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A SIMPLE PROOF OF A THEOREM OF CALABI

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Dedicated to John D'Angelo on the occasion of his 60th birthday

ABSTRACT. We give a simple and more or less elementary proof of a classical result of E. Calabi on the global extension of a local holomorphic isometry into a complex space form.

1. Introduction

Let (M,ω) be a (connected) complex manifold equipped with a real analytic Kähler metric ω. Write (S,ω_{st}) for one of the three complex space forms equipped with the standard canonical metrics ω_{st}. More precisely, when $S = \mathbb{C}^n$, write ω_{st} for the Euclidean metric; when S is the unit ball in $\mathbb{C}^n$, ω_{st} is the Poincaré metric; and when S is the complex projective space $\mathbf{CP}^n$ ω_{st} stands for the Fubini–Study metric. In this short paper, we give a simple, self-contained and, more or less, elementary proof of the following celebrated theorem of Calabi [Ca] (see Theorems 2, 6, 9, 12 in [Ca]).

THEOREM 1.1 (Calabi [Ca]). *Assume the above notation. Let $U \subset M$ be a connected open subset and let $F : U \to S$ be a holomorphic isometric embedding in the sense that $F^*(\omega_{st}) = \omega$ over U. Then F extends holomorphically along any continuous curve $\gamma : [0,1] \to M$ with $\gamma(0) \in U$. In particular, when M is simply connected, F extends to a globally defined holomorphic map from M into S. Moreover, let $G : U \to S$ be another holomorphic isometric embedding. Then F and G are different by a rigid motion in the sense that there is a holomorphic isometry T of (S,ω_{st}) such that $G = T \circ F$.*

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2. Proof of the theorem

Step I. We first prove the part for uniqueness (up to a rigid motion). We need only to consider the case when $S = \mathbf{CP}^N$ equipped with the standard Fubini–Study metric and the other case can be done in the same way. Let $p \in U$ and consider a holomorphic chart near p with holomorphic coordinates $z = (z_1, \ldots, z_n)$ and $0 \leftrightarrow p$. Let $F, G : U \to S$ be holomorphic isometric embeddings. Composing F and G with isometries of $S = \mathbf{CP}^N$ if necessary, we can assume that $F(0) = G(0) = [1, 0, \ldots, 0]$. For a small neighborhood U' of 0 in U, we have $F(U'), G(U') \subset \{[1, w_1, \ldots, w_N] \in \mathbf{CP}^N\}$, which is identified in a nature way with $\mathbb{C}^N$ with holomorphic coordinates $(w_1, \ldots, w_N)$. In what follows, we also identify a Kähler metric tensor with its associated positive $(1,1)$-Kähler form. Then in the $(w_1, \ldots, w_n)$-coordinates, the Fubini–Study metric can be written as $\omega_{st} = i\partial\bar{\partial}\log(1 + \sum_{j=1}^N |w_j|^2)$. Now, near 0, we can write $F = [1, f_1, \ldots, f_N], G = [1, g_1, \ldots, g_N]$ with $F(0), G(0) = [1, 0, \ldots, 0]$ and f_j, g_j holomorphic near 0.

The assumption that $F^*(\omega_{st}) = G^*(\omega_{st}) = \omega$ gives immediately that

$$\sum_{j=1}^N \left|f_j(z)\right|^2 = \sum_{j=1}^N \left|g_j(z)\right|^2, \quad \text{or} \quad \sum_{j=1}^N f_j(z) \cdot \overline{f_j(\xi)} = \sum_{j=1}^N g_j(z) \cdot \overline{g_j(\xi)} \tag{2.1}$$

for z, ξ in a neighborhood of $0 \in U$. By a lemma of D'Angelo [DA] (see also Calabi [Ca] Bochner–Martin [Section 5 of Chapter 2, BM] for some related results), from (2.1) we conclude F and G differ by a unitary matrix. More precisely, there is an $N \times N$ unitary matrix V such that $(g_1, \ldots, g_N) = (f_1, \ldots, f_N) \cdot V$. Hence, we see that $F = G \cdot \mathrm{diag}(1, V)$, which proves that F and G differ by a rigid motion. Next, for completeness, we include in the following paragraph a detailed proof of the above mentioned lemma:

