S}$ denote any of the sheaves of germs of smooth differential forms $\mathcal{I}_k(E)$, $0 \leq k \leq r$, and $\hat{\mathcal{S}} \subset \mathcal{S}$ the subsheaf of ∂-closed forms. Then we define the *primary obstruction* (to finding a global Godbillon–Vey (G–V) form Γ in $\hat{\mathcal{S}}^{(2r+1,0)}$) to be the sheaf cohomology class

$$[\xi] \in H^2_\delta\big(M, \hat{\mathcal{S}}^{(2r,0)}\big), \tag{4.3}$$

represented by the above ξ relative to $\mathcal{U}$. We must show that it is well defined, and plays the appropriate role.

For this, we consider the usual exact sequence of sheaves of differential forms,

$$0 \longrightarrow \hat{\mathcal{S}}^{(p,0)} \longrightarrow \mathcal{S}^{(p,0)} \longrightarrow \hat{\mathcal{S}}^{(p+1,0)} - - - > 0. \tag{4.4}$$

Here the first two arrows are inclusion, the third one is by the operator ∂. For $\mathcal{S} = \mathcal{I}_0(E) = \mathcal{A}(M)$, the last arrow can be replaced with a solid one, by the

Dolbeault lemma. For $\mathcal{S}=\mathcal{I}_k(E)$, $k>0$, it may be missing, as indicated in Section 2. We have the corresponding commutave diagram of cochain groups, where the notation $\mathcal{U}$ is omitted,

$$\begin{array}{ccccc} \mathcal{C}^0(\hat{\mathcal{S}}^{(2r,0)}) & \longrightarrow & \mathcal{C}^0(\mathcal{S}^{(2r,0)}) & \longrightarrow & \mathcal{C}^0(\hat{\mathcal{S}}^{(2r+1,0)}) \\ \downarrow\delta & & \downarrow\delta & & \downarrow\delta \\ \mathcal{C}^1(\hat{\mathcal{S}}^{(2r,0)}) & \longrightarrow & \mathcal{C}^1(\mathcal{S}^{(2r,0)}) & \longrightarrow & \mathcal{C}^1(\hat{\mathcal{S}}^{(2r+1,0)}) \\ \downarrow\delta & & \downarrow\delta & & \downarrow\delta \\ \mathcal{C}^2(\hat{\mathcal{S}}^{(2r,0)}) & \longrightarrow & \mathcal{C}^2(\mathcal{S}^{(2r,0)}) & \longrightarrow & \mathcal{C}^2(\hat{\mathcal{S}}^{(2r+1,0)}). \end{array} \tag{4.5}$$

To show that the primary obstruction is well defined, we first change Γ to Γ', $\Gamma'_\mu=\Gamma_\mu+\partial\phi_\mu$ in $\mathcal{C}^0(\mathcal{U},\hat{\mathcal{S}}^{(2r+1,0)})$, $\phi=\{\phi_\mu\}\in\mathcal{C}^0(\mathcal{U},\mathcal{S}^{(2r,0)})$. Then, $\delta\Gamma'=\delta(\Gamma+\partial\phi)=\partial\eta+\partial\delta\phi=\partial\eta'$, $\eta'=\eta+\delta\phi$. Hence, $\xi'=\delta\eta'=\delta\eta=\xi$, so there is no change in ξ.

Next, we change $\eta\in\mathcal{C}^1(\mathcal{U},\mathcal{S}^{(2r,0)})$. $\Gamma=\partial\eta=\partial\eta'$; $\zeta=\eta'-\eta\in\mathcal{C}^1(\mathcal{U},\hat{\mathcal{S}}^{(2r,0)})$. Then $\xi'=\delta\eta'=\delta(\eta+\zeta)=\xi+\delta\zeta$. Hence, $[\xi']=[\xi]$ in $H^2_\delta(\mathcal{U},\hat{\mathcal{S}}^{(2r,0)})$.

