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QUASIHYPERBOLIC BOUNDARY CONDITION: COMPACTNESS OF THE INNER BOUNDARY

PÄIVI LAMMI

Abstract. We prove that if a metric space satisfies a suitable growth condition in the quasihyperbolic metric and the Gehring–Hayman theorem in the original metric, then the inner boundary of the space is homeomorphic to the Gromov boundary. Thus, the inner boundary is compact.

1. Introduction

Let (Ω, d) and (Ω', d') be metric spaces. Recall that a homeomorphism $f : \Omega \to \Omega'$ is quasisymmetric, or η-quasisymmetric, if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right)$$

for all triples of distinct points x, y, z in X. Bonk, Heinonen and Koskela proved in [3, Theorem 1.11] that a bounded domain in $\mathbb{R}^n$ is uniform if and only if it is Gromov hyperbolic in the quasihyperbolic metric and its Euclidean boundary is quasisymmetric to the Gromov boundary, see Section 2 for definitions. It is also well known that in a bounded uniform domain $\Omega \subset \mathbb{R}^n$ the quasihyperbolic metric k satisfies a logarithmic growth condition

$$k(w, x) \leq C \log\left(\frac{d(w)}{d(x)}\right) + C', \tag{1.1}$$

where w is a fixed base point in Ω and constants $C \geq 1$ and $C' < \infty$ depend on the constant of uniformity and the diameter of the domain (cf. [8]). Here

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$d(x)$ is an abbreviation for the Euclidean distance from the point x to the boundary $\partial\Omega$.

However, this growth condition (1.1) does not necessarily guarantee that the boundaries are quasisymmetric. Let us construct a simply connected planar domain Ω for which the quasihyperbolic metric satisfies the growth condition (1.1), but the inner boundary $\partial_I\Omega$ in the Euclidean length metric is not quasisymmetric to the Gromov boundary $\partial_G\Omega$. To construct Ω, we "weld" the sequence of squares $Q_j = (a_j - l_j, a_j) \times (1, l_j)$, where $a_j = 1 - 2^{-j}$ and $l_j = 2^{-j}$, $j = 0, 1, 2, \ldots$, to the square $(-1,1)^2$ via the intervals $(a_j - l_j/2 - l_j^2, a_j - l_j/2 + l_j^2) \times \{1\}$, $j = 2, 3, 4, \ldots$. Let the origin be the base point. The quasihyperbolic metric k satisfies the condition (1.1), but taking two boundary points from the "throat" of the small square and the third from the top middle of the same square shows that the Gromov boundary cannot be quasisymmetric to $\partial_I\Omega$. Nevertheless, the inner boundary $\partial_I\Omega$ is homeomorphic to the Gromov boundary $\partial_G\Omega$, see [2] and [13].

Moreover, not every growth condition guarantees that the boundaries can be identified even as sets. For example, let

$$\Omega = \big\{(x,y) \in \mathbb{R}^2 : x \geq -1, |y| < \exp\{-x\}\big\}$$

and let the origin be the base point. Now the quasihyperbolic metric k satisfies the growth condition

$$k(0,z) \leq C\frac{d(0)}{d(z)} + C \tag{1.2}$$

for some $C \geq 1$, but the the ray $[0,\infty)$ cannot be identified with any point in $\partial\Omega$.

Thus in order that the inner boundary and the Gromov boundary be homeomorphic, we need a condition stronger than (1.2). Suppose we are given the growth condition

$$k(w,x) \leq \phi\left(\frac{d(w)}{d(x)}\right), \tag{1.3}$$

where $\phi : (0,\infty) \to (0,\infty)$ is an increasing function and w is a fixed base point in a metric space (Ω, d). It is more convenient to write

$$d(x) \leq \frac{d(w)}{\phi^{-1}(k(w,x))} \tag{1.4}$$

and let us assume, motivated by [10], that the function ϕ satisfies

$$\sum_{j=1}^{\infty} \frac{1}{\phi^{-1}(j)} < \infty. \tag{1.5}$$

Condition (1.5) is sufficient for the inner boundary and the Gromov boundary to be homeomorphic in a rather general setting. This is the content of our main theorem.