First, we define $\mathcal{H}_{F,p} := \mathrm{span}_{\mathbf{C}}\{D^\alpha(f_1, \ldots, f_N)|_0\}$, which is regarded as a Hermitian subspace of the stand complex Euclidean space $\mathbb{C}^N$. Here, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with each α_j a nonnegative integer, as usual, we define $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n}$. Notice that $F(z) = \sum_{|\alpha|>0} \frac{D^\alpha F(0)}{\alpha!} z^\alpha$ for $z \approx 0$. Hence, $\mathcal{H}_{F,p}$ is the smallest linear subspace of $\mathbb{C}^N$ containing $F(z)$ for $z \approx 0$ and thus containing $F(z)$ for all $z \in U$ by the uniqueness of holomorphic functions. We can now similarly define $\mathcal{H}_{G,p}$. Write $u_\alpha = D^\alpha(f_1, \ldots, f_N)(0)$ and $v_\alpha = D^\alpha(g_1, \ldots, g_N)(0)$. Define $\mathcal{I} : \mathcal{H}_F \to \mathcal{H}_G$ by linearly extending the map sending u_α to v_α. To see $\mathcal{I}$ is indeed a well-defined linear operator, it suffices to show that $\sum_j a_j u_{\alpha_j} = 0$ if and only if $\sum_j a_j v_{\alpha_j} = 0$ for $a_j' s \in \mathbb{C}$. But this follows trivially from (2.1), for (2.1) gives that $u_\alpha \cdot \overline{u_\beta} = v_\alpha \cdot \overline{v_\beta}$ and thus $\|\sum_j a_j u_{\alpha_j}\|^2 = \|\sum_j a_j v_{\alpha_j}\|^2$. Moreover, this also demonstrates that $\mathcal{I}$ is a linear isometry from $\mathcal{H}_F$ to $\mathcal{H}_G$, which of course can be extended to a unitary self-transformation of $\mathbb{C}^N$. By the Taylor expansion and by the linearity of

$\mathcal{I}$, we have $\mathcal{I}(F(z)) = G(z)$. This thus shows that there is a $N \times N$ unitary matrix V such that $G(z) = F(z) \cdot V$ for $z \approx 0$.

Step II. We next present the proof for the extension part. We use all the notation set up above.

For any $q \in U$, we can find a holomorphic isometry T_q of S such that $T_q(F(q)) = [1, 0, \ldots, 0]$. Also choose a holomorphic coordinates near q with $q \leftrightarrow 0$. We can then similarly define $\mathcal{H}_{F,q}$. Though $\mathcal{H}_{F,q}$ depends only on the choice of T_q, it is easy to see, from the uniqueness of holomorphic functions, that the complex dimension of $\mathcal{H}_{F,q}$ is independent of the choice of T_q, the holomorphic coordinates near q and the point $q \in U$ itself. We write this dimension as d_F. Again, from the uniqueness property for holomorphic functions, it is clear that if F^* is a holomorphic map from a domain $U^* \subset M$ into S, that is obtained by holomorphically continuing F along a certain curve, then $d_{F^*} = d_F$. Also, if $d_F < N$, then we can compose F with a certain holomorphic isometry T of S such that $T \circ F = [1, f_1, \ldots, f_{d_F}, 0, \ldots, 0]$ for $z \approx 0$. Now, to prove the theorem, we need only consider the map $[1, f_1, \ldots, f_{d_F}]$ from a small neighborhood of $p \in U$ into $\mathbf{CP}^{d_F}$ equipped with the Fubini–Study metric. Hence, without loss of generality, we can assume at the beginning that $N = d_F$.

Let $\gamma : [0,1] \to M$ be a continuous curve with $\gamma(0) \in U$. Seeking a contradiction, suppose F does not extend holomorphically along γ. Then there is a point $c \in [0,1]$ such that F extends holomorphically along $\gamma([0,t])$ for $t < c$ but not along $\gamma([0,c])$. Choose $t_j (\in [0,c)) \to c^-$ and let F_{t_j} be a holomorphic map from a small neighborhood of $\gamma(t_j)$ in M into S, obtained by holomorphically continuing F along $[0, t_j]$. Let T_j be a holomorphic isometry of S such that $G_j := T_j \circ F_{t_j}$ maps $\gamma(t_j)$ to $[1, 0, \ldots, 0]$. Choose a holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighborhood of $\gamma(c)$ with $\gamma(c) \leftrightarrow 0$ and write $G_j = [1, g_{j,1}, \ldots, g_{j,N}]$ in a small neighborhood of $\gamma(t_j)$. Also, let $\phi(z, \overline{z})$ be a positive-valued real analytic function in a certain fixed neighborhood U_0 of 0 such that $\omega = i\partial\overline{\partial} \log \phi(z, \overline{z})$ near $\gamma(c)$. Since $G_j^*(\omega_{st}) = \omega$ over a small neighborhood of $\gamma(t_j)$ for $j \gg 1$, we get, near $\gamma(t_j)$, the following:

$$\partial\overline{\partial} \log \phi(z, \overline{z}) = \partial\overline{\partial} \log\left(1 + \sum_{k=1}^{N} |g_{j,k}|^2\right). \tag{2.2}$$

Now, we write $\phi(z, \overline{z}) = \Re(h_j(z)) + \phi_j(z, \overline{z})$, where $h_j(z)$ is holomorphic in a certain fixed neighborhood U^* of $\gamma(c)$ (independent of j for $j \gg 1$) with $\Re(h_j(z)) > 0$ over U^*. Also, over U^*, we have the following convergent power series expansion:

$$\phi_j(z, \overline{z}) = \sum_{|\alpha|, |\beta| > 0} a_{\alpha\overline{\beta}} \big(z - z(\gamma(t_j))\big)^{\alpha} \overline{\big(z - z(\gamma(t_j))\big)^{\beta}}.$$

Then, one easily derives the following equation

$$\sum_{k=1}^{N} g_{j,k}(z)\overline{g_{j,k}(\xi)} = \psi_j(z,\overline{\xi}) = \frac{2\phi_j(z,\overline{\xi})}{h_j(z)+\overline{h_j(\xi)}}. \tag{2.3}$$

Since $d_{F_j} = N$, there are $\{\alpha_1,\ldots,\alpha_N\}$ such that

$$u_{\alpha_k,j} := D^{\alpha_k}(g_{j,1},\ldots,g_{j,N})|_{z=\gamma(t_j)}$$

for $j=1,\ldots,N$ form a linearly independent family. Write A_j for the constant $N\times N$ matrix with $u_{\alpha_k,j}$ as its kth-row. Applying D^{α_j} to (2.3) and then substituting z by $\gamma(t_j)$, we get the following equation:

$$A_j \cdot \begin{pmatrix} \overline{g_{j,1}(\xi)} \\ \vdots \\ \overline{g_{j,N}(\xi)} \end{pmatrix} = \begin{pmatrix} D^{\alpha_1}\psi_j(z,\overline{\xi})|_{z=\gamma(t_j)} \\ \vdots \\ D^{\alpha_N}\psi_j(z,\overline{\xi})|_{z=\gamma(t_j)} \end{pmatrix}. \tag{2.4}$$

Aprio, (2.4) only holds for $\xi \approx \gamma(t_j)$. However, since $\psi_j(z,\overline{\xi})$ is real-analytic for $z,\xi \in U^*$, the left-hand side of (2.4) is well defined and is holomorphic for $\xi \in U^*$. The crucial point for our simple proof is that the left-hand side is linear in $\overline{g_{j,k}}$ and thus no implicit function theorem is needed for solving $\overline{g_{j,k}}$. Hence, the solution is well defined over the same defining domain of the right-hand side, whenever the coefficient matrix A_j is invertible. A_j is indeed invertible by our arrangement, though its determinant may approach to 0 as $j\to\infty$. Hence, from (2.4), we conclude that $g_{j,k}$ extends to a holomorphic function, for each k and $j \gg 1$, to the fixed neighborhood U^* of $\gamma(c)$. That shows that F also admits a holomorphic continuation along γ all the way across $\gamma(c)$. This is a contradiction. The extension part is also proved.

REMARK. (1) The same argument above (with a little more care on the convergence) also applies to the case when S is a Hilbert space form as obtained in the original paper of Calabi. (2) For late work along the lines of Calabi in [Ca], we refer the reader to [MN] and many references therein. (3) The simplicity for the argument here makes it applicable in other more complicated settings. We refer the reader to a recent preprint [HY].

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