

EXISTENCE OF REGULAR NEIGHBORHOODS FOR H -SURFACES

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ABSTRACT. In this paper, we study the global geometry of complete, constant mean curvature hypersurfaces embedded in n -manifolds. More precisely, we give conditions that imply properness of such surfaces and prove the existence of fixed size one-sided regular neighborhoods for certain constant mean curvature hypersurfaces in certain n -manifolds.

1. Introduction

In this paper, we present some useful results on the geometry of a complete H -hypersurface M in an n -manifold N , where by H -hypersurface we mean that M is an *embedded* (injectively immersed) hypersurface of constant mean curvature $H \geq 0$. When $H > 0$, N has bounded absolute sectional curvature and M is connected, has bounded second fundamental form and it is proper and separating in N , then we prove that M has a fixed size regular neighborhood on its mean convex side (see Remark 3.4 and Theorem 3.5). This result is useful for obtaining local $(n - 1)$ -dimensional volume estimates for such an M in terms of local volume estimates in N . This existence of a fixed size one-sided neighborhood for certain H -surfaces M is a cornerstone in proving the *CMC* Dynamics and Minimal Elements Theorems in [14], and more generally, the *CMC* Dynamics and Minimal Elements Theorems in other homogenous n -manifolds such as \mathbb{R}^n and hyperbolic n -space \mathbb{H}^n [13]. Theorem 3.5 also plays a key role in the classification of H -spheres in the 3-dimensional solvable group Sol_3 with respect to its usual homogeneous structure; see [8].

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In Proposition 2.4, we demonstrate that complete embedded H -hypersurfaces M with locally bounded second fundamental form in certain complete n -manifolds N are properly embedded. For example, when $n = 3$ this proposition implies that if for some $\varepsilon > 0$, M and N satisfy $H^2 \geq S_N + \varepsilon$, where S_N is the average sectional curvature (scalar curvature) of N , then M is properly embedded in N ; this special case of Proposition 2.4 was found independently by Harold Rosenberg and this case is essentially the statement of Theorem 1.1 in [18] (also see [10]). If $N = \mathbb{H}^3$ and $H > 1$, it then follows that such an M is properly embedded in \mathbb{H}^3 ; see [15] for the same properness result for connected, embedded $H = 1$ surfaces in \mathbb{H}^3 .

Finally, in the last section we observe how using results by Grosse-Brauckmann in [5] make it possible to construct examples of complete disconnected H -surfaces in a fixed horizontal slab in \mathbb{R}^3 , each of which is properly embedded in the open slab but not properly embedded in \mathbb{R}^3 .

2. Background on H -laminations

In order to help understand the results described in this paper, we make the following definitions. For further discussion on the general theory of weak H -laminations, whose definition appears below, and related CMC foliations, see the survey [10].

DEFINITION 2.1. Let M be a complete surface embedded in a three-manifold N . A point $p \in N$ is a *limit point* of M if there exists a sequence of points $\{p_n\}_n \subset M$ which diverges to infinity on M with respect to the intrinsic Riemannian topology on M but converges in N to p as $n \rightarrow \infty$. Let $L(M)$ denote the set of all limit points of M in N . In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where \overline{M} denotes the closure of M in N .

DEFINITION 2.2. A *weak H -lamination* \mathcal{L} of a three-manifold N is a collection of immersed surfaces $\{L_\alpha\}_{\alpha \in I}$ of constant positive mean curvature H called *leaves* of \mathcal{L} satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} L_\alpha$ is a closed subset of N .
2. For each leaf L_α of \mathcal{L} , considered to be the zero section Z_α of its normal bundle $N_b(L_\alpha)$, there exists a closed one-sided neighborhood $\Delta(Z_\alpha) \subset N_b(L_\alpha)$ of Z_α such that:
 - (a) The exponential map $\exp : \Delta(Z_\alpha) \rightarrow N$ is a submersion.
 - (b) With respect to the pull-back metric on $\Delta(Z_\alpha)$, $Z_\alpha \subset \partial\Delta(Z_\alpha)$ is mean convex.
 - (c) The inverse image $\exp^{-1}(\mathcal{L}) \cap \Delta(Z_\alpha)$ is a lamination of $\Delta(Z_\alpha)$.