THEOREM 1.1. *Let* (Ω, d) *be a locally compact and non-complete quasi-convex space. Assume that* (Ω, k) *is Gromov hyperbolic and that the Gehring–Hayman theorem holds in* (Ω, d). *Let* $w \in \Omega$ *be a base point and suppose that the quasihyperbolic metric* k *satisfies* (1.3) *and* (1.5). *Under these assumptions, the identification map* $\mathrm{id} : \partial_G\Omega \to \partial_d\Omega$ *is a homeomorphism and moreover,* $\partial_d\Omega$ *is compact.*

It is natural to assume that the Gehring–Hayman theorem holds in (Ω, d) because Balogh and Buckley proved in [1] that a bounded domain in $\mathbb{R}^n$ equipped with the Euclidean length metric satisfies the Gehring–Hayman theorem if the domain is Gromov hyperbolic in the quasihyperbolic metric.

The conclusion of Theorem 1.1 appears to follow from previously known results only when $\Omega \subset \mathbb{R}^n$ is a quasiconformally equivalent to a bounded uniform domain. In this case, (1.3) and (1.5) imply that $\partial_d\Omega$ is homeomorphic to the boundary of the uniform domain by results of Koskela and Nieminen [12] and Koskela, Onninen and Tyson [13]. Moreover, $\partial_d\Omega$ is homeomorphic to the Gromov boundary $\partial_G\Omega$ by Bonk, Heinonen and Koskela [3].

Our proof of Theorem 1.1 borrows some ideas from [10], but our arguments are necessarily very different.

2. Definitions

Let (Ω, d) be a metric space. The *boundary* $\partial_d\Omega$ of Ω is $\partial_d\Omega = \overline{\Omega} \setminus \Omega$, where $\overline{\Omega}$ is the metric completion of Ω. By a *curve* we mean a continuous mapping $\gamma : I \to \Omega$ from an interval $I \subset \mathbb{R}$ to Ω. We also denote the image set of γ by γ. For $I = [a, b]$ the *length*, $\ell_d(\gamma)$, of γ with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup \sum_{i=0}^{m-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a, b]$. If $\ell_d(\gamma) < \infty$, then γ is said to be a *rectifiable curve.* When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c,d]}),$$

where supremum is taken over all compact subintervals $[c, d]$. If $\gamma : [a, b) \to \Omega$ is rectifiable, then there is a continuous extension $\gamma : [a, b] \to \overline{\Omega}$, and γ has a terminal *endpoint* $\gamma(b) := \lim_{t \nearrow b} \gamma(t) \in \overline{\Omega}$.

The metric d is called a *quasi-convex metric* and the metric space (Ω, d) a *quasi-convex space* if there is a constant $A \geq 1$ such that every pair of points $x, y \in \Omega$ can be joined with a curve γ which satisfies

$$\ell_d(\gamma) \leq Ad(x, y). \tag{2.1}$$

Added to this, a locally compact and non-complete quasi-convex space (Ω, d) is called a *uniform space* if there is a constant $B \geq 1$ such that for every pair of points $x, y \in \Omega$ the curve γ in the previous condition (2.1) satisfies also the *twisted cone condition*—that is, for every $a \in \gamma$

$$\min\{\ell_d(\gamma(x,a)), \ell_d(\gamma(a,y))\} \le B \operatorname{dist}_d(a, \partial_d \Omega), \tag{2.2}$$

where $\gamma(x,a)$ is the subcurve of γ from x to a and $\gamma(a,y)$ is the rest of the curve.

If for every $x, y \in \Omega$ it holds that

$$d(x,y) = \ell_d(x,y) := \inf \ell_d(\gamma), \tag{2.3}$$

where the infimum is taken over all curves joining points x and y, then the metric $d = \ell_d$ is a *length metric* and the space (Ω, d) is a *length space.* If $d(x,y) = \ell_d(\gamma)$ for some curve γ joining points $x, y \in \Omega$, then γ is said to be a *geodesic.* If every pair of points in Ω can be joined with a geodesic, then Ω is called a *geodesic space.*

For a rectifiable curve $\gamma : [a,b] \to \Omega$ we define the *arc length* $s : [a,b] \to [0,\infty)$ along γ by

$$s(t) = \ell_d(\gamma|_{[a,t]}).$$

It is not hard to see that for every rectifiable curve γ it holds that

$$\ell_d(\gamma) = \int_\gamma ds = \int_a^b ds(t).$$

Let (Ω, d) be a geodesic metric space and let $\delta \ge 0$. Denote by $[x,y]$ any geodesic joining two points x and y in Ω. If for all triples of geodesics $[x,y], [y,z], [z,x]$ in Ω every point in $[x,y]$ is within distance δ from $[y,z] \cup [z,x]$, the space Ω is called *δ-hyperbolic.* In other words, it means that *geodesic triangles* in Ω are *δ-thin.* Moreover, we say that a space is *Gromov hyperbolic* if it is δ-hyperbolic for some δ. All Gromov hyperbolic spaces in this paper are assumed to be unbounded.

