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THE SPACE OF COMMUTING n-TUPLES IN SU(2)

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ABSTRACT. Let $Y := \mathrm{Hom}(\mathbb{Z}^n, \mathrm{SU}(2))$ denote the space of commuting n-tuples in SU(2). We determine the homotopy type of the suspension ΣY, and compute the integral cohomology groups of Y for all positive integers n.

1. Introduction

It is interesting to study representations of discrete groups into compact Lie groups. For example, if X is a smooth manifold then the space of homomorphisms $\mathrm{Hom}(\pi_1(X), G)$ may be identified with the space of flat G-connections on X modulo based gauge transformations, which has diverse applications in geometry.

An interesting special case is $X = (S^1)^n$, which has fundamental group $\mathbb{Z}^n$. The space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ is identified with

$$Y_G[n] := \{(g_1, \dots, g_n) \in G^n | g_i g_j = g_j g_i \text{ for all } i, j\}.$$

(Usually n and G will be understood and omitted.)

This space was first studied by Adem and Cohen [1] (and has been further investigated with collaborators Torres-Giese [3] and Gomez [4] in connection with canonical filtrations of the classifying space BG). They considered the problem in greater generality and obtained results even in the more complicated case where G is not compact. One of their main results ([1, Theorem 1.6]) is a decomposition formula for the suspension of $Y_G[n]$

$$\Sigma Y_G[n] \simeq \Sigma\left(\bigvee_{k=1}^{n} \binom{n}{k} Y_G[k] \Big/ S_k(G)\right), \tag{1}$$

where $S_k(G) \subset Y_G[k]$ consists of those k-tuples with some entry equal to the identity. Although their papers focused on the many interesting aspects of

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these spaces rather than explicit cohomology computations, they also used their methods to explicitly work out the cohomology groups of $Y_G[n]$ for $G = \mathrm{SU}(2)$ in the cases when $n=2$ and $n=3$.

Later, the first author [5] gave a concise description of the rational cohomology ring as a ring of invariants under the action of the Weyl group, and explicitly worked out Poincaré polynomials for the cases $G = \mathrm{SU}(2)$ and $\mathrm{SU}(3)$. In the case $G = \mathrm{SU}(2)$ and $n = 2$ or 3, the cohomology computed by [1] agrees rationally with that given by the first author. The purpose of this paper is to calculate the stable homotopy type of $Y_{\mathrm{SU}(2)}[n]$ for all n. Our formula is expressed in the form of (1), in terms explicit enough to compute (co)homology groups. Our answer agrees rationally with that of the first author [5]. It also agrees with [1] in the case $n=2$. However, for $n=3$ our results disagree with the published version of [1] although they agree rationally. The authors would like to thank Alejandro Adem and Fred Cohen for the large volume of email discussion during the interval since March 2006 when we first emailed them our results pointing out the conflict with their paper. Now that the issue has been resolved, we are pleased to publish our paper.

Since posting, we have been informed of an erratum [2] to [1], and of a 2008 preprint by M. C. Crabb [7] giving results similar to ours.

We do not obtain the ring structure, although some information concerning the multiplication can be deduced from the rational calculation in [5].

2. Wedge decomposition

Let $G = \mathrm{SU}(2)$ and let G act on itself by conjugation.

Let $T = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) | \lambda \in S^1 \right\} \subset G$ be a maximal torus. The coset space $G/T = \{gT\}$ is homeomorphic to $\mathbb{C}P^1 \cong S^2$. Let $\bar{g}$ denote the coset gT.

Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $W = \{T = \bar{e}, wT = \bar{w}\}$ is the Weyl group of G. The action of W on T is given by $\bar{w} \cdot t := wtw^{-1} = t^{-1}$. There is also an action of W on G/T given by $\bar{w} \cdot \bar{g} := \overline{wgw^{-1}}$, corresponding to the antipodal action of $\mathbb{Z}/2$ on S^2.

We use the subscripts $(\cdot)_r$ and $(\cdot)_s$ for the regular and singular subsets.

