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OKA'S LEMMA, CONVEXITY, AND INTERMEDIATE POSITIVITY CONDITIONS

A.-K. HERBIG AND J. D. MCNEAL

Dedicated to J. P. D'Angelo, whose work celebrates positivity in many forms

ABSTRACT. A new proof of Oka's lemma is given for smoothly bounded, pseudoconvex domains $\Omega \subset\subset \mathbb{C}^n$. The method of proof is then also applied to other convexity-like hypotheses on the boundary of Ω.

1. Introduction

If Ω is a domain of holomorphy in $\mathbb{C}^n$, Oka's lemma states that $\phi(z) = -\log d_{b\Omega}(z)$ is plurisubharmonic for z in Ω, where $d_{b\Omega}(z)$ denotes the Euclidean distance from z to $\Omega^c = \mathbb{C}^n \setminus \Omega$. This is a foundational result in several complex variables, with ϕ serving as the initial building block in various constructions of holomorphic functions on Ω, for example, Theorems 4.2.2, 4.4.3, and 4.4.4 in [9], Theorem 3.18 in [13], Theorems 3.4.5 and 5.4.2 in [11], and Theorem D.4 in Chapter IX of [7], among others, hinge on Oka's lemma.

The aim of this paper is to give a new proof of Oka's lemma when Ω has smooth boundary $b\Omega$, and to examine the result as an instance where positivity conditions on the Hessian of a function f are "spread" to a wider set of points and vectors by taking functional combinations of f of the form $\chi \circ f$, for $\chi : \mathbb{R} \to \mathbb{R}$.

This point of view is easiest to describe via the signed distance-to-the-boundary function $\delta = \delta_{b\Omega}$; see (3.1) below. If Ω is a smoothly bounded

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domain of holomorphy, then Ω is Levi pseudoconvex, see, for example, Theorem 2.6.12 in [9]. Since δ is a defining function for Ω, it follows that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \delta}{\partial z_j\, \partial \bar{z}_k}(p) V_j \overline{V}_k \geq 0, \quad \text{if } p \in b\Omega, V \in \mathbb{C}T_p(b\Omega). \tag{1.1}$$

Oka's lemma says that (1.1) implies $\phi = -\log(-\delta)$ is plurisubharmonic on Ω, that is,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \delta}{\partial z_j\, \partial \bar{z}_k}(z) W_j \overline{W}_k + \frac{1}{d(z)} \left| \sum_{j=1}^{n} \frac{\partial \delta}{\partial z_j}(z) W_j \right|^2 \geq 0, \quad \text{if } z \in \Omega, W \in \mathbb{C}^n. \tag{1.2}$$

Notice that the quadratic form in (1.1) is only nonnegative-definite at a small set of points in $\overline{\Omega}$ (namely, $p \in b\Omega$) and in certain directions (namely, $V \in \mathbb{C}T_p(b\Omega)$), while the form in (1.2) is nonnegative-definite at all points in Ω and in all directions. Thus, Oka's lemma asserts that the positivity (on its complex Hessian) ϕ inherits from δ is more widespread than condition (1.1) implies at first glance.

This paper grew out of our desire to find a direct proof of Oka's lemma. The standard proof, see Theorems 2.6.12 in [9], Theorem 3.3.5 in [11], E.5.11 in [13], is by contradiction: assuming (1.2) is violated at some $z \in \Omega$ and in some direction W, a boundary point p and a direction $V \in \mathbb{C}T_p(b\Omega)$ are found where (1.1) cannot hold. The advantage of the canonical approach is the usual one: negation of the nonstrict inequalities result in strict inequalities and these are easier to deal with than (1.1) and (1.2) themselves.

Our proof deals with the semi-definite inequalities (1.1) and (1.2) directly which, we believe, has intrinsic interest. The proof given here re-casts the semi-definite conclusion (1.2) as another nonstrict inequality on the *square* of the distance function, see (4.1), then uses simple Taylor analysis to show that (1.1) implies (4.1). Variational arguments often fail when one tries to pass from one nonstrict inequality to another, so their success in this instance merits mention. The local constancy of $\|\nabla\delta\|$ plays a key role in our approach to this issue.

Once Oka's lemma (Theorem 4.1) is proved in this way, it is illuminating to apply this method to other convexity-like hypotheses on $b\Omega$ besides pseudoconvexity. The most natural hypotheses of this kind are: (i) the real Hessian of δ nonnegative on the real tangent space to $b\Omega$ (convexity), (ii) the real Hessian of δ nonnegative on the complex tangent space to $b\Omega$ ($\mathbb{C}$-convexity), and (iii) the complex Hessian of δ nonnegative on the real tangent space to $b\Omega$ (δ plurisubharmonic "on the boundary"). We examine how these hypotheses yield widespread nonnegativity on the Hessians of δ or $-\log(-\delta)$ in Sections 5 and 6. We follow the method used to prove Theorem 4.1 quite closely in these sections, in order to identify how the different hypotheses lead to different conclusions. After our paper was written, we learned that [1] earlier gave a proof of the $\mathbb{C}$-convex case along these lines, so our proof of Theorem 6.1 merely reprises their proof.

In the final part of Section 6, we examine nonnegativity of the complex Hessian of δ on cones of vectors containing the complex tangent space and lying in the real tangent space. Under this hypothesis, we show (Theorem 6.3) how the size of the Diederich–Fornæss exponent ([12])—but only for the fixed defining function δ—is determined by the angle of the cone of nonnegativity. Theorem 6.3 gives a spectrum of results that naturally interpolate between the conclusion given in Theorem 4.1 and that given in Theorem 6.2. This result explains an example given in [2], where no $\eta > 0$ exists such that $-(-\delta)^\eta$ is plurisubharmonic, and is also related to results in [4], [5] which deals with situations where η can be chosen close to 1 (but for defining functions other than δ).

2. Tangent spaces and Hessians

Succinct notation for Hessians (real and complex) of smooth functions and tangent spaces (real and complex) will make the arguments in Sections 4–6 transparent. We present these objects using global coordinates for brevity, mentioning only the invariance needed in the subsequent proofs.

Let $\Omega \subset \mathbb{C}^n$ denote a domain with smooth boundary $b\Omega$. A local defining function for Ω in a neighborhood U of $p \in b\Omega$, is a real-valued function $r \in C^\infty(U)$ satisfying $U \cap \Omega = \{z \in U : r(z) < 0\}$ and $\nabla r(z) \neq 0$ for $z \in U$.