When $H = 0$, by the maximum principle, each leaf of \mathcal{L} is embedded; thus \mathcal{L} is an actual lamination and we call \mathcal{L} a *minimal* lamination. When $H > 0$, the maximum principle implies that each nonembedded leaf of a weak H -lamination is almost-embedded in the following sense.

DEFINITION 2.3. We call an immersed hypersurface in an n manifold N , which is the image of an immersion $f : M \rightarrow N$, *almost-embedded* if whenever $p, q \in M$ and $f(p) = f(q)$, then there exist disjoint neighborhoods U_p, U_q such that $f(U_p)$ and $f(U_q)$ lie locally on opposite sides of $f(p)$ and f is injective on an open dense set of M .

The reader not familiar with the subject of minimal or weak H -laminations should think about a geodesic on a Riemannian surface. If the geodesic is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination of the surface. When this geodesic has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics through the accumulation points forming the leaves of the geodesic lamination of the surface. The similar result is true for a complete H -surface of locally bounded second fundamental form (curvature is bounded in compact extrinsic balls) embedded in a Riemannian three-manifold, that is, the closure of a complete, embedded H -surface of locally bounded second fundamental form has the structure of a weak H -lamination. For the sake of completeness, we now sketch the proof of this elementary fact for H -surfaces in 3-manifolds, see [11] for the proof in the minimal case.

Consider a complete H -surface M of locally bounded second fundamental form embedded in a three-manifold N . Consider a limit point p of M , which is a limit of a sequence of divergent points p_n in M . Since M has bounded curvature near p and M is embedded, then for some small $\varepsilon > 0$, a subsequence of the intrinsic disks $B_M(p_n, \varepsilon)$ converges C^2 to an embedded H -disk $D(p, \varepsilon) \subset N$ of intrinsic radius ε , centered at p and of constant mean curvature H . Since M is embedded, any two such limit disks, say $D(p, \varepsilon)$, $D'(p, \varepsilon)$, do not intersect transversally (in fact, two such intersecting disks lie locally on one side of each other). By the maximum principle for H -surfaces, we conclude that if a second disk $D'(p, \varepsilon)$ exists, then $D(p, \varepsilon)$, $D'(p, \varepsilon)$ are the only such limit disks and they are oppositely oriented at p .

Now consider any sequence of embedded disks E_n of the form $B_M(q_n, \frac{\varepsilon}{4})$ or $D(q_n, \frac{\varepsilon}{4})$ such that q_n converges to a point in $D(p, \frac{\varepsilon}{2})$ and such that E_n locally lies on the mean convex side of $D(p, \varepsilon)$. For ε sufficiently small and for n, m large, E_n and E_m must be graphs over domains in $D(p, \varepsilon)$ such that when oriented as graphs, they have the same mean curvature (see also the proof of Lemma 3.1). By the maximum principle, the graphs E_n and E_m are disjoint. It follows that near p and on the mean convex side of $D(p, \varepsilon)$, \overline{M} has the structure of a lamination with leaves with constant mean curvature H . This proves that \overline{M} has the structure of a weak H -lamination.

The next proposition is a simple consequence of the fact that the closure \overline{M} of a complete, H -hypersurface M of locally bounded second fundamental form in a complete n -manifold N has the structure of a weak H -lamination

and the fact that if M is not proper in N , then \overline{M} contains a limit leaf L which is stable by the main theorem in [9].

PROPOSITION 2.4. *Suppose M is a complete connected H -hypersurface with locally bounded second fundamental form in a complete n -manifold N . If the only complete, stable, almost-embedded, H -hypersurfaces in N are compact with finite fundamental group, then M is properly embedded in N .*

Proof. After possibly lifting to a 2-sheeted cover of N , we may assume that M is 2-sided. If M is not proper in N , then $\mathcal{L} = \overline{M}$ has the structure of a weak H -lamination with a limit leaf L . By the Theorem 1 in [9], L is stable and since it is almost-embedded, our hypotheses imply that L is compact with finite fundamental group.

