L^2 ESTIMATES FOR $\bar\partial$ ACROSS A DIVISOR WITH POINCARÉ-LIKE SINGULARITIES

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Dedicated to John D'Angelo, who really enjoys life in a suspended fourth

ABSTRACT. We present results on L^2 estimates for solutions of $\bar\partial$-equations on a Stein manifold with a divisor. The structure of the divisor allows us to introduce weights with certain types of singularities, and the geometry of the manifold near the divisor allows us, by exploiting twisted techniques, to weaken the usual curvature hypotheses that guarantee a solution. We investigate two situations; one in which the weights are not locally integrable, and another in which they can be.

1. Introduction

The present paper concerns the problem of solving the $\bar\partial$-equation with L^2-estimates, under curvature conditions that are less restrictive than those required to obtain the usual L^2 estimate commonly known as *Hörmander's theorem*. The subject matter takes its place in a long tradition of research started by J. J. Kohn, where John D'Angelo has made, and continues to make, creative, fundamental and lasting contributions to the area. I have learned a lot from John in the time I spent in Champaign-Urbana and since then, and the friendship that John and I have developed goes well beyond anything I could say with mathematics or other forms of prose. Nevertheless, it is with pleasure and humility that this paper is dedicated to John, on the occasion of his 60th birthday.

Let (X, ω) be a Stein Kähler manifold and $L \to X$ a holomorphic line bundle equipped with a possibly singular Hermitian metric $e^{-\varphi}$. Let $D \subset X$ be a smooth complex hypersurface, which we identify with a smooth divisor.

Fix a smooth Hermitian metric $e^{-\eta}$ for the line bundle L_D associated to D, and a holomorphic section $w \in H^0(X, L_D)$ such that

 (i) D is exactly the zero divisor of w, and
(ii) $\sup_X |w|^2 e^{-\eta} = 1$.

For a pair of real numbers $s \in (0,1)$ and $\mu \geq 1$, define the metric

$$e^{-\wp_{s,\mu}} = \frac{1}{|w|^2 (\log(\frac{e^\mu}{|w|^2 e^{-\eta}}))^{1-s}}$$

for L_D. The above data define Hilbert spaces of L^2 sections of $L + L_D$, and more generally $(L + L_D)$-valued (p, q)-forms, on X. As usual, there is an extension of the $\bar{\partial}$ operator, initially defined on smooth $(L + L_D)$-valued (p, q)-forms, to a densely-defined operator. We can now state our first main result.

THEOREM 1. *Assume there exists a Kähler form Θ on X such that*

$$dd^c\varphi + \mathrm{Ric}(\omega) \geq \Theta \quad and \quad dd^c\varphi + \mathrm{Ric}(\omega) - \mu^{-1}dd^c\eta \geq \Theta$$

for some positive number μ. Then for any $(L + L_D)$-valued $(0,1)$-form θ with L^2_{loc} coefficients such that $\bar{\partial}\theta = 0$ there exists an L^2_{loc} section u of $L + L_D$ such that

$$\bar{\partial}u = \theta \quad and \quad \int_{X-D} |u|^2 e^{-\varphi} e^{-\wp_{s,\mu}} \, dV_\omega \leq \frac{\mu^{1-s}}{s^2(1-s)} \int_{X-D} |\theta|^2_\Theta e^{-\varphi} e^{-\wp_{s,\mu}} \, dV_\omega,$$

provided the right-hand side is finite.

REMARK 1.1. We emphasize that the form $dd^c\wp_{s,\mu}$, which is smooth on $X - D$, becomes negative and infinite as one approaches D. Thus, for many interesting singular Hermitian metrics $e^{-\varphi}$, the $(1,1)$-current

$$dd^c\varphi + \mathrm{Ric}(\omega) + dd^c\wp_{s,\mu}$$

is never positive on all of X. In particular, Theorem 1 does not follow from Hörmander's theorem, as the hypotheses of the former may hold when those of the latter do not. This is the case, for example, if $e^{-\varphi}$ is any everywhere-smooth Hermitian metric for $L \to X$ with sufficiently positive curvature form. There are many other interesting and useful examples in which Theorem 1 holds while Hörmander's theorem does not.

We illustrate Theorem 1 with the following example.

EXAMPLE 1.2. Let $X := \mathbb{B} \times \mathbb{D}$ where $\mathbb{D} \subset \mathbb{C}$ is the unit disk with Euclidean coordinate w and $\mathbb{B} \subset \mathbb{C}^{n-1}$ the unit ball with coordinates ζ, $D = \{0\} \times \mathbb{B}$, $\eta \equiv 0$ (and $\mu = 1$), $\omega = dd^c|z|^2$ the Euclidean metric, where $z = (\zeta, w)$, and $\Theta = c\omega$. Let $\varphi = \psi + c|z|^2$, where ψ is any plurisubharmonic function. Then

Theorem 1 implies that for any $(0,1)$-current f on $\mathbb{B} \times \mathbb{D}$ with measurable components, such that

$$\int_{\mathbb{B} \times \mathbb{D}} \frac{|f(z)|^2 e^{-\psi(z)} \, dV(z)}{|w|^2 (\log \frac{e}{|w|^2})^{1-s}} < +\infty$$

there exists a locally integrable function u such that $\bar{\partial} u = f$ (in the sense of distributions) and

$$\int_{\mathbb{B} \times \mathbb{D}} \frac{|u(z)|^2 e^{-\psi(z)} \, dV(z)}{|w|^2 (\log \frac{e}{|w|^2})^{1-s}} \leq \frac{e^c}{cs^2(1-s)} \int_{\mathbb{B} \times \mathbb{D}} \frac{|f(z)|^2 e^{-\psi(z)} \, dV(z)}{|w|^2 (\log \frac{e}{|w|^2})^{1-s}}.$$