Let $(z_1, \dots, z_n)$ denote the standard coordinates on $\mathbb{C}^n$, with $z_k = x_{2k-1} + ix_{2k}$ for $k = 1, \dots, n$. The usual Cauchy–Riemann vector fields are written

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2k-1}} - i\frac{\partial}{\partial x_{2k}}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2k-1}} + i\frac{\partial}{\partial x_{2k}}\right)$$

and differentiation of a smooth function will be denoted with subscripts, e.g., $f_{z_j} = \frac{\partial f}{\partial z_j}$. The real tangent space to $b\Omega$ at $q \in b\Omega$, $\mathbb{R}T_q(b\Omega)$, is

$$\mathbb{R}T_q(b\Omega) = \left\{W \in \mathbb{R}^{2n} : \sum_{k=1}^{2n} r_{x_k}(q) W_k = 0\right\}, \tag{2.1}$$

where $W = \sum W_k \frac{\partial}{\partial x_k}$. Note that if $(y_1, \dots, y_{2n})$ is another, smooth coordinate system in a neighborhood of q, then $\sum r_{x_k}(q) W_k = \sum r_{y_k}(q) \widetilde{W}_k$ if $W = \sum W_k \frac{\partial}{\partial x_k} = \sum \widetilde{W}_k \frac{\partial}{\partial y_k}$, so (2.1) is invariant of coordinate change. The complex tangent space to $b\Omega$ at $q \in b\Omega$, $\mathbb{C}T_q(b\Omega)$, is

$$\mathbb{C}T_q(b\Omega) = \left\{V \in \mathbb{C}^n : \sum_{k=1}^{n} r_{z_k}(q) V_k = 0\right\}, \tag{2.2}$$

where $V = \sum V_k \frac{\partial}{\partial z_k}$. If $(w_1, \dots, w_n)$ is an arbitrary local holomorphic coordinate system near q, the vector fields $\frac{\partial}{\partial w_k}$ are given by the chain rule, and

$V=\sum \widetilde{V}_k \frac{\partial}{\partial w_k}$ is decomposed with respect to the frame $\{\partial/\partial w_1,\dots,\partial/\partial w_n\}$, it is easy to see (2.2) is an invariant definition. Both (2.1) and (2.2) are independent of the choice of local defining function for Ω.

The Hessian of a smooth function $f:\mathbb{C}^n \longrightarrow \mathbb{C}$ can be viewed as a bilinear form on vectors in $\mathbb{R}^{2n}$ or on vectors in $\mathbb{C}^n$. We invert the usual presentation by considering its action on complex vectors first. The *real Hessian* of f at a point p acting on the pair of vectors $(A,B)\in\mathbb{C}^n\oplus\mathbb{C}^n$ is

$$\mathcal{H}_{f(p)}(A,B)=2\operatorname{Re}\left(\sum_{k,\ell=1}^{n} f_{z_k z_\ell}(p)A_kB_\ell\right)+2\sum_{k,\ell=1}^{n} f_{z_k\bar{z}_\ell}(p)A_k\overline{B}_\ell. \tag{2.3}$$

Checking that (2.3) agrees with the more familiar definition of the Hessian using the underlying real coordinates requires a small computation. We first fix a specific identification of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$; if $A=(a_1+ia_2,\dots,a_{2n-1}+ia_{2n})$ and $B=(b_1+ib_2,\dots,b_{2n-1}+ib_{2n})$ are in $\mathbb{C}^n$, then the corresponding vectors in $\mathbb{R}^{2n}$, $(a_1,a_2,\dots,a_{2n-1},a_{2n})$ and $(b_1,b_2,\dots,b_{2n-1},b_{2n})$, will also be denoted by the symbols A and B. The definition of the operators $\frac{\partial}{\partial z_k}$, $\frac{\partial}{\partial \bar{z}_k}$ and straightforward linear algebra shows

$$\mathcal{H}_{f(p)}(A,B)=\sum_{k,\ell=1}^{2n}\frac{\partial^2 f}{\partial x_k\,\partial x_\ell}(p)a_kb_\ell.$$

The *complex Hessian* of f at p is (one-half of) the second term on the right-hand side of (2.3):

$$\mathcal{L}_{f(p)}(A,B)=\sum_{k,\ell=1}^{n} f_{z_k\bar{z}_\ell}(p)A_k\overline{B}_\ell. \tag{2.4}$$

One-half the first term on the right-hand side of (2.3)—henceforth, the *complement of the Levi form*—will be denoted

$$\mathcal{Q}_{f(p)}(A,B)=\operatorname{Re}\left(\sum_{k,\ell=1}^{n} f_{z_k z_\ell}(p)A_kB_\ell\right). \tag{2.5}$$

The forms $\mathcal{L}$ and $\mathcal{Q}$ transform differently under multiplication of their arguments by $i=\sqrt{-1}$. Indeed, (2.4) immediately yields $\mathcal{L}_{f(p)}(A,B)=\mathcal{L}_{f(p)}(iA,iB)$, while (2.5) shows $\mathcal{Q}_{f(p)}(A,B)=-\mathcal{Q}_{f(p)}(iA,iB)$. Consequently, for all pairs $(A,B)\in\mathbb{C}^n\oplus\mathbb{C}^n$

$$\mathcal{H}_{f(p)}(A,B)+\mathcal{H}_{f(p)}(iA,iB)=4\mathcal{L}_{f(p)}(A,B).$$

It is also convenient to have notation for first-derivative expressions of f. The complex gradient of f acting on a vector in $\mathbb{C}T(\mathbb{C}^n)$ will be denoted

$$\left\langle \partial f(p),V\right\rangle=\sum_{k=1}^{n} f_{z_k}(p)V_k, \tag{2.6}$$

when $V = \sum V_k \frac{\partial}{\partial z_k}$. The symbol $\langle \bar{\partial} f(p), V\rangle$ is defined analogously. The real gradient of f acting on a vector $W \in \mathbb{R}^{2n}$ will be denoted

$$\big\langle \nabla f(p), W\big\rangle = \sum_{k=1}^{2n} f_{x_k} W_k, \tag{2.7}$$

if $W = \sum_{k=1}^{2n} W_k \frac{\partial}{\partial x_k}$.

If $f \in C^3(U)$, U open in $\mathbb{C}^n$, and $p = (p_1 + ip_2, \dots, p_{2n-1} + ip_{2n}), q = (q_1 + iq_2, \dots, q_{2n-1} + iq_{2n})$ are two points in U, Taylor's theorem to second-order in real notation says

$$f(q) = f(p) + \big\langle \nabla f(p), V\big\rangle + \frac{1}{2}\mathcal{H}_{f(p)}(V,V) + \mathcal{O}\big(\|V\|^3\big),$$

where $V = (p_1 - q_1, \dots, p_{2n} - q_{2n}) \in \mathbb{R}^{2n}$. In complex notation, the same result is expressed

$$\begin{aligned} f(q) = f(p) &+ 2\,\mathrm{Re}\big\langle \partial f(p), W\big\rangle + \mathcal{Q}_{f(p)}(W,W) + \mathcal{L}_{f(p)}(W,W) \\ &+ \mathcal{O}\big(\|W\|^3\big), \end{aligned} \tag{2.8}$$

where $W = p - q \in \mathbb{C}^n$.