It follows that the obstruction class is well defined.

Now suppose that $[\xi]=0$ in the group $H^2_\delta(\mathcal{U},\hat{\mathcal{S}}^{(2r,0)})$. Then $\xi=\delta\zeta$, $\zeta=\{\zeta_{\mu\nu}\}\in\mathcal{C}^1(\mathcal{U},\hat{\mathcal{S}}^{(2r,0)})$. Thus, $0=\xi-\delta\zeta=\delta(\eta-\zeta)$, or $\eta-\zeta\in Z^1(\mathcal{U},\mathcal{S}^{(2r,0)})$. But $\mathcal{S}$ is a fine sheaf (admits smooth partitions of unity), so its sheaf cohomology vanishes in positive degree.

It follows that $\eta-\zeta=\delta\kappa$, $\kappa=\{\kappa_\mu\}\in\mathcal{C}^0(\mathcal{U},\mathcal{S}^{(2r,0)})$. Hence, $\delta\Gamma=\partial\eta=\partial(\eta-\zeta)=\partial\delta\kappa=\delta\,\partial\kappa$, or $(\delta(\Gamma-\partial\kappa))_{\mu\nu}=0$. In other words, $\Gamma_\mu-\partial\kappa_\mu=\Gamma_\nu-\partial\kappa_\nu$ on $U_\mu\cap U_\nu$. Thus, there exists a global $(2r+1,0)$-form on M, again denoted by Γ, such that $\Gamma=\Gamma_\mu-\partial\kappa_\mu$ on U_μ. Clearly $\partial\Gamma=0$, and Γ is a global GV-form. Notice that this Γ belongs to $\hat{\mathcal{S}}^{(2r+1,0)}$.

Taking $\mathcal{S}=\mathcal{I}_r(E)$ gives Theorem 0.2.

In the special case $\mathcal{S}=\mathcal{I}_0(E)=\mathcal{A}(M)$, we have a Poincaré lemma by the anti-Dolbeault–Grothendieck theorem. Then we have anti-Dolbeault isomorphisms $H^2_\delta(M,\hat{\mathcal{A}}^{(2r,0)})\cong H^1_\delta(M,\hat{\mathcal{A}}^{(2r+1,0)})\cong H^{(2r+2)}_\partial(M,\mathcal{A})$. Thus, $[\xi]$ corresponds to a ∂-closed $(2r+2,0)$-form, modulo exact such forms.

In the other direction, $H^2_\delta(M,\hat{\mathcal{A}}^{(2r,0)})\cong\cdots\cong H^{2r+2}_\delta(M,\hat{\mathcal{A}}^{(0,0)})=H^{2r+2}_\delta(M,\overline{\mathcal{O}})$. Here $[\xi]$ corresponds to an anti-holomorphic $(2r+2)$-δ-cocycle.

5. Local differential invariants

We consider a fixed $(1,0)$-form $\theta\neq0$, on a complex n-manifold, $n\geq2$, satisfying the foregoing ∂-integrability and Levi nondegeneracy conditions. We want to derive a complete system of local differential invariants for θ.

This invariant theory is analogous to that of [10]. The invariant theory of the family of multiples of θ, that is, of the line bundle E, is analogous to the Chern–Moser theory [3]. It is developed in [11].

For this, we use local $(1,0)$-coframe fields of the form $\{\theta^\alpha, \theta^n = \theta\}$, $1 \le \alpha \le n-1$. Greek indices will run over this range, and the summation convention will be used. Thus, we have

$$\partial\theta = \theta \wedge \phi', \qquad \phi' = a_\alpha \theta^\alpha, \tag{5.1}$$

$$\overline{\partial}\theta = \chi + h_\alpha \theta^\alpha \wedge \overline{\theta} + \theta \wedge \phi'', \tag{5.2}$$

$$\phi'' = a_{\overline{\alpha}} \theta^{\overline{\alpha}} + a_{\overline{n}} \overline{\theta}, \tag{5.3}$$

$$\chi = h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{5.4}$$

where χ is the Levi form of θ, with $\det(h_{\alpha\overline{\beta}}) \neq 0$; but $h_{\alpha\overline{\beta}}$ has no assumed symmetry. We may use the matrix $h_{\alpha\overline{\beta}}$ to lower, and its inverse $h^{\overline{\beta}\alpha}$ to raise greek indices. Note that $a_{\overline{\alpha}}$ is not necessarily the complex conjugate of a_α.