Using this convention, we set $T_s := Z(G) = \{e, -e\} \subset T = T^W$ and $T_r := T \setminus T_s$. The action of W on T restricts to a free action on T_r and the trivial action on the fixed point set $T_s = T^W$. Similarly, set

$$Y_s := \left\{ y \in Y | y_j \in Z(G) \ \forall j \right\} = \left(T^n\right)_s$$

and $Y_r := Y \setminus Y_s$. Note that $(T_s)^n = (T^n)_s = Y_s = \{\pm e\}^n$ is a collection of 2^n isolated points in Y.

Any set of commuting elements in G must lie in a common maximal torus [6] and all maximal tori are conjugate. Consequently, the map

$$\phi = \phi[n] : (G/T) \times T^n \to Y[n]$$

satisfying

$$\phi(\bar{g}, t_1, \ldots, t_n) := g \cdot (t_1, \ldots, t_n) = \big(gt_1g^{-1}, \ldots, gt_ng^{-1}\big)$$

is a G-equivariant surjection. The principal orbit type of Y_r is G/T, so it follows that the restriction π of ϕ,

$$\pi : (G/T) \times \big(T^n\big)_r \to Y_r$$

is a covering map. Because conjugacy classes in G intersect T in a W orbit, we deduce that π is a Galois cover, with deck transformation group W acting diagonally on the product $G/T \times (T^n)_r$.

We thus obtain a homeomorphism,

$$Y_r \cong \big((G/T) \times \big(T^n\big)_r\big)/W. \tag{2}$$

Another way to look at this homeomorphism is as follows. As noted above, for any $y \in Y$ there exists $\bar{g} \in G/T$ such that $y \in \phi(\bar{g}, T^n)$. If $y \in Y_r$ then the class of $\bar{g}$ in $(G/T)/W \cong \mathbb{R}P^2$ is uniquely determined by y so there is a well defined map $q : Y_r \to (G/T)/W$. The map q is a fibration with fibre $F = q^{-1}(*) = (T_r)^n$. The Weyl group W acts diagonally on T^n, and the inclusion $F = (T_r)^n \subset T^n$ induces the W-action on F.

Taking the pullback of q with the universal covering projection $\pi : S^2 \to \mathbb{R}P^2$ gives a fibration $\tilde{q} : \tilde{Y}_r \to S^2$.

$$\begin{tikzcd}
 & W \arrow[r, equal] \arrow[d] & W \arrow[d] \\
F \arrow[r] \arrow[d, equal] & \tilde{Y}_r \arrow[r, "\tilde{q}"] \arrow[d, "\pi"'] & S^2 \arrow[d, "\pi"'] \\
F \arrow[r] & Y_r \arrow[r, "q"] & \mathbb{R}P^2
\end{tikzcd}$$

The action of W on $\tilde{Y}_r \subset Y_r \times (G/T)$ is given by $\bar{w} \cdot (y, \bar{g}) = (y, \bar{w} \cdot \bar{g}) = (y, \overline{wgw^{-1}})$. This pullback fibration is trivial with retraction $r : \tilde{Y}_r \to F$ given by $r(y) = (t_1, \ldots, t_n)$ where for all j, $y_j = gt_jg^{-1}$ with $\bar{g} = \tilde{q}(y) \in G/T = S^2$. Thus, $(r, q) : \tilde{Y}_r \to F \times S^2$ is a homeomorphism. If $y_j = gt_jg^{-1}$ then we also have $y_j = gw(w \cdot t_j)(gwg^{-1})$. Hence, if $r(y, g) = (t_1, \ldots, t_r)$, then $r(w \cdot (y, g)) = (w \cdot t_1, \ldots, w \cdot t_r)$. Therefore, $\tilde{Y}_r \cong F \times S^2$ is a W-equivariant homeomorphism, and so we obtain (2).

Recall that the set $Y_s = \{\pm 1\}^n$ is a collection of 2^n isolated points. The following proposition shows that each point has a contractible neighbourhood in Y.

