Illinois Journal of Mathematics
Volume 56, Number 1, Spring 2012, Pages 1–6
S 0019-2082

JOHN D'ANGELO'S CONTRIBUTIONS TO THE $\bar{\partial}$-NEUMANN PROBLEM

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Dedicated to John D'Angelo on his 60th birthday

Abstract. In this note, I recall various important contributions made by D'Angelo to the $\bar{\partial}$-Neumann problem. I briefly describe some of the issues that D'Angelo addressed and comment on the significance of his results.

The $\bar{\partial}$-Neumann problem is a fundamental problem, both in the theory of several complex variables and in the theory of partial differential equations. It is also closely connected to analysis on CR manifolds and has led to research in significant areas of harmonic analysis. Here we state the problem briefly, for detailed accounts we refer to the expositions in [FK], [H], and [CS]. These also describe several important applications and contain other references. D'Angelo's original ideas, which we outline here, together with the necessary background topics, are exposed in [D2].

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain (or, more generally, a compact complex manifold with boundary) and let $\bar{\partial} : \mathcal{A}^{p,q}(\Omega) \to \mathcal{A}^{p,q+1}(\Omega)$ be the Cauchy–Riemann operator on $\mathcal{A}^{p,q}(\Omega)$, the space of (p,q)-forms with coefficients in $C^\infty(\Omega)$. We denote the by $\bar{\partial}$ the maximal L_2 closure of $\bar{\partial}$ with domain $Dom^{p,q}(\bar{\partial},\Omega)$. Further, we denote its L_2 adjoint by $\bar{\partial}^*$, with domain $Dom^{p,q}(\bar{\partial}^*,\Omega)$, so that $\bar{\partial}^* : Dom^{p,q}(\bar{\partial}^*,\Omega) \to L^{p,q-1}(\Omega)$. Let $\Box$ denote the operator on $Dom^{p,q}(\Box,\Omega) = \{\varphi \in Dom^{p,q}(\bar{\partial},\Omega) \cap Dom^{p,q}(\bar{\partial}^*,\Omega) \mid \bar{\partial}\varphi \in Dom^{p,q+1}(\bar{\partial}^*)$ and $\bar{\partial}^*\varphi \in Dom^{p,q-1}(\bar{\partial},\Omega)\})$, defined by

$$\Box = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*.$$

Received March 22, 2012.
2010 *Mathematics Subject Classification.* 32V35, 32W05, 35H20.

Let $\mathcal{H}^{p,q}(\Omega)$ denote the nullspace of $\Box$. Then the $\bar\partial$-Neumann problem is the following. Given $\alpha \in L_2^{p,q}(\Omega)$ with $\alpha \perp \mathcal{H}^{p,q}(\Omega)$, does there exist $\varphi \in Dom^{p,q}(\Box,\Omega)$ such that $\Box\varphi = \alpha$? What are the regularity properties of φ?

The method of analyzing these problems on domains with smooth boundaries is to obtain *a-priori* estimates of the "energy" form

$$Q(\varphi,\varphi) = (\bar\partial\varphi, \bar\partial\varphi) + \big(\bar\partial^*\varphi, \bar\partial^*\varphi\big),$$

where $\varphi \in \dot{\mathcal{D}}^{p,q}(\bar\Omega)$ and is defined by

$$\dot{\mathcal{D}}^{p,q}(\bar\Omega) = \big\{\varphi \in C^\infty(\bar\Omega) \text{ whose restriction to } \Omega \\ \text{is in } Dom^{p,q}(\bar\partial,\Omega) \cap Dom^{p,q}\big(\bar\partial^*,\Omega\big)\big\}.$$

The basic *a-priori* estimates are as follows:

There exists a constant $C > 0$ such that

$$\|\varphi\|^2 \le CQ(\varphi,\varphi), \tag{1}$$

for all $\varphi \in \dot{\mathcal{D}}^{p,q}(\Omega)$ with $\varphi \perp \mathcal{H}^{p,q}(\Omega)$.