The admissible coframe changes, $\tilde{\theta}^\alpha = \theta^\beta U_\beta^\alpha + \theta v^\alpha$, define a G-structure on M. With $U_\beta^\alpha = \delta_\beta^\alpha$, we get new coefficients $\tilde{h}_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}}$, and $\tilde{h}_\alpha = h_\alpha - h_{\alpha\overline{\beta}} v^{\overline{\beta}}$. Hence, we can choose the v^α uniquely to get $h_\alpha = 0$. Then restricting to such *adapted* coframes, we get

$$\overline{\partial}\theta = \chi + \theta \wedge \phi'', \tag{5.5}$$

in place of (5.2), and the reduced structure group $(U_\beta^\alpha) \in Gl(n-1, \mathbf{C})$, $v^\alpha = 0$. In terms of the dual $(1,0)$-frame $\{X_\alpha, X_n \equiv V\}$, the adapted condition is equivalent to

$$\iota_{\overline{V}}(\overline{\partial}\theta) \in \mathcal{I}(\theta). \tag{5.6}$$

Since $\theta(V) = 1$, this uniquely determines the transverse vector V. Note that now $a_{\overline{n}} = 0$ in (5.3) is equivalent to the condition $\overline{\partial}\Theta = 0$ of Section 1, which is a reason for this particular normalization. For future purposes [11], we note that if we replace θ by a multiple $v\theta$, $v \neq 0$, then, since $\theta(\overline{V}) = 0$, the transversal V retains its direction, and is only scaled.

Together with the integrability condition for the (almost) complex structure, these normalizations give the structure equations,

$$\begin{aligned} d\theta &= \chi + \theta \wedge \phi, \qquad \phi = \phi' + \phi'', \\ d\theta^\alpha &= \theta^\beta \wedge \omega_\beta^\alpha + \theta \wedge \tau^\alpha, \\ \tau^\alpha &= \theta^{\overline{\beta}} A_{\overline{\beta}}^\alpha + \overline{\theta} A_{\overline{n}}^\alpha. \end{aligned} \tag{5.7}$$

In this arrangement χ, ϕ, and the *torsion* forms τ^α are uniquely determined by the coframe; whereas the 1-forms ω_β^α are determined up to changes,

$$\tilde{\omega}_\beta^\alpha = \omega_\beta^\alpha + B_{\beta\gamma}^\alpha \theta^\gamma, \qquad B_{\beta\gamma}^\alpha = B_{\gamma\beta}^\alpha. \tag{5.8}$$

To begin the process of determining these latter forms precisely, we take the d-derivative of the first equation in (5.7). This gives

$$d\chi + \chi \wedge \phi = \theta \wedge d\phi. \tag{5.9}$$

To compute the left-hand side, we introduce the covariant derivative notation,

$$Dh_{\alpha\overline{\beta}} = dh_{\alpha\overline{\beta}} - \omega_\alpha^\gamma h_{\gamma\overline{\beta}} - h_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}^{\overline{\gamma}} \tag{5.10}$$
$$= h_{\alpha\overline{\beta},\gamma}\theta^\gamma + h_{\alpha\overline{\beta},n}\theta + h_{\alpha\overline{\beta},\overline{\gamma}}\theta^{\overline{\gamma}} + h_{\alpha\overline{\beta},\overline{n}}\overline{\theta}.$$