Basic convexity notions, on both functions and domains, are easily expressed using the above notation.

Definition 2.1. Let $U \subset \mathbb{C}^n$ be an open set and $f \in C^2(U)$. Then

(a) f is convex at $p \in U$ if

$$\mathcal{H}_{f(q)}(W,W) \geq 0 \quad \forall q \in U', W \in \mathbb{C}^n$$

for some neighborhood $U' \subset U$ containing p.

(b) f is plurisubharmonic at $p \in U$ if

$$\mathcal{L}_{f(q)}(W,W) \geq 0 \quad \forall q \in U', W \in \mathbb{C}^n$$

for some neighborhood $U' \subset U$ containing p.

If f is convex at p, then so is $f \circ L$ for any $\mathbb{R}$-affine coordinate change of the standard coordinates. This follows easily from the chain rule. Plurisubharmonicity is not invariant under a general $\mathbb{R}$-affine coordinate change. But it is invariant under an arbitrary, local biholomorphic map (again, by the chain rule), in particular under a $\mathbb{C}$-affine coordinate change.

Definition 2.2. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded open set, $p_0 \in b\Omega$, and r is a local defining function for Ω in a neighborhood of p_0. Then

(a) Ω is convex near p_0 if

$$\mathcal{H}_{r(p)}(V,V) \geq 0 \quad \forall p \in U \cap b\Omega, V \in \mathbb{R}T_p(b\Omega)$$

for some neighborhood U containing p_0.

(b) Ω is pseudoconvex near p_0 if

$$\mathcal{L}_{r(p)}(V,V) \geq 0 \quad \forall p \in U \cap b\Omega, V \in \mathbb{C}T_p(b\Omega)$$

for some neighborhood U containing p_0.

Both conditions in Definition 2.2 are independent of the choice of local defining function. The conditions are also invariant under a $\mathbb{C}$-affine coordinate change, as mentioned above.

3. Distance to the boundary

The other ingredient in Oka's lemma is the distance-to-the-boundary function, which we denote by $d = d_{b\Omega}$:

$$d(z) = \inf_{q \in b\Omega} \|z - q\|.$$

The signed distance to $b\Omega$ will be denoted by $\delta = \delta_{b\Omega}$:

$$\delta(z) = \begin{cases} -d(z), & z \in \Omega, \\ d(z), & z \in \Omega^c. \end{cases} \tag{3.1}$$

We collect some basic facts about δ on a smoothly bounded domain in $\mathbb{C}^n$.

Proposition 3.1. *If $\Omega \subset \mathbb{C}^n$ is a smoothly bounded domain, then there exists a neighborhood U of $b\Omega$ such that:*

(a) *The map $b_{b\Omega} : U \longrightarrow b\Omega$ satisfying $\|b_{b\Omega}(z) - z\| = |\delta_{b\Omega}(z)|$ is well-defined.*
(b) *The functions $b_{b\Omega}$ and $\delta_{b\Omega}$ are smooth on U.*
(c) *For each $p \in b\Omega$, let ν_p be the real outward unit normal to $b\Omega$ at p. Then there exists a coordinate system $(w_1, \ldots, w_n)$, $w_k = y_{2k-1} + iy_{2k}, k = 1, \ldots, n$, which is a $\mathbb{C}$-affine coordinate change of the standard coordinates on $\mathbb{C}^n$, such that for all $q = t\nu_p \in U \cap \Omega$, $t \in \mathbb{R}$,*

$$\delta_{y_j}(q) = \begin{cases} 0, & j \neq 2n-1, \\ 1, & j = 2n-1. \end{cases} \tag{3.2}$$

For a proof of (a) see, for example, [3], Lemma 4.1.1 on pp. 444–445. Proofs of (b) and (c) follow from Corollary 5.2 in [8] after using Lemma 1, p. 382, in [6].

4. A proof of Oka's lemma

The significant content of Oka's lemma is that $-\log d_{b\Omega}$ is plurisubharmonic *near* $b\Omega$, if Ω is pseudoconvex. Since the new feature in our proof also occurs near $b\Omega$, we shall focus on proving the following theorem.

Theorem 4.1 (Version of Oka's lemma). *Let Ω be a smoothly bounded, pseudoconvex domain in $\mathbb{C}^n$. There exists a neighborhood U of $b\Omega$ such that $-\log(-\delta(z))$ is plurisubharmonic for $z \in U \cap \Omega$.*

Proof. For the expansion of a normed expression below (see (4.4)), it is convenient to consider the square of the function d rather than d (or δ) itself; let

$$D(z) = d^2_{b\Omega}(z) = \inf\big\{ \|z-q\|^2 : q \in b\Omega \big\}.$$

Obviously, $-\log(-\delta(z))$ is plurisubharmonic iff $-2\log(-\delta(z))$ is plurisubharmonic, and $-2\log(-\delta(z)) = -\log D(z)$ if $z \in \Omega$. Thus, it suffices to show there exists a neighborhood U of $b\Omega$ such that

$$\mathcal{L}_{D(z)}(V,V) \le \frac{|\langle \partial D(z), V\rangle|^2}{D(z)} \tag{4.1}$$

for all $z \in U \cap \Omega$ and all $V \in \mathbb{C}^n$.

Let U be a small enough neighborhood of $b\Omega$ so that the projection map b is well-defined and smooth. For a given $q \in U \cap \Omega$, make the $\mathbb{C}$-affine coordinate change in Proposition 3.1 to achieve

(i) $b(q) = 0$ $(= (0,\dots,0) = (0',0) \in \mathbb{C}^{n-1} \times \mathbb{C}))$,
(ii) $q = (0', a)$, $a < 0$,
(iii) $\mathbb{R}T_0(b\Omega) = \{\operatorname{Re} z_n = 0\}$.

We shall continue to denote the changed coordinates as $(z_1,\dots,z_n)$, with $z_k = x_{2k-1} + ix_{2k}$, and do all subsequent computations with respect to these coordinates.

In a neighborhood U_q of 0, the Implicit Function theorem says that $b\Omega$ can be viewed as a smooth graph over $\mathbb{R}T_0(b\Omega)$. Explicitly, we can find a defining function, $r(z)$, of the form

$$r(z) = \operatorname{Re} z_n + h\big(z', \operatorname{Im} z_n\big), \tag{4.2}$$

where $h \in C^2(U_q)$, $h(0) = 0$ and $\nabla h(0) = (0,\dots,0)$.