The special case $n = 1$ (so that $X = \mathbb{D}$) may be rephrased equivalently in the setting of hyperbolic geometry as follows. Let $\varkappa_s(w) := \varphi(w) - (1 + s) \log \log \frac{e}{|w|^2}$ and denote by ω_P the Poincaré metric

$$\omega_P(w) = \frac{\sqrt{-1} \, dw \wedge d\bar{w}}{|w|^2 (\log \frac{e}{|w|^2})^2}.$$

If for some $s \in (0,1)$ one has

$$dd^c \varkappa_s + \text{Ric}(\omega_P) \geq -(1-s)\omega_P + cdd^c |w|^2,$$

then by saying that one can solve $\frac{\partial u}{\partial \bar{w}} = f$ with the estimate

$$\int_{\mathbb{D}} |u(w)|^2 e^{-\varkappa_s(w)} \omega_P(w) \leq C_s \int_{\mathbb{D}} |f(w)|^2 e^{-\varkappa_s(w)} \omega_P(w)$$

as soon as the right-hand side is finite. (Note that the metric ω_P is normalized so that

$$\text{Ric}(\omega_P) = -2\omega_P,$$

which is different from the usual normalization for Kähler–Einstein metrics.) This shows again that the result is stronger than Hörmander's theorem, as the latter requires that $dd^c \varkappa + \text{Ric}(\omega_P) \geq cdd^c |w|^2$. Thus we have lowered the curvature lower bounds by "anything bigger than $\frac{1}{2} \text{Ric}(\omega_P)$."

REMARK 1.3. As we will see, Theorem 1 holds because there is a nice function on a Stein (or essentially stein) manifold in the complement of a smooth divisor. The sort of function to which we are referring has been named *function with self-bounded gradient* by McNeal: f is such a function if it is plurisubharmonic and satisfies

$$|\partial f|^2_{dd^c f} \in L^\infty.$$

The existence of such a function in a given smoothly bounded pseudoconvex domain is linked to the geometry of the boundary of that domain. It would be very interesting to link the existence of such a function to the geometry of a Stein Kähler manifolds with a holomorphic line bundle having nontrivial Bergman structure, for instance, or other analytic geometric properties. Such geometric results have been obtained by McNeal in the case of domains, as

well as other authors in some different situations (see the survey [M-2005] for an excellent discussion and many references), but there is as yet, to the best of the author's knowledge, no systematic understanding of the phenomenon from the point of view of the modern theory of complex analytic geometry of Hermitian holomorphic line bundles on Kähler manifolds.

Theorem 1 has numerous uses in problems of analytic geometry, since it forces the solution u to vanish along the divisor D as soon as u is continuous. The reason for such vanishing is basically that the volume of a neighborhood of any point of D in X is infinite.

There are situations when one wants the volume to be finite. So far as we can tell, the finite volume case is not as symmetric as the infinite volume case of Theorem 1. To state our second main theorem, we will need two metrics. The first is locally integrable, but the second is not (cf. Proposition 4.3).

For a pair of real numbers $s > 1$ and $\mu > 0$, define the metrics

$$e^{-\gamma_{s,\mu}} = \frac{1}{|w|^2 (\log(\frac{e^{\mu}}{|w|^2 e^{-\psi}}))^{1+s}} \quad \text{and} \quad e^{-\sigma_{s,\mu}} = \frac{1}{|w|^2 H_s(\log(\frac{e^{\mu}}{|w|^2 e^{-\psi}}))}$$

for L_D, where

$$H_s(x) := x + \int_1^x \frac{dt}{(1+\sqrt{s})t^s - 1}.$$

One can compute H_s more explicitly for integer values of s; for example, $H_1(x) = x + \frac{1}{2}\log(2x - 1)$. (Of course, we assumed $s > 1$.)

We can now state our second main theorem.

THEOREM 2. *Assume there exists a Kähler form Θ on X such that*

$$dd^c\varphi \geq \Theta \quad \text{and} \quad dd^c\varphi + \mathrm{Ric}(\omega) - \frac{1 + s^{-1/2}}{\mu} dd^c\eta \geq \Theta$$

for some positive number μ. Then for any $(L + L_D)$-valued $(0,1)$-form θ with $L^2_{\ell oc}$ coefficients such that $\bar\partial\theta = 0$ there exists an $L^2_{\ell oc}$ section u of $L + L_D$ such that $\bar\partial u = \theta$ and

$$\int_{X-D} |u|^2 e^{-\varphi} e^{-\gamma_{s,\mu}} \, dV_\omega \leq \mu^{1+s}(1+\sqrt{s})^2 \int_{X-D} |\theta|^2_\Theta e^{-\varphi} e^{-\sigma_{s,\mu}} \, dV_\omega,$$

provided the right-hand side is finite.

As we have already mentioned, the metric $e^{-\sigma_{s,\mu}}$ is singular. However, it is sometimes possible to choose the $(1,1)$-form Θ in a way that improves things.