Then (5.9) takes the form,

$$(Dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi)\wedge\theta^\alpha\wedge\theta^{\overline{\beta}} = \theta\wedge\big(d\phi - \tau^\alpha\wedge\theta_\alpha\big) + \overline{\theta}\wedge\big(\tau^{\overline{\alpha}}\wedge\theta_{\overline{\alpha}}\big); \tag{5.11}$$

$$\tau^\alpha\wedge\theta_\alpha = A_{\overline{\beta}\overline{\alpha}}\theta^{\overline{\beta}}\wedge\theta^{\overline{\alpha}} + A_{\overline{n}\overline{\alpha}}\overline{\theta}\wedge\theta^{\overline{\alpha}}. \tag{5.12}$$

It follows that the terms on the left-hand side of (5.11), free of $\theta,\overline{\theta}$, must vanish. This gives the symmetries,

$$\begin{aligned} h_{\alpha\overline{\beta},\gamma} + h_{\alpha\overline{\beta}}a_\gamma &= h_{\gamma\overline{\beta},\alpha} + h_{\gamma\overline{\beta}}a_\alpha,\\ h_{\alpha\overline{\beta},\overline{\gamma}} + h_{\alpha\overline{\beta}}a_{\overline{\gamma}} &= h_{\alpha\overline{\gamma},\overline{\beta}} + h_{\alpha\overline{\gamma}}a_{\overline{\beta}}. \end{aligned} \tag{5.13}$$

The other terms give

$$\begin{aligned} 0 = \theta\wedge\big(d\phi - \tau^\alpha\wedge\theta_\alpha - h_{\alpha\overline{\beta},n}\theta^\alpha\wedge\theta^{\overline{\beta}}\big)&\\ + \overline{\theta}\wedge\big(\tau^{\overline{\alpha}}\wedge\theta_{\overline{\alpha}} - (h_{\alpha\overline{\beta},\overline{n}} + h_{\alpha\overline{\beta}}a_{\overline{n}})\theta^\alpha\wedge\theta^{\overline{\beta}}\big)&. \end{aligned} \tag{5.14}$$

Substituting the complex conjugate of (5.12) into the second parentheses, shows that we must have

$$A_{\overline{\alpha}\overline{\beta}} = A_{\overline{\beta}\overline{\alpha}}, \qquad h_{\alpha\overline{\beta},\overline{n}} + h_{\alpha\overline{\beta}}a_{\overline{n}} = 0. \tag{5.15}$$

Then (5.14) reduces to

$$d\phi \equiv h_{\alpha\overline{\beta},n}\theta^\alpha\wedge\theta^{\overline{\beta}}, \quad \mathrm{mod}\ \theta. \tag{5.16}$$

If we substitute the change (5.8) into (5.10), we get

$$\tilde{D}h_{\alpha\overline{\beta}} = Dh_{\alpha\overline{\beta}} - B_{\alpha\sigma}^\gamma h_{\gamma\overline{\beta}}\theta^\sigma - h_{\alpha\overline{\gamma}}B_{\overline{\beta}\overline{\sigma}}^{\overline{\gamma}}\theta^{\overline{\sigma}}. \tag{5.17}$$

Hence, $\tilde{h}_{\alpha\overline{\beta},n} = h_{\alpha\overline{\beta},n}$, $\tilde{h}_{\alpha\overline{\beta},\overline{n}} = h_{\alpha\overline{\beta},\overline{n}}$, and

$$\begin{aligned} \tilde{h}_{\alpha\overline{\beta},\gamma} &= h_{\alpha\overline{\beta},\gamma} - B_{\alpha\gamma}^\sigma h_{\sigma\overline{\beta}},\\ \tilde{h}_{\alpha\overline{\beta},\overline{\gamma}} &= h_{\alpha\overline{\beta},\overline{\gamma}} - h_{\alpha\overline{\sigma}}B_{\overline{\beta}\overline{\gamma}}^{\overline{\sigma}}. \end{aligned} \tag{5.18}$$