Clearly, $D(q) = a^2$. It follows from Proposition 3.1 that $D_{x_{2n-1}}(q) = 2a$ and that all the other real partial derivatives of D vanish at q. This translates to the following information on the complex partials of D:

$$D_{z_k}(q) = \begin{cases} 0, & \text{if } k \ne n, \\ a, & \text{if } k = n. \end{cases} \tag{4.3}$$

Let $V = (V', V_n)$ denote an arbitrary direction in $\mathbb{C}^n$, with $\|V\|$ small enough so that $q + V$ lies in U_q. Decompose V_n into its real and imaginary parts, $V_n = s + it$, and note

$$q + V = \big(V', a + s + it\big).$$

The form of the defining function r suggests a suitable point on $b\Omega$ with which to estimate $D(q+V)$: (4.2) says that $(V', -h(V',0)) \in b\Omega$ for any $(V',0) \in U_q$. Consequently,

$$\begin{aligned} D(q+V) &\le \big\| \big(V', a + V_n\big) - \big(V', -h\big(V',0\big)\big) \big\|^2 \\ &= \big\| a + V_n + h\big(V',0\big) \big\|^2 \end{aligned} \tag{4.4}$$

$$= a^2 + 2(a+s)\cdot h\big(V',0\big) + 2as + |V_n|^2 + h^2\big(V',0\big)$$
$$= a^2 + 2a\cdot h\big(V',0\big) + 2as + |V_n|^2 + \mathcal{O}\big(\|V\|^3\big).$$

The last equality follows since h vanishes to second order at 0.

Set $\widetilde{V} = (V',0)$ and notice that $h(\widetilde{V}) = r(\widetilde{V})$. Since $\widetilde{V} \in \mathbb{R}T_0(b\Omega)$, Taylor's theorem gives

$$h(\widetilde{V}) = \frac{1}{2}\mathcal{H}_{r(0)}(\widetilde{V},\widetilde{V}) + \mathcal{O}\big(\|\widetilde{V}\|^3\big). \tag{4.5}$$

However, $\widetilde{V}$ actually belongs to $\mathbb{C}T_0(b\Omega)$, so $\mathcal{L}_{r(0)}(\widetilde{V},\widetilde{V}) \geq 0$ by pseudoconvexity. Since $a<0$, it follows that the second-order part of $2a\cdot h(V',0)$ in (4.4) corresponding to the complex Hessian is negligible, that is, that

$$a\mathcal{H}_{r(0)}(\widetilde{V},\widetilde{V}) \leq a2\mathcal{Q}_{r(0)}(\widetilde{V},\widetilde{V}).$$

Returning to (4.4), we obtain the estimate

$$\begin{aligned} D(q+V) &\leq a^2 + 2a\cdot \mathcal{Q}_{r(0)}(\widetilde{V},\widetilde{V}) + 2as + |V_n|^2 + \mathcal{O}\big(\|V\|^3\big) \\ &= D(q) + 2a\cdot \mathcal{Q}_{r(0)}(\widetilde{V},\widetilde{V}) + 2\operatorname{Re}\big(\langle \partial D(q), V\rangle\big) \\ &\quad + \frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big), \end{aligned}$$

where (4.3) and the fact that $D(q) = a^2$ are used to obtain the last equality. A similar estimate holds in the direction iV, the only changes occurring in the second and third terms:

$$\begin{aligned} &D(q+iV) \\ &\quad \leq D(q) - 2a\cdot \mathcal{Q}_{r(0)}(\widetilde{V},\widetilde{V}) - 2\operatorname{Im}\big(\langle \partial D(q), V\rangle\big) + \frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big). \end{aligned}$$

Adding these two estimates yields, for $S = D(q+V) + D(q+iV)$,

$$S \leq 2D(q) + F(q,V) + 2\frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big), \tag{4.6}$$

where $F(q,V) = 2\operatorname{Re}(\langle \partial D(q), V\rangle) - 2\operatorname{Im}(\langle \partial D(q), V\rangle)$.

On the other hand, expanding $D(q+V)$ and $D(q+iV)$ about q by Taylor's theorem gives

$$\begin{aligned} &D(q+V) \\ &\quad = D(q) + 2\operatorname{Re}\big(\langle \partial D(q), V\rangle\big) + \mathcal{Q}_{D(q)}(V,V) + \mathcal{L}_{D(q)}(V,V) + \mathcal{O}\big(\|V\|^3\big) \end{aligned}$$

and

$$\begin{aligned} &D(q+iV) \\ &\quad = D(q) - 2\operatorname{Im}\big(\langle \partial D(q), V\rangle\big) - \mathcal{Q}_{D(q)}(V,V) + \mathcal{L}_{D(q)}(V,V) + \mathcal{O}\big(\|V\|^3\big). \end{aligned}$$

Adding these two equations yields

$$S = 2D(q) + F(q,V) + 2\mathcal{L}_{D(q)}(V,V) + \mathcal{O}\big(\|V\|^3\big). \tag{4.7}$$

Estimating (4.7) from above by (4.6) and making the obvious cancellations yields

$$2\mathcal{L}_{D(q)}(V,V) \le 2\frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big).$$

Homogeneity considerations in V then show (4.1) holds for $q \in U_q \cap \Omega$. Since the argument above can be given for every $q \in U \cap \Omega$, the proof is complete. □

5. Convex domains

If $\Omega \subset \mathbb{C}^n$ is convex, it is not necessary to compose δ with a function like $\chi(x) = -\log(-x)$ in order to get a conclusion related to Theorem 4.1. Indeed, if $\mathcal{H}_{\delta(p)} \ge 0$ on $\mathbb{R}T_p(b\Omega)$ for $p \in b\Omega$, then δ itself inherits widespread positivity on its real Hessian.

THEOREM 5.1. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^n$. There exists a neighborhood U of $b\Omega$ such that $\delta(z)$ is a convex function for $z \in U \cap \Omega$.*

Different proofs of Theorem 5.1 are known, see pp. 354–357 in [6], pp. 57–60 in [10] and Corollaries 5.7 and 5.12 in [8]. In fact, δ is convex on a full neighborhood of $b\Omega$, not just on $U \cap \Omega$; see Remark 5.1 below. As mentioned in the Introduction, the proof below is parallel to the proof of Theorem 4.1, to trace how the stronger hypothesis in Theorem 5.1 leads to its stronger conclusion.