A *geodesic ray* in Ω is an isometric image in Ω of the interval $[0,\infty)$, and two rays are *equivalent* if their Hausdorff distance is finite. That is, both rays have finite neighbourhoods such that one ray is contained in the neighbourhood of the other ray. The *Gromov boundary* $\partial_G \Omega$ of a proper (closed balls are compact) geodesic Gromov hyperbolic space Ω is defined to be the set of equivalence classes of geodesic rays. We say that a geodesic ray ends at $\xi \in \partial_G \Omega$ if it represents the point ξ. For each $x \in \Omega$ and $\xi \in \partial_G \Omega$ there is a geodesic ray $[x, \xi]$ issuing from x and ending at ξ. Similarly, for every pair of points $\xi, \eta \in \partial_G \Omega$ there is a *geodesic line* $[\xi, \eta]$ from ξ to η which is an isometric image of $(-\infty, \infty)$ ending at ξ and η in the obvious sense. See [5, §III.H, Lemma 3.1 and Lemma 3.2].

Alternatively, the Gromov boundary of a proper geodesic Gromov hyperbolic space (Ω, d) can be defined as the set of equivalence classes of sequences $(x_n) \subset \Omega$ which *tend to infinity* in the sense that

$$\lim_{n,m \to \infty} (x_n | x_m)_w = \infty,$$

where

$$(x|y)_w = \frac{1}{2}\{d(w,x) + d(w,y) - d(x,y)\} \tag{2.4}$$

is the *Gromov product* between points $x, y \in \Omega$ with respect to a base point $w \in \Omega$. Two sequences $(x_n), (y_n) \subset \Omega$, tending to infinity, are equivalent if

$$\lim_{n\to\infty} (x_n|y_n)_w = \infty.$$

The Gromov product (2.4) extends to the Gromov boundary $\partial_G\Omega$ in a natural way. From the geometric point of view, the Gromov product has the following property:

$$\big|(x|y)_w - \mathrm{dist}_d\big(w, [x,y]\big)\big| \le 8\delta \tag{2.5}$$

for any pair of points $x, y \in \Omega \cup \partial_G\Omega$ and any geodesic $[x, y]$ between the points. See [6] for more detailed definitions of Gromov boundaries.

The Gromov boundary $\partial_G\Omega$ does not, in general, posses a preferred metric. However, there is a collection of distance functions

$$\rho_{w,\varepsilon}(\xi,\eta) = \exp\{-\varepsilon(\xi|\eta)_w\}, \quad \xi,\eta \in \partial_G\Omega, \tag{2.6}$$

where $\varepsilon > 0$ and $w \in \Omega$ is a base point. Although, the function in (2.6) does not necessarily satisfy the triangle inequality, and further, does not necessarily define a metric, there is $\varepsilon(\delta) > 0$ such that for $0 < \varepsilon < \varepsilon(\delta)$ one finds a metric $d_{w,\varepsilon}$ in $\partial_G\Omega$ satisfying

$$\frac{1}{2}\rho_{w,\varepsilon}(\xi,\eta) \le d_{w,\varepsilon}(\xi,\eta) \le \rho_{w,\varepsilon}(\xi,\eta) \tag{2.7}$$

for $\xi, \eta \in \partial_G\Omega$. By combining (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned} \frac{1}{C(\delta)} \exp\{-\varepsilon\, \mathrm{dist}_d\big(w, [\xi,\eta]\big)\} &\le d_{w,\varepsilon}(\xi,\eta) \\ &\le C(\delta) \exp\{-\varepsilon\, \mathrm{dist}_d\big(w, [\xi,\eta]\big)\}, \end{aligned} \tag{2.8}$$

whenever $0 < \varepsilon < \varepsilon(\delta)$ and $\xi, \eta \in \partial_G\Omega$. The Gromov boundary equipped with this metric $d_{w,\varepsilon}$ is compact. For more details, see [3], [4] and [9].