EXAMPLE 1.4. As in Example 1.2, let X be the unit disk with Euclidean coordinate w, $D = \{0\}$, $\eta \equiv 0$ (and $\mu = 1$), and let ω be the Euclidean metric. However this time let

$$\varphi = \psi - \log\log\frac{e}{|w|^2} \quad \text{and} \quad \Theta = \frac{\sqrt{-1}\, dw \wedge d\bar{w}}{|w|^2 (\log\frac{e}{|w|^2})^2},$$

where ψ is any subharmonic function. Then $dd^c\varphi - \Theta \geq dd^c\psi$, so since $H_s(x) \geq x$, Theorem 2 implies that for any $(0,1)$-form f with measurable coefficients such that

$$\int_{\mathbb{D}} \left|f(w)\right|^2 e^{-\psi(w)} \left(\log \frac{e}{|w|^2}\right)^s dA(w) < +\infty$$

there exists a locally integrable function u such that $\bar{\partial}u = f$ (in the weak sense of distributions) and

$$\int_{\mathbb{D}} \frac{|u(w)|^2 e^{-\psi(w)} \, dA(w)}{|w|^2 (\log \frac{e}{|w|^2})^{1+s}} \leq (1+\sqrt{s})^2 \int_{\mathbb{D}} \left|f(w)\right|^2 e^{-\psi(w)} \left(\log \frac{e}{|w|^2}\right)^s dA(w).$$

The weight on the right-hand side is locally integrable.

REMARK 1.5. It would be very interesting to find an analog of Example 1.4 in higher dimensions, especially in the setting of for Example 1.2. Such a result, if the volume forms on both sides of the estimate were locally integrable, could be used to prove the coherence of so-called *analytic adjoint ideals* in general. These analytic analogs of (algebraic) adjoint ideals (see [L-04] for more on the algebraic ideals) were introduced by Guenancia [G-2012], who proved their coherence for an important class of weights, namely weights $e^{-\varphi}$ such that e^{φ} is Hölder continuous.

REMARK 1.6. It is straight-forward to extend Theorems 1 and 2 to so-called *essentially Stein manifolds*, namely, manifolds that have a complex subvariety whose complement is Stein. (One needs to assume, in addition, that there is such a subvariety that does not contain D.) In that case, the L^2 estimates are obtained on the Stein subset, and extend across the omitted subvariety when the data is appropriate. Thus, for example, Theorems 1 and 2 hold if X is a projective manifold. There are other versions as well: for example, one can take X to be weakly pseudoconvex if one assumes the metric $e^{-\varphi}$ has only analytic singularities; the proof in this case has to be slightly modified, but it is well-known how to modify it. (See, for example, [D-2001] or references therein.)

2. Twisting

2.1. The basic identity. Let $E \to X$ be a holomorphic line bundle with smooth Hermitian metric $e^{-\phi}$. Let $\Omega \subset\subset X$ be a smoothly bounded domain whose (possibly empty) boundary has real codimension 1, and fix a smooth defining function ρ for Ω, that is, a function such that $\Omega = \{\rho < 0\}$. One can choose such a function ρ so that $|d\rho|_\omega \equiv 1$ on $\partial\Omega$. Then for any smooth E-valued $(0,1)$-form β in the domain of $\bar{\partial}_\phi^*$—the adjoint of $\bar{\partial}$ over Ω, which depends on both ϕ and dV_ω, though we have suppressed the latter in our

notation—one has the following well-known basic identity.

$$(1) \qquad \int_\Omega |\bar{\partial}_\phi^* \beta|^2 e^{-\phi} \, dV_\omega + \int_\Omega |\bar{\partial}\beta|_\omega^2 e^{-\phi} \, dV_\omega$$

$$= \int_\Omega \left(dd^c\phi + \mathrm{Ric}(\omega)\right)^\omega (\beta \wedge \bar{\beta}) e^{-\phi} \, dV_\omega$$

$$+ \int_\Omega |\overline{\nabla}\beta|^2 e^{-\phi} \, dV_\omega + \int_{\partial\Omega} \left(dd^c\rho\right)^\omega (\beta \wedge \bar{\beta}) \, dS_{\omega,\Omega}.$$

The superscript ω means that we raise the appropriate indices using the Kähler metric ω. The exact definition of the last two terms on the right-hand side of (1) is not particularly important. It is only important that the first term is nonnegative, while the second term is nonnegative when Ω has pseudoconvex boundary, which we assume from now on.

2.2. The twist. Let $\tau : \Omega \to (0,\infty)$ be smooth, and define the metric $e^{-\psi}$ for $E|_\Omega$ by the relation

$$e^{-\phi} = \tau e^{-\psi}.$$

Then one computes that

$$\bar{\partial}_\phi^* \beta = \bar{\partial}_\psi^* \beta - \frac{1}{\tau}(\partial\tau)^\omega(\beta) \quad \text{and} \quad dd^c\phi = dd^c\psi - \frac{dd^c\tau}{\tau} + \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{\tau^2}.$$

Substituting these two terms into (1) then yields the following identity.

$$(2) \qquad \int_\Omega \tau |\bar{\partial}_\psi^* \beta|^2 e^{-\psi} \, dV_\omega + \int_\Omega \tau |\bar{\partial}\beta|_\omega^2 e^{-\psi} \, dV_\omega$$

$$= \int_\Omega \left(\tau dd^c\psi + \tau \, \mathrm{Ric}(\omega) - dd^c\tau\right)^\omega (\beta \wedge \bar{\beta}) e^{-\psi} \, dV_\omega$$

$$+ 2\,\mathrm{Re} \int_\Omega \bar{\partial}_\psi^* \beta \overline{(\partial\tau)^\omega(\beta)} e^{-\psi} \, dV_\omega$$

$$+ \int_\Omega \tau |\overline{\nabla}\beta|^2 e^{-\psi} e^{-\wp_{s,\mu}} \, dV_\omega + \int_{\partial\Omega} \tau \left(dd^c\rho\right)^\omega (\beta \wedge \bar{\beta}) \, dS_{\omega,\Omega}.$$

2.3. A twisted estimate. Applying the Cauchy–Schwarz inequality to the term on the second line of (2) yields the following lemma, which is only a slight variant of a result in [M-2005] and [MV-2007].