Let $\Pi : \tilde{L} \rightarrow L$ denote the finite sheeted universal cover of L and consider \tilde{L} to be the zero section of its normal bundle $N_b(\tilde{L})$. For some small $\varepsilon > 0$, the ε -neighborhood $\Delta(\tilde{L})$ in $N_b(\tilde{L})$ submerses to its image in N under the exponential map. With respect to the pulled back lamination $\tilde{\mathcal{L}} = (\exp|_{\Delta(\tilde{L})})^{-1}(\mathcal{L})$, we obtain a lamination of $\Delta(\tilde{L})$ with the compact simply-connected hypersurface \tilde{L} as a limit leaf. By a monodromy argument, the leaves in $\tilde{\mathcal{L}}$ close to \tilde{L} are also compact and naturally diffeomorphic to \tilde{L} , and there is a sequence of them which converges to \tilde{L} . Hence, the images or projections to N under the exponential map of these compact leaves correspond to components of M , which contradicts our assumption that M is connected. This contradiction completes the proof of the proposition. \square

The importance of the above proposition is that in certain cases, the hypotheses in the statement of it follow from simple geometric hypotheses. In the next corollary, we list some of these cases. Item 1 in the next corollary was found independently by Harold Rosenberg and this item is essentially the statement of Theorem 1.1 in [18] (also see [10]).

COROLLARY 2.5. *Let N be an n -manifold with absolute sectional curvature less than or equal to 1 and let M be a 2-sided H -hypersurface with locally bounded second fundamental form in N . If any of the following items holds, then M is properly embedded in N .*

1. $n = 3$ and for some $\varepsilon > 0$, $H^2 \geq -S_N + \varepsilon$, where S_N is the average sectional curvature of N (also called the scalar curvature of N). Note that if $S_N \geq \delta > 0$, then this condition holds for all $H \geq 0$.
2. $n = 4$ and $H > \frac{10}{9}$.
3. $n = 5$ and $H > \frac{7}{4}$.
4. $n = 3, 4, 5$, N is flat and $H > 0$.

Proof. Any of the above listed conditions is sufficient to guarantee that any stable H -hypersurface in N is compact with finite fundamental group. The

condition in item 1 appears in [3], [7], [17] and the idea behind it originates from the techniques developed by Fischer-Colbrie and Schoen in [4] (see also [10] for a complete discussion). See [1], [3] for the conditions in items 2 and 3 and see [1], [3], [7], [17] for item 4.

Applying Proposition 2.4 completes the proof of the corollary. \square

3. The existence of a one-sided regular neighborhood

In [12], Meeks and Rosenberg proved that a complete H -surface M of bounded second fundamental form and embedded in \mathbb{R}^3 has a fixed size regular neighborhood on its mean convex side (on both sides when $H = 0$). In other words, for such an M there exists an $\varepsilon > 0$ such that for any $p \in M$, the normal line segment l_p of length ε based at p and contained on the mean convex side of M , forms a collection of pairwise disjoint embedded line segments. An immediate consequence of the existence of such a regular neighborhood is that the surface is properly embedded and for some $c > 0$, the area of the surface is at most cR^3 in any ball of radius $R \geq 1$. The next lemma and theorem generalize this result in the case that the mean curvature is positive.

LEMMA 3.1. *Suppose N is a complete n -manifold with absolute sectional curvature bounded by a constant S_0 and with injectivity radius at least $I_0 > 0$. Suppose $W \subset N$ is a proper smooth mean convex subdomain whose boundary M has constant mean curvature $H_0 > 0$ and $|A_M| \leq A_0$, where $|A_M|$ is the norm of the second fundamental form of M . Then there exists a positive number $\tau \in (0, I_0)$, depending on A_0, H_0, I_0, S_0 , such that M has a regular neighborhood of width τ in W .*

REMARK 3.2. Since a geodesic triangle W in the hyperbolic plane with 3 vertices at infinity does not have a fixed size regular neighborhood in W for its boundary, we see that Lemma 3.1 does not generalize to the case when the mean curvature vanishes.

Proof of Lemma 3.1. The uniform bound $|A_M| \leq A_0$ together with the bounds given by I_0 and S_0 implies that there exists an $\varepsilon > 0$ sufficiently small so that for any $p \in M$, every component of $B_N(p, 2\varepsilon) \cap M$ is a graph over its projection to the disk of radius 3ε in T_pM ; here we are considering the tangent plane to be a plane in normal coordinates and for the orthogonal projection map to be well defined in these coordinates. Moreover, we can choose ε sufficiently small and a smaller positive δ , depending on the bounds A_0, I_0, S_0 , such that every component of $B_N(p, 2\varepsilon) \cap M$ intersecting $B_N(p, \delta)$ contains a graph over $D(p, \varepsilon)$, that is the disk of radius ε in T_pM . Also, we may assume that in our coordinates, these graphs all have gradient of norm less than 1.