Assume that (Ω, d) is a locally compact, rectifiably connected, and noncomplete metric space. Then the boundary $\partial_d\Omega$ is nonempty and for $z \in \Omega$ we denote

$$d(z) = \mathrm{dist}_d(z, \partial_d\Omega) = \inf\{d(z,x) : x \in \partial_d\Omega\}.$$

Next we define the *quasihyperbolic metric* k in Ω by introducing the length element

$$\frac{ds}{d(z)}.$$

Thus for $x, y \in \Omega$,

$$k(x,y) = \inf \int_\gamma \frac{ds}{d(z)} = \int_a^b \frac{ds(t)}{d(\gamma(t))}, \tag{2.9}$$

where the infimum is taken over all rectifiable curves γ joining points x and y. Bonk, Heinonen and Koskela proved in [3, Proposition 2.8] that if the identity map $(\Omega, d) \to (\Omega, \ell_d)$ is a homeomorphism, then it is a homeomorphism $(\Omega, d) \to (\Omega, k)$ and (Ω, k) is complete. Furthermore, as a complete locally compact length space (Ω, k) is geodesic and proper (cf. [5, §I.3]). See [1] for a geometric characterisation of Gromov hyperbolicity of (Ω, k) in $\mathbb{R}^n$.

From now on, we denote by $[x, y]$ a quasihyperbolic geodesic joining points x and y in Ω. By $[x, y](t)$ we mean the image of $t \in [0, k(x, y)] \subset \mathbb{R}$ in the quasihyperbolic geodesic $[x, y]$, where $[x, y] : [0, k(x, y)] \to \Omega$ is a mapping parametrized by the arc length with respect to the metric k.

There are also two elementary inequalities, valid for all $x, y \in \Omega$:

$$\left|\log \frac{d(x)}{d(y)}\right| \le k(x, y), \tag{2.10}$$

$$\log\left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right) \le k(x, y). \tag{2.11}$$

From inequality (2.11) it immediately follows that if a sequence $(x_j) \subset \Omega$ satisfies $k(x_j, x_{j+1}) \le 1$, then

$$d(x_j, x_{j+1}) \le 2d(x_j). \tag{2.12}$$

When we say that the *Gehring–Hayman theorem* holds in (Ω, d), we mean that there is a constant $C \ge 1$ such that for every pair of points $x, y \in (\Omega, d)$,

$$\ell_d\big([x, y]\big) \le C\ell_d(\gamma), \tag{2.13}$$

where γ is any other curve in Ω joining points x and y (cf. [7], [11]).

3. Proof of the theorem

Let us first show that from the growth condition (1.3) and (1.5) it follows that the metric space (Ω, d) is bounded. It suffices to prove that for some $M > 0$ it holds that $d(w, x) \le M$ for every $x \in \Omega$. Let $x \in \Omega$. If $k(w, x) < 1$, then the claim is obvious. Suppose that $k(w, x) \ge 1$. Let $(x_j) \subset \Omega$ be a sequence such that $k(w, x_j) = j$ for each $j \in \mathbb{N}$ and $x_1, \ldots, x_{[k(w,x)]} \in [w, x]$, where $[k(w, x)]$ is the integer part of the quasihyperbolic distance from w to x. Then by (2.12) and (1.5) we obtain

$$\begin{aligned} d(w, x) &\le d(w, x_1) + \sum_{j=1}^{[k(w,x)]-1} d(x_j, x_{j+1}) + d(x_{[k(w,x)]}, x)[-1pt] \\ &\le 2d(w) + 2\sum_{j=1}^{\infty} d(x_j)[-1pt] \\ &\le 2d(w) + 2d(w)\sum_{j=1}^{\infty} \frac{1}{\phi^{-1}(j)} =: M. \end{aligned}$$

Thus, the constant M depends only on the base point w.

Let us start to prove the theorem, step by step. The first step is to prove that an identification map $\mathrm{id}\colon \partial_G\Omega \to \partial_d\Omega$ exists and it is well defined. Let $\xi \in \partial_G\Omega$. There is a geodesic ray $[w,\xi]$ which represents the point ξ. To show that $[w,\xi]$ ends at some point $\xi' \in \partial_d\Omega$, and thus it represents some point ξ' in the boundary $\partial_d\Omega$, we show that $\ell_d([w,\xi]) < \infty$. Let $(x_j) \subset [w,\xi]$ be a sequence such that $k(w,x_j) = j$ for each $j \in \mathbb{N}$. Now, from the Gehring–Hayman theorem, quasi-convexity, and inequalities (2.12) and (1.5) it follows that

$$\begin{aligned}\ell_d\big([w,\xi]\big) &= \ell_d\big([w,x_1]\big) + \sum_{j=1}^{\infty} \ell_d\big([x_j,x_{j+1}]\big)\\ &\le CAd(w,x_1) + CA\sum_{j=1}^{\infty} d(x_j,x_{j+1})\\ &\le 2CAd(w) + 2CA\sum_{j=1}^{\infty} d(x_j)\\ &\le 2CAd(w) + 2CAd(w)\sum_{j=1}^{\infty} \frac{1}{\phi^{-1}(j)} < \infty.\end{aligned}$$

Hence, the geodesic ray $[w,\xi]$ ends in the sense of the metric d, and an endpoint ξ' exists which clearly has to be on the boundary $\partial_d\Omega$.