LEMMA 2.1. *Let $A > 0$ be any continuous function. Then for any smooth $(0,1)$-form β in the domain of $\bar{\partial}_\psi^*$*

$$\int_\Omega (\tau + A)|\bar{\partial}_\psi^* \beta|^2 e^{-\psi} \, dV_\omega + \int_\Omega \tau |\bar{\partial}\beta|^2 e^{-\psi} \, dV_\omega$$

$$\geq \int_\Omega \left(\tau \left(dd^c\psi + \mathrm{Ric}(\omega)\right) - dd^c\tau - A^{-1}\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau\right)^\omega (\beta \wedge \bar{\beta}) e^{-\psi} \, dV_\omega$$

$$+ \int_\Omega \tau |\overline{\nabla}\beta|^2 e^{-\psi} e^{-\wp_{s,\mu}} \, dV_\omega + \int_{\partial\Omega} \tau \left(dd^c\rho\right)^\omega (\beta \wedge \bar{\beta}) \, dS_{\omega,\Omega}.$$

3. The proof of Theorem 1

3.1. The auxiliary functions τ and A, and the metric $e^{-\psi}$. Fix three real numbers $\gamma > 1$, $\varepsilon > 0$ and $\mu \geq 1$. Eventually we will let $\varepsilon \to 0$ and $\gamma \to 1$. Let

$$v := \log |w|^2 e^{-\eta} \quad \text{and} \quad a := \gamma - \frac{1}{\mu} \log\left(e^v + \varepsilon^2\right).$$

If $\varepsilon > 0$ is given, we choose γ so that

$$a > 1.$$

Let

$$h(x) := x^\alpha$$

for some number $\alpha \in (0, 1)$ to be specified later. We define

$$\tau := h(a) \quad \text{and} \quad A = \frac{\tau}{\delta},$$

where $\delta > 0$ is a positive number to be chosen below. While it is straightforward to compute the quantities $\partial \tau \wedge \bar{\partial} \tau$ and $\partial \bar{\partial} \tau$, the resulting formula must be organized correctly in order to be useful to us.

In the next few computations, it is sometimes helpful to think of e^v as the square length of the section w, so that $e^v = (e^{v/2})^2$. With this in mind, we compute that

$$\partial a = -\frac{2e^{v/2}\,\partial e^{v/2}}{\mu(e^v + \varepsilon^2)} \quad \text{and} \quad \bar{\partial} a = -\frac{2e^{v/2}\,\bar{\partial}e^{v/2}}{\mu(e^v + \varepsilon^2)} = -\frac{e^v}{\mu(e^v + \varepsilon^2)}\,\bar{\partial}v,$$

and therefore that

$$\partial a \wedge \bar{\partial} a = \frac{4e^v\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu^2(e^v + \varepsilon^2)^2}.$$

To compute $\partial \bar{\partial} a$ we use the second formula for $-\bar{\partial} a$. Then

$$
\begin{aligned}
-\partial\bar{\partial}a &= \frac{e^v}{\mu(e^v + \varepsilon^2)}\,\partial\bar{\partial}v + \frac{\partial(e^v) \wedge \bar{\partial}v}{\mu(e^v + \varepsilon^2)} - \frac{e^v\,\partial(e^v) \wedge \bar{\partial}v}{\mu(e^v + \varepsilon^2)^2} \\
&= \frac{e^v}{\mu(e^v + \varepsilon^2)}\,\partial\bar{\partial}v + \frac{\varepsilon^2\,\partial(e^v) \wedge \bar{\partial}v}{\mu(e^v + \varepsilon^2)^2} \\
&= \frac{e^v}{\mu(e^v + \varepsilon^2)}\left(\frac{2\pi}{\sqrt{-1}}[D] - \partial\bar{\partial}\eta\right) + \frac{4\varepsilon^2\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2} \\
&= -\frac{e^v}{\mu(e^v + \varepsilon^2)}\,\partial\bar{\partial}\eta + \frac{4\varepsilon^2\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2},
\end{aligned}
$$

where $[D]$ denotes the current of integration over D, and we have use the facts that

$$e^v|_D \equiv 0 \quad \text{and} \quad \partial\left(e^v\right) \wedge \bar{\partial}v = 4\partial\left(e^{v/2}\right) \wedge \bar{\partial}\left(e^{v/2}\right).$$