Proposition 2.1. *The inclusion $Y_s \to Y$ is an absolute neighbourhood deformation retract pair.*

Proof. For $t \in T$ and $\varepsilon > 0$, let $B_\varepsilon(e) = \exp(-\varepsilon, \varepsilon) \subset T$ be a small interval about e in T. For arbitrary $t \in T$, set $B_\varepsilon(t) = tB_\varepsilon(e)$. For $y \in Y$, if $(x,t) \in \phi^{-1}(y)$ is any particular preimage of y, the set

$$V_\varepsilon(y) = \bigcup_{g \in G} \big(gB_\varepsilon(t_1)g^{-1}, \dots, gB_\varepsilon(t_n)g^{-1}\big),$$

is independent of the choice of (x,t) and forms an open neighbourhood of y in Y.

$$\bigcap_{\varepsilon > 0} V_\varepsilon(y) = \{y\} \cup \{\bar{w} \cdot y\} = W \cdot y.$$

In particular, if $y \in Y_s$ then

$$\bigcap_{\varepsilon > 0} V_\varepsilon(y) = \{y\}.$$

Set $V_s = \bigcup_{y \in Y_s} Y_y$, an open subset of Y.

For small ε, define $H : B_\varepsilon(e) \times I \to B_\varepsilon(e)$ by $H(\exp(x), s) = \exp(sx)$. For arbitrary $t \in T$, this induces a contraction $H_t : B_\varepsilon(t) \times I \to B_\varepsilon(t)$ given by $H_t(x,s) = tH(t^{-1}x, s)$ of $B_\varepsilon(t)$ to $\{t\}$. Set $H' := H_{-e} : B_\varepsilon(-e) \times I \to B_\varepsilon(-e)$. Notice that H and H' are W-equivariant. Suppose y belongs to Y_s. Then for all j, $y_j = \pm e \in T$. For $v \in V_\varepsilon(y)$, write $v = g \cdot (x_1, \dots, x_n)$, where $x_j \in B_\varepsilon(y_j)$. Define $H_y(v,s) = g \cdot (H_1(x_1,s), \dots, H_n(x_n,s))$ where $H_j = H$ or H' according to whether $y_j = e$ or $-e$. Since H and H' are W-equivariant, the result is independent of the choice of g and $x_1, \dots, x_n$, so produces a well-defined contraction $H_y : V_\varepsilon(y) \times I \to V_\varepsilon(y)$ of $V_\varepsilon(y)$ to $\{y\}$. Therefore, $V_s \simeq Y_s$. Thus we have shown that V_s is a neighbourhood deformation retract of Y_s. $\square$

For a locally compact Hausdorff space X, let X^+ denote its one-point compactification. Notice that Y is the pushout $Y = Y_r \bigcup_{Y_r \cap V_s} V_s$. Therefore,

$$Y/V_s \simeq Y_r/(Y_r \cap V_s) \simeq Y_r^+. \tag{3}$$

From equation (2),

$$Y_r^+ = \big(F \times S^2\big)^+/W = \left(\frac{F^+ \times S^2}{* \times S^2}\right)\Big/W. \tag{4}$$

Since

$$F = (T_r)^n = T^n \smallsetminus Y_s,$$

it follows that

$$F^+ = T^n/Y_s. \tag{5}$$

In general,

$$\frac{B \times Z}{A \times Z} \cong \frac{(B/A) \times Z}{* \times Z} \tag{6}$$

and

$$(A/B)/W = (A/W)/(B/W). \tag{7}$$

Therefore from (4) and (5) using (6) and (7) we have

$$Y/V_s \cong \left(\frac{T^n \times S^2}{Y_s \times S^2}\right)\Big/W = \frac{((T^n \times S^2)/(* \times S^2))/W}{((Y_s \times S^2)/(* \times S^2))/W} \tag{8}$$

and after suspending,

$$\Sigma(Y/V_s) \cong \frac{\Sigma((T^n \times S^2)/(* \times S^2))/W}{\Sigma((Y_s \times S^2)/(* \times S^2))/W}. \tag{9}$$

REMARK 2.2. There is an ambiguity in the notation X/K—this might mean either the quotient by the action of a group K, or the topological quotient where the subspace K is collapsed to a point. Unfortunately both notations are standard. In the above equations, the quotients by W are those of group actions and the others are quotients of spaces.