Noting the symmetry (5.8) required for the $B_{\beta\gamma}^\alpha$, we rewrite these equations as

$$\begin{aligned} (\tilde{h}_{\alpha\overline{\beta},\gamma} + h_{\alpha\overline{\beta}}a_\gamma) &= (h_{\alpha\overline{\beta},\gamma} + h_{\alpha\overline{\beta}}a_\gamma) - B_{\alpha\gamma}^\sigma h_{\sigma\overline{\beta}},\\ (\tilde{h}_{\alpha\overline{\beta},\overline{\gamma}} + h_{\alpha\overline{\beta}}a_{\overline{\gamma}}) &= (h_{\alpha\overline{\beta},\overline{\gamma}} + h_{\alpha\overline{\beta}}a_{\overline{\gamma}}) - h_{\alpha\overline{\sigma}}B_{\overline{\beta}\overline{\gamma}}^{\overline{\sigma}}. \end{aligned} \tag{5.19}$$

It follows from (5.8), (5.13) and (5.19) that the coefficients $B_{\alpha\gamma}^\sigma$ can be uniquely determined by making either $\tilde{h}_{\alpha\overline{\beta},\gamma} + h_{\alpha\overline{\beta}}a_\gamma = 0$, or $\tilde{h}_{\alpha\overline{\beta},\overline{\gamma}} + h_{\alpha\overline{\beta}}a_{\overline{\gamma}} = 0$.

At this point, the forms ω_β^α are completely determined, and the normalization procedure is finished. We state the results as follows.

THEOREM 5.1. *Let $\theta \neq 0$ be a fixed ∂-integrable* $(1,0)$*-form, with nondegenerate Levi form χ, on a complex n-manifold, $n \geq 2$. Let $\{\theta^\alpha, \theta\}$, $\{X_\alpha, V\}$ be dual adapted coframe and frame fields for θ. Then there are unique forms, ϕ, τ^α, ω_β^α, satisfying* (5.7), (5.4), (5.10) *and either*

$$h_{\alpha\overline{\beta},\gamma} + h_{\alpha\overline{\beta}} a_\gamma = 0 \quad \textit{or} \quad h_{\alpha\overline{\beta},\overline{\gamma}} + h_{\alpha\overline{\beta}} a_{\overline{\gamma}} = 0. \tag{5.20}$$

We want to interpret the one-forms ω_β^α as the connection forms of a connection ∇ on the sub-bundle $F = \{\theta = 0\} \subset T_{(1,0)}M$, via

$$\nabla X_\alpha = \omega_\alpha^\beta \otimes X_\beta, \tag{5.21}$$

relative to an adapted frame $\{X_\alpha, V\}$ and dual coframe $\{\theta^\alpha, \theta\}$. For this we need to see how the forms ω_α^β change with a change of adapted frame; $\hat{\theta} = \theta$, $\hat{V} = V$,

$$\hat{\theta}^\alpha = \theta^\beta U_\beta^\alpha, \qquad X_\alpha = U_\alpha^\beta \hat{X}_\beta, \qquad \hat{h}_{\rho\overline{\sigma}} U_\alpha^\rho U_{\overline{\beta}}^{\overline{\sigma}} = h_{\alpha\overline{\beta}}. \tag{5.22}$$

Taking the exterior d-derivative of the first equation in (5.22) gives

$$d\hat{\theta}^\alpha = \hat{\theta}^\beta \wedge \hat{\omega}_\beta^\alpha + \theta \wedge \hat{\tau}^\alpha; \tag{5.23}$$

where $\hat{\omega}_\beta^\alpha$ and $\hat{\tau}^\alpha$ are determined by

$$dU_\beta^\alpha = \omega_\beta^\gamma U_\gamma^\alpha - U_\beta^\gamma \hat{\omega}_\gamma^\alpha, \qquad \hat{\tau}^\alpha = \tau^\beta U_\beta^\alpha. \tag{5.24}$$