Let $[y,\xi]$ be another geodesic ray which represents the point $\xi \in \partial_G\Omega$. Even though the starting point $y \in \Omega$ may differ from the base point $w \in \Omega$, the ray still represents some point in $\partial_d\Omega$. Indeed, take a sequence $(x_j) \subset [y,\xi]$ such that $k(y,x_j) = j$. Now $k(w,x_j) \ge k(y,x_j) - k(y,w)$ and thus, when $j > k(y,w)$,

$$d(x_j) \le \frac{d(w)}{\phi^{-1}(k(w,x_j))} \le \frac{d(w)}{\phi^{-1}(j - k(w,y))}.$$

Moreover, because (Ω,d) is bounded, quasi-convex and the Gehring–Hayman theorem holds, for $j > k(y,w)$ it is true that

$$\begin{aligned}\ell_d\big([y,\xi]\big) &= \ell_d\big([y,x_j]\big) + \sum_{l=j}^{\infty} \ell_d\big([x_l,x_{l+1}]\big)\\ &\le CAd(y,x_j) + CA\sum_{l=j}^{\infty} d(x_l,x_{l+1})\\ &\le CAM + 2CAd(w)\sum_{l=j}^{\infty} \frac{1}{\phi^{-1}(l - k(w,y))} < \infty.\end{aligned}$$

In the quasihyperbolic sense, geodesic rays $[w,\xi]$ and $[y,\xi]$ are equivalent which means, by definition, that their Hausdorff distance is finite. Let us show that there is a constant $M>0$ such that

$$\sup\bigl\{k\bigl([w,\xi](t),[y,\xi](t)\bigr) : t\geq 0\bigr\}\leq M<\infty. \tag{3.1}$$

The proof is adapted from the book [9, §7, Proposition 2]. Let $N>0$ be the Hausdorff distance between geodesics. For $t\geq 0$ let $s_t\geq 0$ be such that $k([y,\xi](t),[w,\xi](s_t))<N$. By the triangle inequality we obtain that, for each $t\geq 0$,

$$k\bigl([w,\xi](s_0),[w,\xi](s_t)\bigr)=|s_0-s_t|\leq t+2N \tag{3.2}$$

and

$$k\bigl([w,\xi](s_0),[w,\xi](s_t)\bigr)=|s_0-s_t|\geq t-2N. \tag{3.3}$$

Thus inequalities (3.2) and (3.3) together yield

$$|s_t-t|\leq s_0+2N,$$

from which by the triangle inequality we obtain that

$$k\bigl([y,\xi](t),[w,\xi](t)\bigr)\leq N+|s_t-t|\leq 3N+s_0=:M.$$

From the basic estimate (2.11) and from (3.1) we obtain that, for each $t\geq 0$,

$$\log\left(1+\frac{d([w,\xi](t),[y,\xi](t))}{\min\{d([w,\xi](t)),d([y,\xi](t))\}}\right)\leq k\bigl([w,\xi](t),[y,\xi](t)\bigr)\leq M$$

and further, for each $t\geq 0$,

$$\frac{d([w,\xi](t),[y,\xi](t))}{\min\{d([w,\xi](t)),d([y,\xi](t))\}}\leq e^M-1.$$

Because both rays have an endpoint on the boundary $\partial_d\Omega$, both $d([w,\xi](t))\searrow 0$ and $d([y,\xi](t))\searrow 0$, when $t\to\infty$. Then also $d([w,\xi](t),[y,\xi](t))\searrow 0$. Hence, in the sense of the metric d, the curve $[y,\xi]$ also ends at the point $\xi'\in\partial_d\Omega$ and the identification map is well defined.

The next step is to show that the identification map is injective. Let $\xi\neq\eta$ be points on $\partial_G\Omega$. Then there is a geodesic line $[\xi,\eta]\subset\Omega$ and also unique identification points ξ' and η' on the boundary $\partial_d\Omega$. The Gehring–Hayman theorem gives us that

$$\ell_d\bigl([\xi,\eta]\bigr)\leq C\ell_d(\gamma)$$

for any other curve γ joining points ξ' and η'. Thus, if it were true that $\xi'=\eta'$, then because the metric d is a quasi-convex metric, it would hold that $\ell_d([\xi,\eta])=0$. Furthermore, this implies that $\xi=\eta$, which is a contradiction.