Since W acts trivially on Y_s,

$$\big((Y_s \times S^2)/(* \times S^2)\big)/W = \big(Y_s \times \mathbb{R}P^2\big)/\big(* \times \mathbb{R}P^2\big) \tag{10}$$

and after suspending we get

$$\begin{aligned}\Sigma\big(\big(Y_s \times \mathbb{R}P^2\big)/\big(* \times \mathbb{R}P^2\big)\big) &\simeq \Sigma\big(\big(Y_s \wedge \mathbb{R}P^2\big) \vee Y_s\big) \\ &= \Sigma\bigg(\bigvee_{2^n-1} \mathbb{R}P^2 \vee Y_s\bigg).\end{aligned} \tag{11}$$

This is the denominator of (9).

We also have

$$\begin{aligned}&\Sigma\big(\big(\big(T^n \times S^2\big)/\big(* \times S^2\big)\big)/W\big) \\ &\quad\simeq \Sigma\left(\bigvee_{k=1}^{n} \binom{n}{k} \big(\big(S^k \times S^2\big)/\big(* \times S^2\big)\big)\Big/W\right),\end{aligned} \tag{12}$$

where w acts by the antipodal map on S^2 and by the product of n reflections on $T^n = (S^1)^n$. The left hand side of (12) is the numerator of (9). Since the quotient map $(S^1)^k \to S^k$ is compatible with the W-action, the action of w on S^k has degree $(-1)^k$ so is homotopic to the negative of the antipodal map.

Next, we need to identify the right hand side of (12). Given a vector bundle ξ, let $D(\xi)$ and $S(\xi)$ be its disk and sphere bundles, and let $T(\xi) := D(\xi)/S(\xi)$ denote its Thom space.

LEMMA 2.3. *Let W act on S^k and S^2 as above. Then*

$$\big(\big(S^k \times S^2\big)/\big(* \times S^2\big)\big)/W \simeq T(kL),$$

where $T(\xi)$ denotes the Thom space of the bundle ξ, and L is the canonical line bundle over $\mathbb{R}P^2$.

Proof. As W spaces, $\frac{S^k \times S^2}{* \times S^2} = \frac{D^k \times S^2}{\partial D^k \times S^2} = T(k\tilde{L})$, where $\tilde{L}$, a trivial line bundle, is the pullback of L to S^2 with w acting by reflection. Therefore $(\frac{S^k \times S^2}{* \times S^2})/W = T(k\tilde{L})/W = T(kL)$. □

Putting this all together gives

$$\Sigma(Y/V_s) \simeq \Sigma\left(\frac{\bigvee_{k=1}^{n} \binom{n}{k} T(kL)}{\bigvee_{2^n-1} \mathbb{R}P^2 \vee Y_s}\right). \tag{13}$$

Equation (13) is produced from (9) with replacements from (10), (11), (12) and Lemma 2.3.

However

$$\begin{aligned}\frac{\bigvee_{k=1}^{n} \binom{n}{k} T(kL)}{\bigvee_{2^n-1} \mathbb{R}P^2 \vee Y_s} &= \bigvee_{k=1}^{n} \binom{n}{k} \frac{T(kL)}{\mathbb{R}P^2} \Big/ Y_s \\ &\simeq \bigvee_{k=1}^{n} \binom{n}{k} \frac{T(kL)}{D(kL)} \Big/ Y_s \\ &\simeq \bigvee_{k=1}^{n} \binom{n}{k} \Sigma S(kL) \Big/ Y_s.\end{aligned} \tag{14}$$

Therefore,

$$\Sigma(Y/V_s) \simeq \Sigma\left(\bigvee_{k=1}^{n} \binom{n}{k} \Sigma S(kL) \Big/ Y_s\right) \tag{15}$$

and since $V_s \simeq Y_s$ we get

$$\Sigma Y \simeq \Sigma\left(\bigvee_{k=1}^{n} \binom{n}{k} \Sigma S(kL)\right). \tag{16}$$

3. Description of $\Sigma S(kL)$

The goal of this section is to prove the following

PROPOSITION 3.1. *Decompose $\mathbb{R}^{k+3} \cong \mathbb{R}^3 \oplus \mathbb{R}^k$, inducing disjoint embeddings of $\mathbb{R}P^2$ and $\mathbb{R}P^{k-1}$ into $\mathbb{R}P^{k+2}$. Then we have a homeomorphism*

$$\Sigma S(kL) \cong \mathbb{R}P^2 \setminus \mathbb{R}P^{k+2}/\mathbb{R}P^{k-1},$$

where the notation means that we contract $\mathbb{R}P^2$ and $\mathbb{R}P^{k-1}$ to distinct points.