Now, since

$$h'(x) = \frac{\alpha}{x^{1-\alpha}} \quad \text{and} \quad h''(x) = \frac{-\alpha(1-\alpha)}{x^{2-\alpha}},$$

we see that

$$
\begin{aligned}
-dd^c\tau &= h'(a)\big(-dd^c a\big) - h''(a)\,\partial a \wedge \bar{\partial} a \\
&= \frac{\alpha}{\mu a^{1-\alpha}} \left[\frac{e^v}{e^v + \varepsilon^2}\big(-dd^c\eta\big) + \frac{4\varepsilon^2\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{(e^v + \varepsilon^2)^2} \right] \\
&\quad + \frac{\alpha(1-\alpha)}{a^{2-\alpha}} \frac{4e^v\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu^2(e^v + \varepsilon^2)^2}
\end{aligned}
$$

while

$$(3) \qquad \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A} = \delta \frac{(h'(a))^2}{h(a)}\,\partial a \wedge \bar{\partial} a = \frac{\delta \alpha^2}{a^{2-\alpha}} \frac{4e^v\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu^2(e^v + \varepsilon^2)^2}.$$

Consequently,

$$
\begin{aligned}
-dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A} &= \frac{-\alpha e^v}{\mu a^{1-\alpha}(e^v + \varepsilon^2)} dd^c\eta + \frac{\alpha}{a^{1-\alpha}} \frac{4\varepsilon^2\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2} \\
&\quad + \frac{\alpha(1-(1+\delta)\alpha)}{\mu^2 a^{1-\alpha}} \frac{4e^v\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{(e^v + \varepsilon^2)^2}.
\end{aligned}
$$

At this point, to shrink the formulas a bit, we set

$$\delta = \frac{1-\alpha}{\alpha},$$

so that the last term of (3) vanishes.

Finally, we define the metric $e^{-\psi}$ for $L + L_D$ by the formula

$$\psi = \varphi + \log|w|^2.$$

3.2. A priori estimate. One computes that $\tau dd^c\psi = a^\alpha(dd^c\varphi + 2\pi[D])$, and we find that

$$
\begin{aligned}
\tau dd^c\psi &+ \tau \operatorname{Ric}(\omega) - dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A} \\
&= 2\pi a^\alpha[D] + a^\alpha\big(dd^c\varphi + \operatorname{Ric}(\omega)\big) - \frac{\alpha e^v}{\mu a^{1-\alpha}(e^v + \varepsilon^2)} dd^c\eta \\
&\quad + \frac{\alpha}{a^{1-\alpha}} \frac{4\varepsilon^2\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2} \\
&= 2\pi a^\alpha[D] + \frac{a^\alpha \varepsilon^2}{e^v + \varepsilon^2}\big(dd^c\varphi + \operatorname{Ric}(\omega)\big) \\
&\quad + \frac{\alpha a^\alpha e^v}{e^v + \varepsilon^2}\big(dd^c\varphi + \operatorname{Ric}(\omega) - \mu^{-1}dd^c\eta\big)
\end{aligned}
$$

$$+\frac{\alpha}{a^{1-\alpha}}\frac{4\varepsilon^2\,\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{\mu(e^v+\varepsilon^2)^2}$$

$$\geq\alpha\tau\Theta.$$

Define the operator T, mapping sections of $L+L_D$ to $L+L_D$-valued $(0,1)$-forms, and S, mapping $L+L_D$-valued $(0,1)$-forms to $L+L_D$-valued $(0,2)$-forms, by the formulas

$$Tf=\frac{1}{1-\alpha}\bar{\partial}(\sqrt{\tau}f)\quad\text{and}\quad S\beta:=\sqrt{\tau}\,\bar{\partial}\beta.$$

Then Lemma 2.1 gives us the following lemma.

LEMMA 3.1. *Let $\Omega\subset X$ be a pseudoconvex domain with smooth boundary of real codimension* 1. *Then for any $(0,1)$-form β in the domain of $\bar{\partial}^*_\psi$*

$$(4)\qquad\int_\Omega\left|T^*_\psi\beta\right|^2e^{-\psi}\,dV_\omega+\int_\Omega|S\beta|^2e^{-\psi}\,dV_\omega\geq\alpha\int_\Omega\langle\Theta^\omega,\beta\wedge\bar{\beta}\rangle\tau e^{-\psi}\,dV_\omega.$$

Proof. For smooth forms β, the result follows from Lemma 2.1 and the work of the previous paragraph. For general forms (4) holds by the well-known density of smooth forms in the domain of $\bar{\partial}^*$ in the graph norm. □

3.3. Conclusion of the proof of Theorem 1. Let θ be any $\bar{\partial}$-closed $(0,1)$-form with $L^2_{\ell oc}$ coefficients and such that

$$\int_{X-D}|\theta|^2_\Theta e^{-\varphi}e^{-\wp_{s,\mu}}\,dV_\omega<+\infty.$$

Then for any β in the domain of $\bar{\partial}^*$, one has

$$\left|\int_{X-D}\langle(1-\alpha)^{-1}\theta,\beta\rangle_\omega e^{-\varphi}|w|^{-2}\,dV_\omega\right|^2$$

$$\leq\left(\int_{X-D}\left|(1-\alpha)^{-1}\theta\right|^2_{\tau\Theta}e^{-\varphi}|w|^{-2}\,dV_\omega\right)\left(\int_{X-D}\langle\Theta^\omega,\beta\wedge\bar{\beta}\rangle\tau e^{-\psi}\,dV_\omega\right)$$

$$\leq\frac{1}{\alpha}\left(\int_{X-D}\left|(1-\alpha)^{-1}\theta\right|^2_\Theta e^{-\varphi}\frac{dV_\omega}{\tau|w|^{-2}}\right)\left(\|T^*_\psi\beta\|^2+\|S\beta\|^2\right).$$