Next, we take the exterior derivative of the third equation in (5.22) and use (5.10) and (5.24). With obvious notation, this gives, after some cancellation,

$$Dh_{\alpha\overline{\beta}} = \hat{D}h_{\rho\overline{\sigma}} U_\alpha^\rho U_{\overline{\beta}}^{\overline{\sigma}}. \tag{5.25}$$

It follows that the normalizations on ω_α^β carry to the corresponding normalizations of $\hat{\omega}_\beta^\alpha$. Hence, the $\hat{\omega}_\beta^\alpha$ are the normalized (connection) forms relative to the frame $\hat{X}_\alpha$. It follows that (5.21) defines a linear connection on the sub-bundle F. Actually we have two intrinsic connections ∇^1, ∇^2 on F, according to the choice of normalization (5.20).

To bring in the curvature matrix Ω of either connection ∇, we take the exterior d-derivative of the second equation in (5.7). This leads to the first Bianchi identity,

$$0 = \theta^\beta \wedge \Omega_\beta^\alpha + \theta \wedge D\tau^\alpha + \chi \wedge \tau^\alpha, \tag{5.26}$$

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha, \tag{5.27}$$

$$D\tau^\alpha = d\tau^\alpha - \tau^\beta \wedge \left(\omega_\beta^\alpha - \delta_\beta^\alpha \phi\right). \tag{5.28}$$

We also take the exterior derivative of the equation defining $Dh_{\alpha\overline{\beta}}$. This gives

$$D^2 h_{\alpha\overline{\beta}} \equiv d(Dh_{\alpha\overline{\beta}}) - \omega_\alpha^\gamma \wedge Dh_{\gamma\overline{\beta}} + Dh_{\alpha\overline{\gamma}} \wedge \omega_{\overline{\beta}}^{\overline{\gamma}}, \tag{5.29}$$

$$D^2 h_{\alpha\overline{\beta}} = -\Omega_\alpha^\gamma h_{\gamma\overline{\beta}} - h_{\alpha\overline{\gamma}} \Omega_{\overline{\beta}}^{\overline{\gamma}}. \tag{5.30}$$

The detailed consequences of these relations can be worked out as needed.

References

[1] R. Bott, *Collected papers*, vol. **3**, Foliations, Birkhäuser, Boston, 1995. MR 1321886

[2] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271. MR 0425155

[3] C. Godbillon and J. Vey, *Un invariant des feuilletages de codim 1*, C. R. Acad. Sci. Paris Sér. A **273** (1971), 92–95. MR 0283816

[4] R. Harvey and J. Polking, *Fundamental solutions in complex analysis I, II*, Duke Math. J. **46** (1979), 253–340. MR 0534054

[5] F. Kamber and Ph. Tondeur, *Foliated bundles and characteristic classes*, Lect. Notes in Math., vol. **493**, Springer-Verlag, Berlin, 1975. MR 0402773

[6] M. Kuranishi, *Strongly pseudoconvex CR structures over small balls III*, Ann. of Math. (2) **116** (1982), 249–330. MR 0672837

[7] H. B. Lawson, Jr., *The quantitative theory of foliations*, CBMS Regional Conference Series, vol. **27**, AMS, 1977.

[8] L. Nirenberg, *Lectures on linear partial differential equations*, CBMS Regional Conf. No. 17, AMS, Providence, RI, 1973. MR 0450755

[9] F. Treves, *Hypo-analytic structures*, Princeton Univ. Press, Princeton, NJ, 1992. MR 1200459

[10] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41. MR 0520599

[11] S. M. Webster, *Invariants of a complex cotangent line field*, to appear in Bull. Inst. Math. Acad. Sin.

[12] A. Weil, *Sur les theorèmes de de Rham*, Comment. Math. Helv. **26** (1952), 119–145.

Sidney M. Webster, Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: webster@math.uchicago.edu