The theorem will follow from the observation that two disjoint graphs, u_1 and u_2 over $D(p, \varepsilon)$ with bounded gradients and of constant mean curvature

H_0 which are oppositely oriented and such that u_2 lies on the mean convex side of u_1 , cannot be too close at their centers (this is essentially a consequence of the maximum principle for quasi-linear uniformly elliptic PDEs, see for instance [2, Chapter 7]). Let $p \in M$ and let $u_1 \subset M$ be the graph over $D(p, \varepsilon)$ containing p . Let $\delta_1 \in (0, \delta)$ and suppose that $B_N(p, \delta_1) \cap M$ contains a connected component different from $u_1 \cap B_N(p, \delta_1)$ and which lies on the mean convex side of u_1 . Let $u_2 \subset M$ be the graph over $D(p, \varepsilon)$ which contains such a component and choose it to be the closest one. In other words, the region between the graphs is contained in W and so, since W is mean convex, u_2 lies on the mean convex side of u_1 and they are oppositely oriented. This contradicts our observation, if δ_1 is chosen sufficiently small. Hence, when δ_1 is sufficiently small, $B_N(p, \delta_1)$ does not intersect M on the mean convex side of u_1 . Letting τ be such a small δ_1 proves the lemma. \square

DEFINITION 3.3. We call a proper, almost-embedded H -hypersurface $f : M \rightarrow N$ *strongly Alexandrov embedded* if there exists a proper immersion $F : W \rightarrow N$ of a mean convex 3-manifold W with $\partial W = M$, which extends the map f and which is injective on the interior of W .

REMARK 3.4. Note that if M is a connected, properly embedded and separating H -hypersurface in an n -manifold N , then it is strongly Alexandrov embedded.

THEOREM 3.5 (One-sided regular neighborhood). *Suppose N is a complete n -manifold with absolute sectional curvature bounded by a constant S_0 . Let M be a strongly Alexandrov embedded hypersurface with constant mean curvature $H_0 > 0$ and $|A_M| \leq A_0$. Then the following statements hold.*

1. *There exists a positive number $\tau \in (0, \frac{\pi}{S_0})$, depending on A_0, H_0, S_0 , such that M has a regular neighborhood of width τ on its mean convex side.*
2. *There exists a positive number C , depending on A_0, H_0, S_0 , such that the $(n - 1)$ -dimensional volume of M in balls of radius 1 in N is less than C .*

Proof. We will first prove the existence of a one-sided regular neighborhood of M in N with width a certain $\tau \in (0, \frac{\pi}{S_0})$. We will prove item 1 in the case when M is embedded; with minor modifications, the same arguments demonstrate the case when M is almost-embedded.

Let $\Pi : \widetilde{N} \rightarrow N$ be the universal cover of N . Since M is strongly Alexandrov embedded, there is a proper subdomain $W \subset N$ with $\partial W = M$ that lies on the mean convex side of M . Without loss of generality, we can assume that W is connected. Let $\widetilde{W} \subset \widetilde{N}$ be one of the connected components of $\Pi^{-1}(W)$. Note that $\partial \widetilde{W}$ is mean convex with respect to the pull-back metric. Furthermore, it consists of a collection of surfaces $\{\widetilde{M}_\alpha\}_{\alpha \in I}$ such that for each $\alpha \in I$, the restriction $\Pi : \widetilde{M}_\alpha \rightarrow M$ is a covering space.

Recall that any complete simply-connected manifold with sectional curvature bounded from above by $S_0 \geq 0$ has positive injectivity radius at least $\frac{\pi}{S_0}$. Therefore, by Lemma 3.1, there exists a one-sided regular τ -neighborhood, $N(\partial\widetilde{W}, \tau)$ for $\partial\widetilde{W}$, where $\tau \in (0, \frac{\pi}{S_0})$ depends on A_0, H_0 and S_0 . Note that for any point $p \in N(\partial\widetilde{W}, \tau) - \partial\widetilde{W}$, there exists a unique geodesic in $N(\partial\widetilde{W}, \tau)$ joining p to $\partial\widetilde{W}$ of least length.

For each $x \in \partial\widetilde{W}$, let $\gamma_x \subset \widetilde{W}$ be the unit speed geodesic starting at x , perpendicular to $\partial\widetilde{W}$ and parameterized on $[0, \tau)$.