The second is a local estimate, called a *subelliptic* estimate. Given $P \in \bar\Omega$ there exists a neighborhood U of P, constants $\varepsilon > 0$, and $C > 0$ such that

$$\|\varphi\|_\varepsilon^2 \le C\big(Q(\varphi,\varphi) + \|\varphi\|^2\big), \tag{2}$$

for all $\varphi \in \dot{\mathcal{D}}^{p,q}(\Omega) \cap C_0^\infty(U \cap \bar\Omega)$. Then (1) implies that there exists a bounded selfadjoint operator $N : L_2^{p,q}(\Omega) \to L_2^{p,q}(\Omega)$ with $N\mathcal{H}^{p,q}(\Omega) = 0$ and such that if $\alpha \perp H^{p,q}(\Omega)$ then $\Box N\alpha = \alpha$. Moreover, if (2) holds for every $P \in b\Omega$ then (1) holds. Furthermore, (2) implies that N is pseudolocal, that is, if α restricted to $U \cap \bar\Omega \in C^\infty(U \cap \bar\Omega)$, then $N\alpha$ restricted to $U \cap \bar\Omega \in C^\infty(U \cap \bar\Omega)$.

First, we will describe D'Angelo's contribution to understanding the basic subelliptic estimate (2) on pseudoconvex domains with C^∞ boundaries. From now on, we will restrict the discussion to $(0,1)$-forms, that is, $p = 0$ and $q = 1$ on *pseudoconvex* domains $\Omega \subset \mathbb{C}^n$ with $n > 1$ and smooth boundaries. We will denote the boundary of Ω by $b\Omega$. Before D'Angelo's work, the following results were known:

(a) (1) holds whenever the domain is pseudoconvex.
(b) (2) holds with $\varepsilon = \frac{1}{2}$ if and only if the Levi-form is positive definite at P.
(c) If $\Omega \subset \mathbb{C}^2$, then (2) holds if and only if there is a positive integer m so that the order of contact of every non-singular holomorphic curve through P with $b\Omega$ is less than or equal to m. If there is a curve with order of contact at P equal to m, then the largest ε for which (2) holds is $\varepsilon = \frac{1}{m}$.
(d) If $b\Omega$ contains a holomorphic curve through P, then (2) does not hold.
(e) Associated with each $P \in b\Omega$ is an increasing sequence of ideals of germs of functions at P, denoted by $I_k(P,\Omega)$ (called *multiplier ideals*) such there exists $\varepsilon(k) > 0$ so that if $f \in I_k(P,\Omega)$ there is a neighborhood U of P and a $C > 0$ so that

$$\|f\varphi\|_{\varepsilon(k)}^2 \le C\big(Q(\varphi,\varphi) + \|\varphi\|^2\big),$$

for all $\varphi \in \dot{\mathcal{D}}^{p,q}(\Omega) \cap C_0^\infty(U \cap \bar{\Omega})$. If $1 \in I_k(P,\Omega)$ for some k then P is of *finite ideal type*. Hence, if P is of finite ideal type then (2) holds. Furthermore, if $b\Omega$ is real analytic in a neighborhood of P then (2) holds if and only if P is of finite ideal type.

When all the above were established the crucial problem was to find a necessary and sufficient condition for (2) to hold in dimensions higher than two. In the two dimensional case, the result in item (c) deals with non-singular curves. In dimensions greater than two, the following example of Bloom and Graham (see [BG]) points to difficulties. Consider the hypersurface in $\mathbb{C}^3$ given by:

$$r(z,\bar{z}) = \mathrm{Re}(z_3) + |z_1^2 + z_2^3|^2 = 0.$$

Then any non-singular holomorphic curve through the origin has order of contact at most 6 with this hypersurface, but the singular curve given by $\{(z_1,z_2,z_3) \mid z_1^2 + z_2^3 = 0, z_3 = 0\}$ lies in the surface. Thus, in dimension greater than two, the fact that the order of contact of non-singular holomorphic curves is bounded at a point P does not imply that it is bounded at all points in any neighborhood of P. Hence, any attempt to generalize item (c) to n dimensions must take into account singular curves. In particular, note that the set of P for which (2) holds is open. Therefore, if a condition on the set of all holomorphic curves through P is equivalent to (2) then it must be an open condition. It would be natural to conjecture that such a condition would be that the order of contact of all holomorphic curves through P is bounded. To be more precise, let $D \subset \mathbb{C}$ be a neighborhood of the origin and let $z : D \to \mathbb{C}^n$ with $z(0) = P$ be a holomorphic map. Then, if r is a defining function for $b\Omega$, the order of contact of the holomorphic curve through P given by $\{z(t) \in \mathbb{C}^n \mid t \in D\}$ with $b\Omega$ is defined to be

$$\frac{v(z^*r)}{v(z)},$$

where v, applied to a parametrized holomorphic curve, denotes the order of vanishing at the origin. The *D'Angelo type* of P is defined by

$$T_P(r) = \sup_{z \in \mathcal{V}_P} \frac{v(z^*r)}{v(z)},$$

where $\mathcal{V}_P$ is the set of all parametrized holomorphic curves through P. Now consider $b\Omega \subset \mathbb{C}^3$ given by

$$r(z,\bar{z}) = \mathrm{Re}(z_3) + |z_1^2 - z_2 z_3|^2 + |z_2|^2.$$

Then we have $T_0(r) = 4$ and $T_P(r) = 8$ where $P = (0,0,ia)$ with $a \in \mathbb{R}$ and $a \neq 0$. Thus, the function $P \mapsto T_P(r)$ is not upper semicontinuous. Despite this D'Angelo proved (in [D1]) that if $P_0 \in b\Omega$ is of finite type then there is a neighborhood of P_0 on which

$$T_P(r) \le \frac{T_{P_0}(r)^{n-1}}{2^n}.$$

Hence, the set of P for which $T_P(r) < \infty$ is open. D'Angelo's proof consists of an ingenious analysis of the points of finite type. He proves that the Taylor polynomial $J_{k,P}r$ of a defining function r for $b\Omega$ can be written as

$$J_{k,P}r(z,\bar{z}) = \mathrm{Re}\big(h_{k,P}(z)\big) + \sum_j \big(\big|f^j_{k,P}(z)\big|^2 - \big|g^j_{k,P}(z)\big|^2\big),$$

where the $h_{k,P}$, $f^j_{k,P}$, and $g^j_{k,P}$ are holomorphic. Then the local geometry of $b\Omega$ around P is studied using ideals of hololomorphic functions constructed from these. D'Angelo's techniques and proofs were a major break through. These results enabled Catlin to prove (in [C1], [C2], and [C3]) that the estimate (2) holds if and only if P is of finite D'Angelo type. The work of Catlin completely transformed the theory of the $\bar{\partial}$-Neumann problem and it has given new insights into the CR geometry of weakly pseudoconvex domains as well as the resolution of singularities.

The relationship of the ideal type, described above in (e), to the D'Angelo type is still not completely understood. In particular, in [K1], the necessity condition is proved only for real analytic boundaries and this proof relies heavily on the results and methods developed in [DF]. The problem is how to deal with smooth boundaries that are not real analytic. The key is to find an effective algorithm to determine the ideal type. That is, the number of steps should depend only dimension and the D'Angelo type. D'Angelo (in [D3]) uncovered an important class of domains for which there is an effective algorithm but the algorithm given in [K1] is not effective. For more information about the relation between various types and subelliptic estimates, see [DK] and [CD1]. Y.-T. Siu in [S] found an effective algorithm for determining the ideal type when the defining function of a domain in $\mathbb{C}^n$ is of the form:

$$r(z,\bar{z}) = \mathrm{Re}(z_n) + F\big(z',\bar{z}'\big),$$

where $z' = (z_1, \ldots, z_{n-1})$. This result of Siu proves that finite ideal type is necessary for the subelliptic estimate (2) to hold in domains with such defining functions. Siu also outlines a proof of adapting his methods to prove the theorem in general.