Proof of Theorem 5.1. As before, consider the function $D(z) = (\delta(z))^2$. A straightforward computation gives

$$\mathcal{H}_{D(z)}(V,V) = \frac{|\langle \nabla D(z), V\rangle|^2}{2D(z)} + 2\delta(z)\mathcal{H}_{\delta(z)}(V,V) \tag{5.1}$$

for all z near $b\Omega$ and $V \in \mathbb{C}^n$. To prove Theorem 5.1, it therefore suffices to show that there is a neighborhood U of $b\Omega$ such that

$$\mathcal{H}_{D(z)}(V,V) \le \frac{|\langle \nabla D(z), V\rangle|^2}{2D(z)} \quad \forall z \in U \cap \Omega, V \in \mathbb{C}^n. \tag{5.2}$$

Let U be a small enough neighborhood of $b\Omega$ so that the projection map b is well-defined and smooth. Fix $q \in U \cap \Omega$, make the $\mathbb{C}$-affine coordinate change in Proposition 3.1 and obtain

(i) $b(q) = 0$ $(= (0,\dots,0) = (0',0) \in \mathbb{C}^{n-1} \times \mathbb{C}))$,
(ii) $q = (0', a)$, $a < 0$,
(iii) $\mathbb{R}T_0(b\Omega) = \{\operatorname{Re} z_n = 0\}$.

Continue to denote the changed coordinates as $(z_1,\dots,z_n)$, with $z_k=x_{2k-1}+ix_{2k}$, as in the proof of Theorem 4.1.

Apply the Implicit Function theorem as before: there exists a neighborhood U_q of the origin, a function $h\in C^\infty(U_q)$ with $h(0)=0$ and $\nabla h(0)=(0,\dots,0)$, such that

$$r(z)=\operatorname{Re} z_n+h\big(z',\operatorname{Im} z_n\big) \tag{5.3}$$

is a local defining function for Ω in U_q.

Clearly $D(q)=a^2$, while $D_{x_{2n-1}}(q)=2a$ and all the other partial derivatives of D vanish at q, by Proposition 3.1. Let $V=(V',V_n)\in\mathbb{C}^n$ be given, write $V_n=s+it$, and consider $q+V$ as a small perturbation of q. We have that $(V',-h(V',c)+ic)$ lies in $b\Omega$, for any $c\in\mathbb{R}$. Thus,

$$\begin{aligned}
D(q+V)&\le\big\|\big(V',a+V_n\big)-\big(V',-h\big(V',c\big)+ic\big)\big\|^2\\
&=\big\|a+s+h\big(V',c\big)+i(t-c)\big\|^2\\
&=\big(a+s+h\big(V',t\big)\big)^2,
\end{aligned}$$

if c is chosen equal to t. Expanding this square yields

$$\begin{aligned}
D(q+V)&\le a^2+2as+s^2+2(a+s)\cdot h\big(V',t\big)+h^2\big(V',t\big)\\
&=a^2+2as+(\operatorname{Re}V_n)^2+2a\cdot h\big(V',t\big)+\mathcal{O}\big(\|V\|^3\big),
\end{aligned}\tag{5.4}$$

since h vanishes to second order at 0.

Now set $\widetilde{V}=(V',it)$. Note that $h(V',t)=r(\widetilde{V})$ and that $\widetilde{V}\in\mathbb{R}T_0(b\Omega)$ (though not in $\mathbb{C}T_0(b\Omega)$, unless $t=0$). Taylor's theorem gives

$$h\big(V',t\big)=\frac{1}{2}\mathcal{H}_{r(0)}(\widetilde{V},\widetilde{V})+\mathcal{O}\big(\|\widetilde{V}\|^3\big). \tag{5.5}$$

Convexity of Ω implies $H_{r(0)}(\widetilde{V},\widetilde{V})$ is nonnegative, so we have $h(V',t)\ge -C\|\widetilde{V}\|^3$ for some constant $C>0$. Because $a<0$, it follows from (5.4) that

$$\begin{aligned}
D(q+V)&\le a^2+2as+(\operatorname{Re}V_n)^2+\mathcal{O}\big(\|V\|^3\big)\\
&=D(q)+\big\langle\nabla D(q),V\big\rangle+\frac{|\langle\nabla D(q),V\rangle|^2}{4D(q)}+\mathcal{O}\big(\|V\|^3\big).
\end{aligned}\tag{5.6}$$

However, expanding $D(q+V)$ about q by Taylor's theorem gives

$$D(q+V)=D(q)+\big\langle\nabla D(q),V\big\rangle+\frac{1}{2}\mathcal{H}_{D(q)}(V,V)+\mathcal{O}\big(\|V\|^3\big). \tag{5.7}$$

Estimating (5.7) from above by (5.6) leads to

$$\frac{1}{2}\mathcal{H}_{D(q)}(V,V)\le\frac{|\langle\nabla D(q),V\rangle|^2}{4D(q)}+\mathcal{O}\big(\|V\|^3\big).$$

The homogeneity in V then implies that (5.2) holds. $\square$

REMARK 5.1. Under the hypothesis of Theorem 5.1, $\delta(z)$ is also convex for $z \in U \cap \Omega^c$. The same initial part of the proof above is used; however $a > 0$ when $q \in U \cap \Omega^c$. Notice that the conclusion above (5.6) can be improved: Taylor's theorem actually yields

$$h\big(V', t\big) = r(0) + \big\langle \nabla r(0), \widetilde{V} \big\rangle + \frac{1}{2}\mathcal{H}_{r(\alpha)}(\widetilde{V}, \widetilde{V})$$

for some point α on the line segment connecting the origin and the point $\widetilde{V}$. It follows from the proof of Proposition 4.1 in [8] that $\mathcal{H}_{r(\alpha)}(\widetilde{V}, \widetilde{V}) \geq 0$. Therefore, $h(V', t)$ is nonnegative, and the distance of the point $q + V$ to $b\Omega$ is larger or equal to its distance to the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$. But the latter is attained at the point $\widetilde{V}$. It follows that

$$\begin{aligned} D(q+V) &\geq \big\| \big(V', a + V_n\big) - \widetilde{V} \big\|^2 = \|a+s\|^2 \\ &= D(q) + \big\langle \nabla D(q), V \big\rangle + \frac{|\langle \nabla D(q), V \rangle|^2}{4D(q)}. \end{aligned}$$

This yields, by repeating the arguments in the proof of Theorem 5.1, that δ is convex on $\Omega^c \cap U$.

6. Intermediate positivity conditions

In the previous two sections, hypotheses on the Hessians and tangent spaces were "matched" with respect to the real or complex structure: $\mathcal{H}_\delta \geq 0$ on $\mathbb{R}T(b\Omega)$ in Theorem 5.1 and $\mathcal{L}_\delta \geq 0$ on $\mathbb{C}T(b\Omega)$ in Theorem 4.1. In this section, we study "mixed" situations.

Nonnegativity of $\mathcal{H}_{\delta(p)}$ on $\mathbb{C}T_p(b\Omega)$.