The hard part is to prove that the identification map $\mathrm{id}:\partial_G\Omega\to\partial_d\Omega$ is surjective. We have to show that for every $\xi'\in\partial_d\Omega$ there exists an identification point $\xi\in\partial_G\Omega$, and thus there exists a quasihyperbolic geodesic ray $[w,\xi]$ that ends at ξ' in the sense of d. Let $\xi'\in\partial_d\Omega$. In the metric d, there is a rectifiable curve γ joining points w and ξ'. Let $(x_j)\subset\gamma$ be a sequence which converges to the point ξ'. Because (Ω,k) is geodesic, for each $j\in\mathbb{N}$ there is a quasihyperbolic geodesic $[w, x_j]$. By using the Arzela–Ascoli theorem and the properness of (Ω, k) we find a subsequence of $([w, x_j])_j$, still denoted the same, which converges uniformly on compacta to a geodesic ray $[w, \eta]$. It remains to show that $[w, \eta]$ is the ray $[w, \xi]$ we were looking for. We know that the ray $[w, \eta]$ ends in the sense of the metric d at some point $\eta' \in \partial_d \Omega$ and we need to show that $\eta' = \xi'$. In other words, it remains to show that $d(\xi', [w, \eta]) = 0$.

To show that $d(\xi', [w, \eta]) = 0$, we prove that for every $j \in \mathbb{N}$ the sequence of quasihyperbolic geodesics $([x_j, x_{j+p}])_p$ has a subsequence that converges to a quasihyperbolic ray $[x_j, \eta]$. Because (x_j) is a Cauchy sequence in the metric d, we obtain by using this subsequence, denoted the same, with the Gehring–Hayman theorem and quasi-convexity that for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$\ell_d\big([x_j, x_{j+p}]\big) \le CAd(x_j, x_{j+p}) < CA\varepsilon$$

for all $p \ge 1$, when $j \ge N_\varepsilon$. And furthermore,

$$\ell_d\big([x_j, \eta]\big) \le CA\varepsilon,$$

when $j > N_\varepsilon$. Then we conclude that

$$\begin{aligned} d\big(\xi', [w, \eta]\big) &\le d\big(\xi', \eta'\big) \le d\big(\xi', x_j\big) + d\big(x_j, \eta'\big) \\ &\le d\big(\xi', x_j\big) + \ell_d\big([x_j, \eta]\big) \\ &\le \varepsilon + CA\varepsilon, \end{aligned}$$

when $j > N_\varepsilon$, and therefore $\eta' = \xi'$.

Thus, let $j \in \mathbb{N}$. Again, by using Arzela–Ascoli theorem and the properness of (Ω, k), we find a subsequence of $([x_j, x_{j+p}])_p$, still denoted the same, that converges uniformly on compacta to a geodesic ray $[x_j, \eta_j]$. Let us show that $\eta_j = \eta$. It suffices to prove that for some $M > 0$,

$$k\big([w, \eta](t), [x_j, \eta_j](t)\big) \le M \quad \text{for every } t \ge 0. \tag{3.4}$$

Let us parametrize our quasihyperbolic geodesics $[x, y]$ in the following, natural way: $[x, y] : [0, \infty) \to (\Omega, k)$ such that $[x, y](t)$ is isometric, when $t \le k(x, y)$, and $[x, y](t) = y$, when $t > k(x, y)$. Let $\varepsilon > 0$ and let $t \ge 0$. Because the sequence $([w, x_n])_n$ converges uniformly on compacta, there is $N_1 \in \mathbb{N}$ such that

$$k\big([w, x_n](t), [w, \eta](t)\big) < \varepsilon, \quad \text{when } n > N_1. \tag{3.5}$$

Furthermore, the sequence $([x_j, x_n])_{n>j}$ converges uniformly on compacta and therefore there is $N_2 \in \mathbb{N}$ such that

$$k\big([x_j, x_n](t), [x_j, \eta_j](t)\big) < \varepsilon, \quad \text{when } n > N_2. \tag{3.6}$$

If we can prove that there is a constant $M' > 0$ such that for each $n > j$ it holds that

$$k\big([w, x_n](t), [x_j, x_n](t)\big) \le M', \tag{3.7}$$

then choosing $N := \max\{N_1, N_2\}$, letting $n > N$, and using inequalities (3.5), (3.6) and (3.7) we obtain the desired result (3.4):

$$\begin{aligned} k\big([w,\eta](t),[x_j,\eta_j](t)\big) &\le 2\varepsilon + k\big([w,x_n](t),[x_j,x_n](t)\big) \\ &\le 2\varepsilon + M' =: M < \infty. \end{aligned}$$