The proof of the necessity in the real analytic case is based on the following result of Diederich and Fornaess (see [DF]): If P is a point of infinite D'Angelo type in the boundary of a pseudoconvex domain whose boundary is given by a defining function that is real analytic in a neighborhood of P, then there exists a holomorphic curve through P that lies in the boundary and hence P is also of infinite ideal type. In [K2], I give an explicit derivation of the Taylor series that define this curve. In a recent paper (see [N]), A. Nicoara gives an effective calculation of the vanishing of the Levi determinant depending on the D'Angelo type. All these results indicate that we are close to the understanding of the relation of the subelliptic estimate (2) with the finiteness of the ideal type.

The contributions described above were obtained by D'Angelo by first constructing and studying many examples. These examples and many others that D'Angelo constructed and analyzed are in themselves major contributions that illuminate the subject. In addition to his work on the $\bar{\partial}$-Neumann problem, D'Angelo has made many contributions to the theory of several complex variables, CR geometry, and their applications. A striking example is the result obtained together together with Catlin (see [CD2]). This work uses the Bergman kernel to generalize Hilbert's seventeenth problem.

References

[BG] T. Bloom and I. Graham, *A geometric characterization of points of type m on real submanifolds of* $\mathbb{C}^n$, J. Differential Geom. **12** (1977), 171–182. MR 0492369

[C1] D. W. Catlin, *Necessary conditions for the* $\bar{\partial}$*-Neumann problem*, Ann. of Math. (2) **117** (1983), 147–171. MR 0683805

[C2] D. W. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) **120** (1984), 529–586. MR 0769163

[C3] D. Catlin, *Subelliptic estimates for the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 113–191. MR 0898054

[CD1] D. W. Catlin and J. P. D'Angelo, *Subelliptic estimates*, Complex analysis, Trends Math., Birkhäuser, Basel, 2010, pp. 75–94. MR 2885109

[CD2] D. W. Catlin and J. P. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorpic mappings*, Math. Res. Lett. **3** (1996), 149–166. MR 1386836

[CS] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Mathematics, vol. **19**, Amer. Math. Soc., Providence, RI, 2001. MR 1800297

[D1] J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **115** (1982), 615–637. MR 0657241

[D2] J. P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993. MR 1224231

[D3] J. P. D'Angelo, *Finite type conditions and subelliptic estimates*, Modern methods in complex analysis, Ann. of Math. Stud., vol. **137**, Princeton Univ. Press, Princeton, NJ, 1995, pp. 63–78. MR 1369134

[DK] J. P. D'Angelo and J. J. Kohn, *Subelliptic estimates and finite type*, Math. Sci. Res. Inst. Publ. **37** (1999), 199–232. MR 1748604

[DF] K. Diederich and J. E. Fornaess, *Pseudoconvex domains with real analytic boundary*, Ann. of Math. (2) **107** (1978), 371–384. MR 0477153

[FK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy–Riemann complex*, Ann. of Math. Stud., vol. **75**, Princeton Univ. Press, Princeton, NJ, 1972. MR 0461588

[H] L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, NJ, 1966. MR 0203075

[K1] J. J. Kohn, *Subellipticity of the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains: Sufficient conditions*, Acta Math. **142** (1979), 79–122. MR 0512213

[K2] J. J. Kohn, *Multipliers on pseudoconvex domains with real analytic boundaries*, Boll. Unione Mat. Ital. (9) **3** (2010), 309–324. MR 2666360

[N] A. C. Nicoara, *Effective vanishing order of the Levi determinant*, Math. Ann. **354** (2012), 1233–1245. MR 2992996

[S] Y.-T. Siu, *Effective termination of Kohn's algorithm for subelliptic multipliers*; available at `arXiv:0706.4113v3` [math CV], 2008, 1–77.

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