DEFINITION 6.1. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded open set, $p_0 \in b\Omega$, and r a local defining function for Ω in a neighborhood of p_0. Then Ω is $\mathbb{C}$-convex near p_0 if

$$\mathcal{H}_{r(p)}(V, V) \geq 0 \quad \forall p \in U \cap b\Omega, V \in \mathbb{C}T_p(b\Omega)$$

for some neighborhood U containing p_0.

As with the conditions in Definition 2.2, $\mathbb{C}$-convexity is independent of the choice of local defining function as well as invariant under a $\mathbb{C}$-affine coordinate change.

A convex domain is clearly $\mathbb{C}$-convex, since $\mathbb{C}T(b\Omega) \subset \mathbb{R}T(b\Omega)$. Also, the displayed equation below (2.5) shows that a $\mathbb{C}$-convex domain is pseudoconvex. Not surprisingly, a result intermediate to Theorems 4.1 and 5.1 holds for $\mathbb{C}$-convex domains.

THEOREM 6.1. *Let Ω be a smoothly bounded, $\mathbb{C}$-convex domain in $\mathbb{C}^n$. There exists a neighborhood U of $b\Omega$ such that*

$$\mathcal{H}_{\delta(z)}(V,V) \geq \frac{|\langle \nabla\delta(z), iV\rangle|^2}{\delta(z)} \quad \forall z \in U\cap\Omega, V\in\mathbb{C}^n. \tag{6.1}$$

As mentioned in the introduction, Theorem 6.1 is proved in [1], see the implication (ii) to (iii) of Theorem 2.5.18 therein (there is a notational difference between our paper and [1]—compare (2.3) and the first displayed equation on p. 60 in [1]).

Proof of Theorem 6.1. As in the proofs of Theorems 4.1 and 5.1 work with the function $D(z) = (\delta(z))^2$. Note first that for any vector $V \in \mathbb{C}^n$

$$\left|\left\langle \nabla\delta(z), V\right\rangle\right|^2 = \frac{|\langle \nabla D(z), V\rangle|^2}{4D(z)}.$$

It then follows from (5.1) that (6.1) is equivalent to

$$\mathcal{H}_{D(z)}(V,V) \leq \frac{|\langle \nabla D(z), V\rangle|^2}{2D(z)} + \frac{|\langle \nabla D(z), iV\rangle|^2}{2D(z)} \quad \forall z\in U\cap\Omega, V\in\mathbb{C}^n. \tag{6.2}$$

To prove (6.2), proceed as in the proof of Theorem 4.1, starting below (4.1). Take $q=(0',a)$ for $a<0$ and $V=(V',V_n)$ with $V_n = s+it$. Choose again the boundary point $(V',-h(V',0))$ to obtain an upper bound on $D(q+V)$:

$$D(q+V) \leq a^2 + 2a\cdot h\big(V',0\big) + 2as + s^2 + t^2 + \mathcal{O}\big(\|V\|^3\big).$$

Since $(V',0)$ is in $\mathbb{C}T_0(b\Omega)$, it follows from (4.5) and the hypothesis of $\mathbb{C}$-convexity that $h(V',0) \geq -C\|V\|^3$ for some $C>0$. Therefore

$$\begin{aligned} &D(q+V)\\ &\quad\leq a^2 + 2as + s^2 + t^2 + \mathcal{O}\big(\|V\|^3\big)\\ &\quad= D(q) + \big\langle \nabla D(q), V\big\rangle + \frac{|\langle \nabla D(z), V\rangle|^2}{4D(z)} + \frac{|\langle \nabla D(z), iV\rangle|^2}{4D(z)} + \mathcal{O}\big(\|V\|^3\big). \end{aligned}$$

Using (5.7), it then follows that

$$\frac{1}{2}H_{D(q)}(V,V) \leq \frac{|\langle \nabla D(z), V\rangle|^2}{4D(z)} + \frac{|\langle \nabla D(z), iV\rangle|^2}{4D(z)} + \mathcal{O}\big(\|V\|^3\big),$$

which implies (6.2). $\square$

Nonnegativity of $\mathcal{L}_{\delta(p)}$ on $\mathbb{R}T_p(b\Omega)$. Finally, we turn to the case of nonnegativity of the *complex* Hessian of a defining function on the *real* tangent space. Unlike the previous conditions, this condition is not independent of the choice of defining function. We shall only consider this condition on δ as our method of proof is fine-tuned to this defining function.

It is elementary that this positivity spreads to arbitrary directions:

REMARK 6.1. If $\mathcal{L}_{\delta(z)}(V,V) \geq 0$ for all $z \in b\Omega$ and $V \in \mathbb{R}T_z(b\Omega)$, then $\mathcal{L}_{\delta(z)}(W,W) \geq 0$ for all $z \in b\Omega$ and $W \in \mathbb{C}^n$. This can be seen if, e.g., the coordinates in the proof of Theorem 5.1 are used. Then, for $W \in \mathbb{C}^n$, choose $\theta \in [0, 2\pi)$ such that $\mathrm{Re}(e^{i\theta} W_n) = 0$. The complex Hessian of δ is invariant under such rotations and $V := e^{i\theta} W \in \mathbb{R}T_0(b\Omega)$.

Moreover, this positivity spreads off $b\Omega$.

THEOREM 6.2. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose that $\mathcal{L}_{\delta(z)}(V,V) \geq 0$ for all $z \in b\Omega$ and $V \in \mathbb{R}T_z(b\Omega)$. Then there exists a neighborhood U of $b\Omega$ such that δ is plurisubharmonic on $U \cap \Omega$, i.e., $\mathcal{L}_\delta(z)(V,V) \geq 0$ for all $z \in U \cap \Omega$ and $V \in \mathbb{C}^n$.*

Proof. Again, use the function $D(z) = (\delta(z))^2$, and note that

$$\mathcal{L}_{D(z)}(W,W) = \frac{|\langle \partial D(z), W\rangle|^2}{2D(z)} + 2\delta(z)\mathcal{L}_{\delta(z)}(W,W) \tag{6.3}$$

for z near $b\Omega$ and $W \in \mathbb{C}^n$. Thus, to prove Theorem 6.2 it suffices to show that there exists a neighborhood U of $b\Omega$ such that

$$\mathcal{L}_{D(z)}(W,W) \leq \frac{|\langle \partial D(z), W\rangle|^2}{2D(z)} \quad \forall z \in U \cap \Omega, W \in \mathbb{C}^n. \tag{6.4}$$

Proceed as in the proof of Theorem 5.1, starting below (5.2). By Remark 6.1, it suffices to prove (6.4) for vectors that are of the form $V = (V', it)$ for $V' \in \mathbb{C}^{n-1}$ and $t \in \mathbb{R}$. As in the proof of Theorem 5.1, take $q = (0', a)$ for $a < 0$, and choose the boundary point $(V', -h(V', t) + it)$ to obtain an upper bound for $D(q+V)$:

$$\begin{aligned} D(q+V) &\leq \left\| \big(V', a + V_n\big) - \big(V', -h\big(V', t\big) + it\big) \right\|^2 \\ &= D(q) + 2a \cdot h\big(V', t\big) + \mathcal{O}\big(\|V\|^3\big). \end{aligned} \tag{6.5}$$