ASSERTION 3.6. *For each $x, y \in \partial\widetilde{W}$ the following hold.*

1. $\Pi(\gamma_x) \cap M = \Pi(x)$ and $\Pi(\gamma_x(0, \tau)) \cap M = \emptyset$.
2. $\Pi|_{\gamma_x}$ is injective.
3. If $\Pi(x) \neq \Pi(y)$, then $\Pi(\gamma_x) \cap \Pi(\gamma_y) = \emptyset$.

Proof. Item 1 is a consequence of the fact that $\gamma_x(0, \tau) \subset \text{Int}(\widetilde{W})$ and $\Pi(\text{Int}(\widetilde{W})) = \text{Int}(W)$.

In order to prove items 2 and 3, we first note that if $g : \widetilde{N} \rightarrow \widetilde{N}$ is a covering transformation and $g(\widetilde{W}) \cap \widetilde{W} \neq \emptyset$, then $g(\widetilde{W}) = \widetilde{W}$. To see this, suppose that $p \in g(\widetilde{W}) \cap \widetilde{W}$. Since $g : \widetilde{N} \rightarrow \widetilde{N}$ is an isometry, $g : \Pi^{-1}(M) \rightarrow \Pi^{-1}(M)$ and $\partial\widetilde{W}$ is mean convex, we may pick p to be in the interior of $\widetilde{W} \cap g(\widetilde{W})$. We first show that $\widetilde{W} \subset g(\widetilde{W})$. Let $q \in \widetilde{W}$ and choose a path $\gamma \subset \widetilde{W}$ from p to q and such that $\gamma - q \subset \text{Int}(\widetilde{W})$. Note that γ can only intersect $\Pi^{-1}(M)$ at q and only if $q \in \partial\widetilde{W}$; hence $\gamma - q$ is disjoint from $\partial(g(\widetilde{W}))$. Since p is in the interior of $g(\widetilde{W})$ and $\gamma - q$ is disjoint from $\partial(g(\widetilde{W}))$, then $\gamma \subset g(\widetilde{W})$ and thus $q \in g(\widetilde{W})$, which proves that $\widetilde{W} \subset g(\widetilde{W})$. Note that proving $g(\widetilde{W}) \subset \widetilde{W}$ is equivalent to proving that $\widetilde{W} \subset g^{-1}(\widetilde{W})$. Applying g^{-1} to $g(\widetilde{W}) \cap \widetilde{W} \neq \emptyset$ gives $\widetilde{W} \cap g^{-1}(\widetilde{W}) \neq \emptyset$. Arguing as previously, $\widetilde{W} \subset g^{-1}(\widetilde{W})$ and this proves that $\widetilde{W} = g(\widetilde{W})$.

We now prove item 2. Suppose that there exists $x \in \partial\widetilde{W}$ for which $\Pi|_{\gamma_x}$ is not injective. Then there exist points $a, b \in \gamma_x$, $a \neq b$, such that $\Pi(a) = \Pi(b)$. Since \widetilde{N} is the universal covering space of N , there exists a nontrivial covering transformation $g : \widetilde{N} \rightarrow \widetilde{N}$ such that $g(a) = b$. In particular, $g(\widetilde{W}) \cap \widetilde{W} \neq \emptyset$ and by the previous observation, $g(\widetilde{W}) = \widetilde{W}$. In other words, $g|_{\widetilde{W}} : \widetilde{W} \rightarrow \widetilde{W}$ is an isometry and

$$d_{\widetilde{W}}(a, \partial\widetilde{W}) = d_{\widetilde{W}}(g(a), \partial\widetilde{W}) = d_{\widetilde{W}}(b, \partial\widetilde{W}).$$

However, different points in γ_x have different distances to $\partial\widetilde{W}$ and thus

$$d_{\widetilde{W}}(a, \partial\widetilde{W}) \neq d_{\widetilde{W}}(b, \partial\widetilde{W}).$$

This contradiction finishes the proof of item 2.