Let us prove inequality (3.7). Let $n > j$. The metric space (Ω, k) is Gromov hyperbolic, and thus there is some $\delta \ge 0$ such that each triangle is δ-thin. If $\operatorname{dist}_k([w,x_n](t),[w,x_j]) \le \delta$ and $\operatorname{dist}_k([x_j,x_n](t),[w,x_j]) \le \delta$, then

$$k\big([w,x_n](t),[x_j,x_n](t)\big) \le 2\delta + k(w,x_j),$$

and we may choose $M' = 2\delta + k(w,x_j)$. Otherwise, we have $u \in [x_j,x_n]$ such that $k([w,x_n](t),u) \le \delta$, or $v \in [w,x_n]$ such that $k([x_j,x_n](t),v) \le \delta$. Let us consider the first case, the second case can be handled by similar calculations.

Because of the parametrization of geodesics we have that

$$\begin{aligned} t + k\big([w,x_n](t),x_n\big) = k(w,x_n) &\le k(w,x_j) + k(x_j,x_n) \\ &\le k(w,x_j) + t + k\big([x_j,x_n](t),x_n\big). \end{aligned}$$

We have same kind of inequalities for $k(x_j,x_n)$, and therefore

$$\big|k\big([w,x_n](t),x_n\big) - k\big([x_j,x_n](t),x_n\big)\big| \le k(w,x_j).$$

Similarly, we conclude that

$$\big|k(u,x_n) - k\big([w,x_n](t),x_n\big)\big| \le \delta.$$

Thus, (3.7) follows from these inequalities:

$$\begin{aligned} k\big([w,x_n](t),[x_j,x_n](t)\big) &\le \delta + k\big(u,[x_j,x_n](t)\big) \\ &= \delta + \big|k(u,x_n) - k\big([x_j,x_n](t),x_n\big)\big| \\ &\le 2\delta + k(w,x_j). \end{aligned}$$

The penultimate step is to show that the identification map $\mathrm{id} : \partial_G\Omega \to \partial_d\Omega$ is continuous. Let $\varepsilon > 0$ and $\xi \in \partial_G\Omega$. Thus, we have to find $\delta_\varepsilon > 0$ such that

$$d\big(\xi',\eta'\big) < \varepsilon, \quad \text{when } \eta \in \partial_G\Omega \quad \text{and} \quad d_{w,\tilde\varepsilon}(\xi,\eta) < \delta_\varepsilon. \tag{3.8}$$

Because of the second inequality in (2.8) it suffices to prove that there is a constant $\delta_\varepsilon > 0$ such that

$$d\big(\xi',\eta'\big) < \varepsilon, \quad \text{when } \eta \in \partial_G\Omega \quad \text{and} \quad \exp\big\{-\tilde\varepsilon \operatorname{dist}_k\big(w,[\xi,\eta]\big)\big\} < \delta_\varepsilon. \tag{3.9}$$

Here $0 < \tilde\varepsilon < \tilde\varepsilon(\delta)$, and $\delta \ge 0$ is the constant from the δ-hyperbolicity. Because of the properness of (Ω,k) for each $\eta \in \partial_G\Omega, \eta \ne \xi$, there is $w_\eta \in [\xi,\eta]$ such that

$$\operatorname{dist}_k\big(w,[\xi,\eta]\big) = k(w,w_\eta). \tag{3.10}$$

Thus, it actually suffices to prove that there is a constant $\delta_\varepsilon > 0$ such that

$$\max\big\{\ell_d\big([w_\eta,\xi]\big),\ell_d\big([w_\eta,\eta]\big)\big\} < \varepsilon \quad \text{when } \eta\in\partial_G\Omega \quad \text{and} \tag{3.11}$$
$$k(w,w_\eta) > -\frac{\log\delta_\varepsilon}{\tilde{\varepsilon}}.$$