The last inequality follows from Lemma 4. As an application of standard functional analysis shows, there exists an $L^2_{\ell oc}$ section U of $L+L_D$, such that

$$TU=\frac{1}{1-\alpha}\theta$$

and

$$\int_\Omega|U|^2e^{-\varphi}|w|^{-2}\,dV_\omega\leq\frac{1}{\alpha}\left(\int_{X-D}\left|(1-\alpha)^{-1}\theta\right|^2_\Theta e^{-\varphi}\frac{dV_\omega}{\tau|w|^{-2}}\right).$$

But since $TU=\frac{1}{1-\alpha}\bar{\partial}(\sqrt{\tau}U)$, the section $u:=\sqrt{\tau}U$ satisfies

$$\bar{\partial}u=\theta\quad\text{and}\quad\int_\Omega|u|^2e^{-\varphi}\frac{dV_\omega}{\tau|w|^2}\leq\frac{1}{\alpha(1-\alpha)^2}\left(\int_{X-D}|\theta|^2_\Theta e^{-\varphi}\frac{dV_\omega}{\tau|w|^2}\right).$$

Finally, let $\alpha = 1 - s$. Then we have the estimate

$$\int_\Omega \frac{|u|^2 e^{-\varphi} \, dV_\omega}{|w|^2 (\mu\gamma - \log(|w|^2 e^{-\psi} + \varepsilon^2))^{1-s}}$$
$$\leq \frac{\mu^{1-s}}{s^2(1-s)} \left(\int_{X-D} \frac{|\theta|_\Theta^2 e^{-\varphi} \, dV_\omega}{|w|^2 (\mu\gamma - \log(|w|^2 e^{-\psi} + \varepsilon^2))^{1-s}} \right).$$

Since the volume form on the right-hand side is less singular than $e^{-\wp_{s,\mu}} \, dV_\omega$, our hypothesis about θ bounds the right-hand side by

$$\frac{\mu^{1-s}}{s^2(1-s)} \int_{X-D} |\theta|_\Theta^2 e^{-\varphi} e^{-\wp_{s,\mu}} \, dV_\omega.$$

Now, the solutions u obtained above depend on the parameters γ and ε. However, since the right-hand side is uniformly bounded, the weak*-compactness of balls in Hilbert space allows us to extract a weakly convergent subsequence, which we also call u. But by construction, u also lies in the ball in question, which means that u satisfies the estimate claimed in the statement of Theorem 1. The proof of Theorem 1 is finally complete.

4. The proof of Theorem 2

4.1. The key difference between the proofs of Theorem 1 and Theorem 2.
To get some perspective, we make the some observations about what happened in the proof of Theorem 1. In that proof, as in the proof of Theorem 2 which we will present shortly, one uses the twisted technique. This technique requires one to choose two functions, τ and A, and a metric, $e^{-\psi}$. We had decided to choose A to be proportional to τ. Given this initial choice, the next goal is to make the *twisted sub-curvature*

$$-dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar\partial\tau}{A} = -dd^c\tau - \delta\frac{\sqrt{-1}\,\partial\tau \wedge \bar\partial\tau}{\tau}$$

as large as possible. To reduce the complexity of the problem (for better or worse) we took τ to be a composition $H(a)$ of a function function H of one real variable and a function a whose main property is that $-dd^c a$ is an approximation to the current of integration over D. In terms of these functions, we computed that

$$(5) \qquad dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar\partial\tau}{c\tau}$$
$$= H'(a)\left(-dd^c a\right) - \left(H''(a) + \delta\frac{(H'(a))^2}{H(a)} \right) \partial a \wedge \bar\partial a.$$

One way to treat the problem is to make the second term on the right-hand side is vanish. If nothing else, making the second term vanish does simplify

the problem; the condition is a nonlinear ODE for the function H, that, it turns out, is easy to solve. Indeed, we have

$$0 = \frac{H''}{H'} + \delta\frac{H'}{H} = \big(\log(H') + \log(H^\delta)\big)' = \big(\log(H'H^\delta)\big)',$$

and therefore, for some constants c_o and c_1,

$$\big(H(x)\big)^{\delta+1} = (\delta + 1)e^{c_o}x + c_1.$$

This shows why we chose $\delta = (1 - \alpha)/\alpha$.

REMARK 4.1. It is still not clear to the author if the strategy of writing $\tau = H(a)$ is optimal in any sense other than perhaps simplicity, or if requiring the vanishing of the second term on the right-hand side of (5) maximizes the quadratic form in question. However, the heuristic reasoning we used is the following: one is asking for an increasing function H that blows up, but such a condition is likely to encourage the convexity of $H^{\delta+1}$, which controls the second term on the right; the more convex $H^{\delta+1}$, the more negative the second term becomes.

In the proof of Theorem 2, we will take the function A to be different from a multiple of τ. Since the volume form in question is locally integrable at each point of X, one can use an idea first introduced in our joint paper with McNeal [MV-2007]. While we will take an ad hoc approach, a more elaborate version of the idea can be found in the aforementioned paper. Even so, the choice of functions in the twisted estimate of Lemma 2.1 remains mostly ad-hoc, and it would be better to understand it more thoroughly.

4.2. The auxiliary functions τ and A, and the metric $e^{-\psi}$. From the outset we take our metric $e^{-\psi}$ as in the proof of Theorem 1:

$$\psi = \varphi + \log|w|^2.$$

We also adopt the notation

$$v := \log|w|^2 - \eta \quad \text{and} \quad a := \gamma - \frac{1}{\mu}\log\big(e^v + \varepsilon^2\big)$$

from the previous section; once again $\varepsilon > 0$ is small and $\gamma > 1$ is chosen so that $a \geq 1$.