Next, choose the boundary point $(iV', -h(iV', 0))$ to obtain an upper bound for $D(q+iV)$:

$$\begin{aligned} D(q+iV) &\leq \left\| \big(iV', a + iV_n\big) - \big(iV', -h\big(iV', 0\big)\big) \right\|^2 \\ &= D(q) - 2\,\mathrm{Im}\big(\big\langle \partial D(q), V\big\rangle\big) + 2a \cdot h\big(iV', 0\big) \\ &\quad + \frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big). \end{aligned} \tag{6.6}$$

Suppose (temporarily) that there exists some constant $C > 0$ such that

$$h\big(V', t\big) + h\big(iV', 0\big) \geq -C\|V\|^3. \tag{6.7}$$

Then, adding (6.6) to (6.5) would yield

$$\begin{aligned} S &= D(q+V) + D(q+iV) \\ &\leq 2D(q) - 2\,\mathrm{Im}\big(\big\langle \partial D(q), V\big\rangle\big) + \frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big). \end{aligned}$$

Using (4.7) for S and the fact that $\operatorname{Re}(\langle \partial D(q), V\rangle) = 0$, it then would follow that

$$2\mathcal{L}_{D(q)}(V,V) \leq \frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\big(\|V\|^3\big),$$

which implies (6.4).

Thus it remains to show that (6.7) holds. For that write $\widehat{V} = (iV', 0)$, so that $h(iV',0) = r(\widehat{V})$ (and $h(V',t) = r(V)$). Then Taylor's theorem gives

$$h\big(V',t\big) + h\big(iV',0\big) = \frac{1}{2}\big(\mathcal{H}_{r(0)}(V,V) + \mathcal{H}_{r(0)}(\widehat{V},\widehat{V})\big) + \mathcal{O}\big(\|V\|^3\big).$$

Since $\delta_{x_j x_k}(0) = r_{x_j x_k}(0)$ (see for instance part (i) of Remark 4.2 in [8] with $r = \delta$ there), it suffices to show that

$$\mathcal{H}_{\delta(0)}(V,V) + \mathcal{H}_{\delta(0)}(\widehat{V},\widehat{V}) \geq 0.$$

As $\widehat{V} = iV + (0',t)$, the bilinearity of $\mathcal{H}$ yields

$$\mathcal{H}_{\delta(0)}(\widehat{V},\widehat{V}) = \mathcal{H}_{\delta(0)}(iV,iV) + 2\mathcal{H}_{\delta(0)}\big(iV,\big(0',t\big)\big) + \mathcal{H}_{\delta(0)}\big(\big(0',t\big),\big(0',t\big)\big).$$

However, the fact that $\sum_{j=1}^{2n} |\delta_{x_j}|^2 = 1$ holds in a neighborhood of $b\Omega$ implies that

$$\mathcal{H}_{\delta(0)}\big(\cdot,\big(0',1\big)\big) = 0,$$

see, e.g., (5.5) in [8] for details. Thus $\mathcal{H}_{\delta(0)}(\widehat{V},\widehat{V}) = \mathcal{H}_{\delta(0)}(iV,iV)$, so that by (2.3)

$$\mathcal{H}_{\delta(0)}(V,V) + \mathcal{H}_{\delta(0)}(\widehat{V},\widehat{V}) = 4\mathcal{L}_{\delta(0)}(V,V) \geq 0,$$

since the complex Hessian of δ on the boundary is nonnegative definite on the real tangent space. □

Nonnegativity of $\mathcal{L}_{\delta(p)}$ on cones in $\mathbb{R}T_p(b\Omega)$. One may also consider nonnegativity of the real or complex Hessian of δ on cones of vectors, contained in the real tangent space, whose axes are the complex tangent space. We shall only consider the nonnegativity of $\mathcal{L}_\delta$ on such cones here, but mention $\mathcal{H}_\delta \geq 0$ could be considered as well and a result analogous to Theorem 6.3 below obtained for that hypothesis.

Definition 6.2. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded open set, $p \in b\Omega$. Let r be a defining function for Ω in a neighborhood of p and $\gamma \in [0,\infty)$. Then

$$\mathbb{R}T_p^\gamma(b\Omega) := \left\{ V \in \mathbb{R}T_p(b\Omega) : \frac{|\langle i\nabla r(p), V\rangle|}{\|\nabla r(p)\|} \leq \gamma \left\| V - \frac{\langle i\nabla r(p), V\rangle i\nabla r(p)}{\|\nabla r(p)\|^2} \right\| \right\}.$$

Note that the definition of the cone $\mathbb{R}T_p^\gamma(b\Omega)$ is independent of the choice of defining function. Also, $\mathbb{R}T_p^\gamma(b\Omega)$ is invariant under $\mathbb{C}$-affine coordinate changes that are compositions of translations and rotations. Furthermore,

(i) $\mathbb{R}T_p^0(b\Omega)$ equals $\mathbb{C}T_p(b\Omega)$,

(ii) $\lim_{\gamma\to\infty} \mathbb{R}T_p^\gamma(b\Omega)$ equals $\mathbb{R}T_p(b\Omega)$.

DEFINITION 6.3. Let $\Omega\subset\mathbb{C}^n$ be a smoothly bounded open set, $p_0\in b\Omega$, and U a neighborhood of p_0. Then δ is γ-plurisubharmonic on $U\cap b\Omega$ if

$$\mathcal{L}_{\delta(p)}(V,V)\geq 0 \quad \forall p\in U\cap b\Omega, V\in \mathbb{R}T_p^\gamma(b\Omega).$$

Note that the condition of γ-plurisubharmonicity of δ is a condition intermediate to pseudoconvexity of Ω ($\gamma=0$) and plurisubharmonicity of δ on $b\Omega$ ($\gamma=\infty$). The following theorem establishes that the complex Hessian of δ then inherits nonnegativity intermediate to the results of Theorem 4.1 and Theorem 6.2.