We may assume that $\ell_d([w_\eta,\xi]) \ge \ell_d([w_\eta,\eta])$. Let $x_j\in[w_\eta,\xi]$ be such that $k(w_\eta,x_j)=j$ for each $j\in\mathbb{N}$. Let us first show that for each $j\in\mathbb{N}$ there is $p_j\in[w,x_j]$ such that

$$k(w_\eta,p_j)\le 2\delta+1. \tag{3.12}$$

If $k(w_\eta,w)\le 2\delta+1$, then the claim is clear. Let us assume that $k(w_\eta,w) > 2\delta+1$, and pick $z\in[w,w_\eta]$ so that

$$k(w_\eta,z)=\delta+1=\operatorname{dist}_k\big(z,[\xi,\eta]\big).$$

From the δ-hyperbolicity of (Ω,k) it follows that $\operatorname{dist}_k(z,[w,x_j])\le\delta$ for each $j\in\mathbb{N}$. Hence for every $j\in\mathbb{N}$ there is $p_j\in[w,x_j]$ which satisfies $k(z,p_j)\le\delta$. Therefore for every $j\in\mathbb{N}$,

$$k(w_\eta,p_j)\le k(w_\eta,z)+k(z,p_j)\le 2\delta+1$$

and we have the desired inequality (3.12). Furthermore, we estimate that for each $j\in\mathbb{N}$

$$\begin{aligned} k(w,w_\eta)+j &= k(w,w_\eta)+k(w_\eta,x_j)\\ &\ge k(w,x_j)=k(w,p_j)+k(p_j,x_j)\\ &\ge \big(k(w,w_\eta)-k(w_\eta,p_j)\big)+\big(k(w_\eta,x_j)-k(w_\eta,p_j)\big)\\ &\ge k(w,w_\eta)+j-4\delta-2. \end{aligned} \tag{3.13}$$

Hence by the Gehring–Hayman theorem, quasi-convexity, inequality (2.12), the growth condition (1.5) and inequality (3.13) we obtain that

$$\begin{aligned} \ell_d\big([w_\eta,\xi]\big) &= \ell_d\big([w_\eta,x_1]\big)+\sum_{j=1}^{\infty}\ell_d\big([x_j,x_{j+1}]\big)\\ &\le CAd(w_\eta,x_1)+CA\sum_{j=1}^{\infty}d(x_j,x_{j+1})\\ &\le 2CAd(w_\eta)+2CA\sum_{j=1}^{\infty}d(x_j)\\ &\le 2CA\frac{d(w)}{\phi^{-1}(k(w,w_\eta))}+2CA\sum_{j=1}^{\infty}\frac{d(w)}{\phi^{-1}(k(w,x_j))}\\ &\le 2CA\frac{d(w)}{\phi^{-1}(k(w,w_\eta))}+2CA\sum_{j=1}^{\infty}\frac{d(w)}{\phi^{-1}(k(w,w_\eta)+j-4\delta-2)}\\ &< \varepsilon, \end{aligned}$$

when $k(w, w_\eta)$ is sufficiently large. We conclude that the identification map is continuous, even uniformly continuous.

The last step is to prove that also the inverse of the identification map, $\mathrm{id}^{-1} : \partial_d\Omega \to \partial_G\Omega$, is continuous. Let $\varepsilon > 0$ and $\xi' \in \partial_d\Omega$. Let for every $\eta \in \partial_G\Omega$ $w_\eta \in [\xi, \eta]$ be the point such that $\mathrm{dist}_k(w, [\xi, \eta]) = k(w, w_\eta)$. When ξ' and η' are sufficiently close to each other, we may assume that $d(w) \geq d(w_\eta)$. Then we obtain by using the basic estimate (2.10), the Gehring–Hayman theorem and quasi-convexity that

$$\begin{aligned} d_{w,\tilde{\varepsilon}}(\xi, \eta) &\leq C(\delta) \exp\{-\tilde{\varepsilon} k(w, w_\eta)\} \\ &\leq C(\delta) \left(\frac{d(w_\eta)}{d(w)} \right)^{\tilde{\varepsilon}} \\ &\leq C(\delta) \left(\frac{d(w_\eta, \xi')}{d(w)} \right)^{\tilde{\varepsilon}} \\ &\leq C(\delta) \left(\frac{\ell_d([\xi, \eta])}{d(w)} \right)^{\tilde{\varepsilon}} \\ &\leq C(\delta) \left(\frac{CAd(\xi', \eta')}{d(w)} \right)^{\tilde{\varepsilon}} < \varepsilon, \end{aligned}$$

when $\eta' \in \partial_d\Omega$ and $d(\xi', \eta') \leq (\frac{\varepsilon}{C(\delta)})^{1/\tilde{\varepsilon}} \frac{d(w)}{CA} =: \delta_\varepsilon$.

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Päivi Lammi, Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland

E-mail address: paivi.e.lammi@jyu.fi