For the functions τ and A, we exploit the ideas in [MV-2007, Section 3.3]. Indeed, we take

$$\tau = a + h(a) \quad \text{and} \quad A = \frac{(1 + h'(a))^2}{-h''(a)},$$

which immediately implies that

(6) $$-dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A} = \big(1 + h'(a)\big)\big(-dd^c a\big).$$

We choose the function h according to the formula

$$h(x) = h_\delta(x) := \int_1^x \frac{\delta\, dt}{(1+\delta)t^s - \delta},$$

for $\delta > 0$ to be chosen shortly. We chose this particular function h to satisfy the ODE

$$h''(x) + \frac{s\delta}{(1+\delta)x^{1+s}}\left(1 + h'(x)\right)^2 = 0,$$

and therefore

$$A = \frac{1+\delta}{\delta}\frac{a^{1+s}}{s}.$$

We made this choice because one expects that A is generally larger than τ, as is the case in our situation. Indeed, for $x \geq 1$ we have

$$(7) \qquad\qquad h(x) \leq \int_1^x \delta\, dt \leq \delta(x-1),$$

and therefore $\tau \leq (1+\delta)a - \delta \leq (1+\delta)a^{s+1}$. In particular, we find that

$$(8) \qquad\qquad \tau + A \leq \frac{(1+\delta)(1+\delta s)a^{1+s}}{\delta s},$$

an estimate we will use later. We note also that, as a function of δ, the term $\frac{(1+\delta)(1+\delta s)}{\delta s}$ has the minimum value $(1+\sqrt{s})^2$, which is achieved at $\delta = 1/\sqrt{s}$; this is our choice for δ.

We will make use of the following lemma below.

LEMMA 4.2. *For all $x \geq 1$ one has the estimate $1 + h'(x) \leq (1+\delta)(x+h(x))$. In particular,*

$$(1+\delta)\tau \geq 1 + h'(a).$$

Proof. Consider the function $F(x) = (1+\delta)(x + h(x)) - (1 + h'(x))$. Then with $Y = x^s$,

$$\begin{aligned}
F'(x) &= (1+\delta)\left(1 + h'(x)\right) - h''(x) \\
&= (1+\delta)\left(1 + \frac{\delta}{(1+\delta)Y - \delta}\right) + \frac{s\delta(1+\delta)Y/x}{((1+\delta)Y - \delta)^2} \\
&\geq 1 + \frac{\delta}{(1+\delta)Y - \delta} + \frac{s\delta(1+\delta)Y/x}{((1+\delta)Y - \delta)^2} \\
&= \frac{(((1+\delta)Y - \delta)^2 + \delta((1+\delta)Y - \delta) + s\delta(1+\delta)Y/x)}{((1+\delta)Y - \delta)^2} \\
&= \frac{((1+\delta)^2 Y^2 - \delta(1+\delta)Y + \frac{s\delta}{x}(1+\delta)Y)}{((1+\delta)Y - \delta)^2} \\
&= (1+\delta)Y\frac{((1+\delta)Y - \delta + \frac{s\delta}{x})}{((1+\delta)Y - \delta)^2} > 0.
\end{aligned}$$

Therefore F is increasing, that is, for all $x \geq 1$, $F(x) \geq F(1) = 0$. The proof is finished. □

We end with the following proposition.

PROPOSITION 4.3. *The metric $e^{-\sigma_{\mu,s}}$ is not locally integrable for any $s > 0$.*

Proof. By (7) we see that the function H_s defined just prior to the statement of Theorem 2 satisfies

$$H_s(x) \leq (1+\delta)x - \delta \leq (1+\delta)x,$$

where $\delta = \frac{1}{\sqrt{s}}$. But then

$$e^{-\sigma_{s,\mu}} \geq \frac{1}{1+\delta} \frac{1}{|w|^2 \log(\frac{e^\mu}{|w|^2 e^{-\eta}})},$$

and the right-hand side is not locally integrable in any neighborhood of any point of D. □

4.3. A priori estimate. From Section 3, we have

$$-dd^c a = -\frac{e^v}{\mu(e^v + \varepsilon^2)} dd^c \eta + \frac{4\varepsilon^2 \sqrt{-1}\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2},$$

and therefore

$$\tau\big(dd^c\psi + \mathrm{Ric}(\omega)\big) - dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A}$$

$$= 2\pi\tau[D] + \tau\big(dd^c\varphi + \mathrm{Ric}(\omega)\big) + \big(1 + h'(a)\big)\big(-dd^c a\big)$$

$$= 2\pi\tau[D] + \left(\tau - \frac{(1 + h'(a))}{1+\delta}\right)\big(dd^c\varphi + \mathrm{Ric}(\omega)\big)$$

$$+ \frac{(1 + h'(a))}{1+\delta}\left(\frac{\varepsilon^2}{e^v + \varepsilon^2}\big(dd^c\varphi + \mathrm{Ric}(\omega)\big)\right)$$

$$+ \frac{(1 + h'(a))}{1+\delta}\frac{e^v}{e^v + \varepsilon^2}\left(dd^c\varphi + \mathrm{Ric}(\omega) - \frac{1+\delta}{\mu}dd^c\eta\right)$$

$$+ \big(1 + h'(a)\big)\left(\frac{4\varepsilon^2 \sqrt{-1}\,\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{\mu(e^v + \varepsilon^2)^2}\right).$$