THEOREM 6.3. *Let $\Omega\subset\mathbb{C}^n$ be a smoothly bounded domain. Suppose δ is γ-plurisubharmonic on $b\Omega$ for some $\gamma>0$. Let $\eta=1-2/(2+\gamma^2)$. Then there exists a neighborhood U of $b\Omega$ such that*

$$\mathcal{L}_{-(-\delta)^\eta(z)}(V,V)\geq 0 \quad \forall z\in U\cap\Omega, V\in\mathbb{C}^n.$$

Proof. Use the function $D(z)=(\delta(z))^2$. First note

$$\begin{aligned}
&\mathcal{L}_{-(-\delta)^\eta(z)}(V,V)\\
&\quad=\eta\big(-\delta(z)\big)^{\eta-2}\big(\big(-\delta(z)\big)\mathcal{L}_{\delta(z)}(V,V)+(1-\eta)\big|\big\langle\partial\delta(z),V\big\rangle\big|^2\big)\\
&\quad=\eta\big(-\delta(z)\big)^{\eta-2}\left(\big(-\delta(z)\big)\mathcal{L}_{\delta(z)}(V,V)+(1-\eta)\frac{|\langle\partial D(z),V\rangle|^2}{4D(z)}\right).
\end{aligned}$$

It follows from (6.3) that it suffices to show that there exists a neighborhood U of $b\Omega$ such that

$$\begin{aligned}
\mathcal{L}_{D(z)}(V,V)&\leq(2-\eta)\frac{|\langle\partial D(z),V\rangle|^2}{2D(z)}\\
&=\left(1+\frac{2}{2+\gamma^2}\right)\frac{|\langle\partial D(z),V\rangle|^2}{2D(z)}
\end{aligned}\tag{6.8}$$

for all $z\in U\cap\Omega$ and $V\in\mathbb{C}^n$.

Proceed as in the proof of Theorem 5.1, starting below (5.2). In particular, let $q=(0',a)$ for $a<0$. By the arguments in Remark 6.1, it suffices to prove (6.8) at $z=q$ for vectors $V=(V',V_n)$ with $V_n=it$ for $t\in\mathbb{R}_0^+$.

Let us first suppose that $V\in\mathbb{R}T_0^\gamma(b\Omega)$. Then, since δ is γ-plurisubharmonic on $b\Omega$, the proof of Theorem 6.2 is applicable so that (6.4) holds (which implies (6.8) for any $\gamma\geq 0$).

Next, suppose that $V\notin\mathbb{R}T_0^\gamma(b\Omega)$, i.e., $|t|>\gamma\|V'\|$. Since $(W',-h(W',c)+ic)$ is a boundary point for any point (W',ic) sufficiently close to the origin, it follows that

$$\begin{aligned}
D(q+V)&\leq\big\|\big(V',a+V_n\big)-\big(W',-h\big(W',c\big)+ic\big)\big\|^2\\
&=\big(a+h\big(W',c\big)\big)^2+\big\|V'-W'\big\|^2+(t-c)^2.
\end{aligned}\tag{6.9}$$

Similarly, since $(iW', -h(iW', 0))$ is a boundary point, it follows that

$$D(q+iV) \leq \left\| \left(iV', a+iV_n\right) - \left(iW', -h\left(iW', 0\right)\right) \right\|^2 = \left(a - t + h\left(iW', 0\right)\right)^2 + \|V' - W'\|^2. \tag{6.10}$$

Adding (6.10) to (6.9) gives

$$\begin{aligned} S &= D(q+V) + D(q+iV) \\ &\leq \left(a + h\left(W', c\right)\right)^2 + \left(a - t + h\left(iW', 0\right)\right)^2 + 2\left\|V' - W'\right\|^2 + (t-c)^2. \end{aligned}$$

Minimizing the function $f(W', c) = 2\|V' - W'\|^2 + (t-c)^2$ subject to the constraint $|c| = \gamma\|W'\|$ (so that $(W', ic) \in \mathbb{R}T_0^\gamma(b\Omega)$ holds), yields the minimal value

$$f\left(W_0', c_0\right) = \frac{2t^2}{2+\gamma^2}\left(1 - \frac{\|V'\|\gamma}{t}\right)^2,$$

where $c_0 = \gamma(t\gamma + 2\|V'\|)/(2+\gamma^2)$ and $W_0' = c_0 V'/(\gamma\|V'\|)$ if $V' \neq 0$. If $V' = 0$, we take $W_0 = 0$. Since $V \notin \mathbb{R}T_0^\gamma(b\Omega)$ it follows that $f(W_0', c_0) \leq 2t^2/(2+\gamma^2)$, and hence

$$\begin{aligned} S \leq{}& 2D(q) - 2\,\mathrm{Im}\left(\langle \partial D(q), V\rangle\right) + 2a\left(h\left(W_0', c_0\right) + h\left(iW_0', 0\right)\right) \\ &+ \left(1 + \frac{2}{2+\gamma^2}\right)\frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\left(\|V\|^3\right), \end{aligned} \tag{6.11}$$

where it was used that $\|(W_0', c_0)\| = \mathcal{O}(\|V\|)$.

Suppose (temporarily) that there exists a constant $C > 0$ such that

$$h\left(W_0', c_0\right) + h\left(iW_0', 0\right) \geq -C\left\|\left(W_0', c_0\right)\right\|^3. \tag{6.12}$$

Then using (4.7) for S in (6.11) and the fact that $\mathrm{Re}(\langle \partial D(q), V\rangle) = 0$, would yield

$$2\mathcal{L}_{D(q)}(V, V) \leq \left(1 + \frac{2}{2+\gamma^2}\right)\frac{|\langle \partial D(q), V\rangle|^2}{D(q)} + \mathcal{O}\left(\|V\|^3\right),$$

which would imply (6.8).

That (6.12) is indeed true may be shown by arguments analogous to the ones in the proof of (6.7), using the facts that δ is γ-plurisubharmonic on $b\Omega$ and that (W_0', ic_0) was chosen to be in $\mathbb{R}T_0^\gamma(b\Omega)$, the cone of nonnegativity of the complex Hessian of δ. □

REMARK 6.2. (a) In [2], pp. 134–137, an example of a pseudoconvex domain is given such that $-(-\delta)^\eta$ is not plurisubharmonic for *any* $\eta > 0$. It is straightforward to check for this example, using the computations in [2], that $\mathcal{L}_{\delta(0)}(V, V) < 0$ for any $V \in \mathbb{R}T_0 \setminus \mathbb{C}T_0$, that is, that δ is *not* γ-plurisubharmonic for *any* $\gamma > 0$.

(b) The condition of γ-plurisubharmonicity, $\gamma > 0$, is less interesting when one tries to find the "best" positivity condition satisfied by *some* defining function. In fact, a careful analysis reveals: whenever δ is γ-plurisubharmonic near some boundary point for some positive γ, then there is a defining function which is plurisubharmonic on the boundary near that boundary point.

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A.-K. Herbig, Department of Mathematics, University of Vienna, Vienna, Austria

E-mail address: anne-katrin.herbig@univie.ac.at

J. D. McNeal, Department of Mathematics, Ohio State University, Columbus, Ohio, USA

E-mail address: mcneal@math.ohio-state.edu