The proof of item 3 is similar. Let $x, y \in \partial\widetilde{W}$ such that $\Pi(x) \neq \Pi(y)$. If $\Pi(\gamma_x) \cap \Pi(\gamma_y) \neq \emptyset$, then there exist $a \in \gamma_x$ and $b \in \gamma_y$ such that $\Pi(a) = \Pi(b)$. As before, there exists a nontrivial covering transformation $g: \widetilde{N} \rightarrow \widetilde{N}$ inducing an isometry $g|_{\widetilde{W}}: \widetilde{W} \rightarrow \widetilde{W}$ and such that $g(a) = b$. Since $g|_{\widetilde{W}}$ preserves distances to $\partial\widetilde{W}$ and $g(a) = b$, then $g(\gamma_x) = \gamma_y$ and so $g(x) = y$, which implies $\Pi(x) = \Pi(y)$. This contradiction finishes the proof of item 3 and the proof of the assertion. \square

It follows from Assertion 3.6 that as $x \in M$ varies, the geodesic segments $\gamma_x \subset W$ starting at x and perpendicular to M of length τ are embedded and form a pairwise disjoint collection. This proves the first statement of the theorem.

Let $N(M, \tau)$ be the one-sided regular neighborhood of M in N given by item 1. For a domain $E \subset M$, let $N(E, \tau) \subset N(M, \tau)$ be the associated one-sided regular neighborhood in $W \subset N$. The uniform bound on $|A_M|$ and on the absolute sectional curvature of N implies that there exists a constant K , depending also on τ , such that the $(n-1)$ -dimensional volume of any compact domain E on M is less than K times the volume of $N(E, \tau)$. Thus, since the volumes of balls in N of radius 2 are uniformly bounded by a constant depending only on S_0 , the $(n-1)$ -dimensional volume of M in such balls is also uniformly bounded. In other words, for such a τ and for any $p \in M$,

$$\begin{aligned} \text{Volume}_{n-1}[M \cap B_N(p, 1)] &\leq K \cdot \text{Volume}_n[N(M \cap B_N(p, 1), \tau)] \\ &\leq K \cdot \text{Volume}_n[B_N(p, 2)] \leq C, \end{aligned}$$

where C depends solely A_0 , S_0 and H_0 . This proves item 2, and thus completes the proof of the theorem. \square

Our proof of the Dynamics Theorem in [16] uses the following corollary to Theorem 3.5.

COROLLARY 3.7. *Suppose N is a complete, simply-connected n -manifold with absolute sectional curvature bounded by a constant S_0 . Let M be a connected, properly immersed, almost-embedded hypersurface with constant mean curvature $H_0 > 0$ and $|A_M| \leq A_0$. Then the following statements hold.*

1. *There exists a positive number $\tau \in (0, \frac{\pi}{S_0})$, depending on A_0, H_0, S_0 , such that M has a regular neighborhood of width τ on its mean convex side.*
2. *There exists a positive number C , depending on A_0, H_0, S_0 , such that the $(n-1)$ -dimensional volume of M in balls of radius 1 in N is less than C .*

Proof. Since a connected, properly embedded surface in N separates N into two connected components, a surface M satisfying the hypotheses of the corollary is seen to be strongly Alexandrov embedded. Applying Theorem 3.5 finishes the proof. \square

4. The existence of H -surfaces in \mathbb{R}^3 which are not properly embedded

In [5], Grosse-Brauckmann constructed a one-parameter family $M(t)$, $0 < t < \infty$, of properly embedded doubly periodic H -surfaces with the (x_1, x_2) -plane P as a plane of Alexandrov symmetry and varying constant mean curvature $H_{M(t)} > 0$. This family contains the doubly periodic surface $\xi_{2,2}$ of Lawson [6]. The rescaled surfaces $\widehat{M}(t) = H_{M(t)}M(t)$ have constant mean curvature 1 and converge pointwise to P with area multiplicity two as $t \rightarrow 0$. In particular, for $n \in \mathbb{N}$, there exists a properly embedded surface $M_n \subset \mathbb{R}^3$ with constant mean curvature 1 and contained in the slab $\{(\frac{1}{2})^{n+1} < x_3 < (\frac{1}{2})^n\}$. The disconnected surface $M_\infty = \bigcup_{n \in \mathbb{N}} M_n$ is a complete 1-surface properly embedded in the slab $\{0 < x_3 < \frac{1}{2}\}$ which is not properly embedded in \mathbb{R}^3 . A simple modification of this argument proves the following proposition when $H > 0$ and when $H = 0$, the proposition follows trivially by using an appropriate infinite collection of horizontal planes in the slab.

PROPOSITION 4.1. *For any $H \geq 0$, there exists a complete, disconnected H -surface properly embedded in the slab $\{0 < x_3 < \frac{1}{2}\}$, which is not properly embedded in \mathbb{R}^3 .*

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