By Lemma 4.2 and the hypotheses of Theorem 2, namely

$$dd^c\varphi + \mathrm{Ric}(\omega) \geq \Theta \quad \text{and} \quad dd^c\varphi + \mathrm{Ric}(\omega) - \frac{1+\delta}{\mu}dd^c\eta \geq \Theta,$$

together with the obvious nonnegativity of certain terms, we conclude that

$$\tau\big(dd^c\psi + \mathrm{Ric}(\omega)\big) - dd^c\tau - \frac{\sqrt{-1}\,\partial\tau \wedge \bar{\partial}\tau}{A} \geq \tau\Theta.$$

From here on, things proceed even more similarly to the proof of Theorem 1. Define the operator T, mapping sections of $L + L_D$ to $L + L_D$-valued $(0,1)$-forms, and S, mapping $L + L_D$-valued $(0,1)$-forms to $L + L_D$-valued $(0,2)$-forms, by the formulas

$$Tf = \bar{\partial}(\sqrt{\tau + A}f) \quad \text{and} \quad S\beta := \sqrt{\tau}\,\bar{\partial}\beta.$$

Then we have the following lemma.

LEMMA 4.4. *Let $\Omega \subset X$ be a pseudoconvex domain with smooth boundary of real codimension* 1. *Then for any* $(0,1)$-*form β in the domain of $\bar{\partial}_{\psi}^{*}$*

$$(9) \qquad \int_{\Omega} |T_{\psi}^{*}\beta|^{2} e^{-\psi}\, dV_{\omega} + \int_{\Omega} |S\beta|^{2} e^{-\psi}\, dV_{\omega} \geq \int_{\Omega} \langle \Theta^{\omega}, \beta \wedge \bar{\beta} \rangle \tau e^{-\psi}\, dV_{\omega}.$$

The proof is directly analogous to that of Lemma 3.1, so we omit it.

4.4. Conclusion of the proof of Theorem 2. Let θ be a $(0,1)$-forms with $L_{\ell oc}^{2}$ coefficients such that

$$\int_{X-D} |\theta|_{\Theta}^{2} e^{-\varphi} e^{-\wp_{s,\mu}}\, dV_{\omega} < +\infty.$$

Then for any β in the domain of $\bar{\partial}^{*}$, one has

$$\left| \int_{X-D} \langle \theta, \beta \rangle_{\omega} e^{-\varphi} |w|^{-2}\, dV_{\omega} \right|^{2}$$

$$\leq \left(\int_{X-D} |\theta|_{\tau\Theta}^{2} e^{-\varphi} |w|^{-2}\, dV_{\omega} \right) \left(\int_{X-D} \langle \Theta^{\omega}, \beta \wedge \bar{\beta} \rangle \tau e^{-\psi}\, dV_{\omega} \right)$$

$$\leq \left(\int_{X-D} |\theta|_{\Theta}^{2} e^{-\varphi} \frac{dV_{\omega}}{\tau |w|^{-2}} \right) \left(\|T_{\psi}^{*}\beta\|^{2} + \|S\beta\|^{2} \right).$$

The last inequality follows from Lemma 9. Again by the usual method, there exists an $L_{\ell oc}^{2}$ section U of $L + L_D$, such that $TU = \theta$ and

$$\int_{\Omega} |U|^{2} e^{-\varphi} |w|^{-2}\, dV_{\omega} \leq \frac{1}{\alpha(1-\alpha)} \left(\int_{X-D} |(1-\alpha)^{-1}\theta|_{\Theta}^{2} e^{-\varphi} \frac{dV_{\omega}}{\tau |w|^{-2}} \right).$$

But since $TU = \bar{\partial}(\sqrt{\tau + A}U)$, the section $u := \sqrt{\tau + A}U$ satisfies

$$\bar{\partial}u = \theta \quad \text{and} \quad \int_{\Omega} \frac{|u|^{2} e^{-\varphi}\, dV_{\omega}}{|w|^{2}(\tau + A)} \leq \left(\int_{X-D} |\theta|_{\Theta}^{2} e^{-\varphi} \frac{dV_{\omega}}{\tau |w|^{2}} \right).$$

But by (8) we have the estimate

$$\int_{\Omega} \frac{|u|^{2} e^{-\varphi}\, dV_{\omega}}{|w|^{2}(\mu\gamma - \log(|w|^{2}e^{-\psi} + \varepsilon^{2}))^{1+s}} \leq \mu^{1+s}(1 + \sqrt{s})^{2} \left(\int_{X-D} \frac{|\theta|_{\Theta}^{2} e^{-\varphi}\, dV_{\omega}}{|w|^{2}\tau} \right).$$

Since the volume form on the right-hand side increases to the more singular measure $e^{-\sigma_{s,\mu}}\, dV_\omega$ as $\varepsilon \to 0$ and then $\gamma \to 1$, the right-hand side is bounded by

$$\mu^{1+s}(1+\sqrt{s})^2 \int_{X-D} |\theta|_\Theta^2 e^{-\varphi} e^{-\sigma_{s,\mu}}\, dV_\omega,$$

As in the proof of Theorem 1, the solutions u obtained above depend on the parameters γ and ε, but by weak*-compactness we can extract a weakly convergent subsequence, which we also call u. But by construction, u also lies in the ball in question, which means that u satisfies the estimate claimed in the statement of Theorem 2. The proof of Theorem 2 is complete.

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