We begin with a lemma.

LEMMA 3.2. *If L is the canonical line bundle over $\mathbb{R}P^m$ then $T(kL) \cong \mathbb{R}P^{k+m}/\mathbb{R}P^{k-1}$.*

Proof. Let $W = \mathbb{Z}/2$ acting via the antipodal action on S^m. Let $\pi : S^m \to \mathbb{R}P^m$ be the quotient map, and let $\tilde{L} = \pi^! L$ be the pullback of L to S^m.

$$\begin{tikzcd}
W \arrow[r, equal] \arrow[d] & W \arrow[d] \\
S(k\tilde{L}) \arrow[r] \arrow[d, "\pi"'] & D(k\tilde{L}) \arrow[d, "\pi"'] \\
S(kL) \arrow[r] & D(k\tilde{L})
\end{tikzcd}$$

$\tilde{L}$ is a trivial bundle, so $D(k\tilde{L}) \simeq I^k \times S^m$ and $S(k\tilde{L}) \simeq \partial(I^k) \times S^m$. Therefore

$$T(kL) = \frac{D(kL)}{S(kL)} = \frac{D(k\tilde{L})}{S(k\tilde{L})}\Big/W = \frac{I^k \times S^m}{\partial(I^k) \times S^m}\Big/W.$$

Claim 3.3.

$$S^{m+k} \smallsetminus S^{k-1} \cong \operatorname{Int}\bigl(D^k\bigr) \times S^m. \tag{17}$$

Proof.

$$\begin{aligned} S^{m+k} \smallsetminus S^{k-1} = \bigl\{&\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k, \\ &\mathbf{y} = (y_1, \dots, y_{m+1}) \in \mathbb{R}^{m+1} \,\big|\, |\mathbf{x}|^2 + |\mathbf{y}|^2 = 1, |\mathbf{x}| < 1\bigr\}. \end{aligned}$$

Define a homeomorphism

$$f : S^{m+k} \smallsetminus S^{k-1} \to \operatorname{Int}\bigl(D^k\bigr) \times S^m$$

by

$$f(\mathbf{x}, \mathbf{y}) = \bigl(\mathbf{x}, \mathbf{y}/|\mathbf{y}|\bigr).$$

The inverse of this homeomorphism is

$$g :\ \operatorname{Int}\bigl(D^k\bigr) \times S^m \to S^{m+k} \smallsetminus S^{k-1}$$

defined by

$$g(\mathbf{x}, \mathbf{w}) = \bigl(\mathbf{x}, \mathbf{w}\sqrt{1 - |\mathbf{x}|^2}\bigr). \qquad \square$$

Taking one point compactifications of (17) followed by W orbits gives

$$\mathbb{R}P^{m+k}/\mathbb{R}P^{k-1} \cong \frac{S^{m+k}}{S^{k-1}}\Big/W \cong \frac{I^k \times S^m}{\partial I^k \times S^m}\Big/W. \qquad \square$$

Proof of Proposition 3.1. We have $\Sigma S(kL) \cong T(kL)/\mathbb{R}P^2$ where $\mathbb{R}P^2$ embeds into $T(kL)$ as the zero section. By Lemma 3.2, we know $T(kL) \cong \mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}$. Under the homeomorphism (17), the embedding of $\mathbb{R}P^{k-1}$ is by the y-coordinates and the embedding of $\mathbb{R}P^2$ is by the x-coordinates, so $\Sigma S(kL) \cong \mathbb{R}P^2 \setminus \mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}$. $\square$

For cohomology calculations, the following corollary is convenient.

Corollary 3.4.

$$\Sigma S(kL) \simeq \begin{cases} S^3 & \textit{if } k=1, \\ S^2 \vee \left(\mathbb{R}P^4/\mathbb{R}P^2\right) & \textit{if } k=2, \\ \Sigma \mathbb{R}P^2 \vee \left(\mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}\right) & \textit{if } k>2. \end{cases}$$

This combined with

$$\Sigma Y[n] \simeq \Sigma \left(\bigvee_{k=1}^{n} \binom{n}{k} \Sigma S(kL) \right) \tag{18}$$

completely describes $\Sigma Y[n]$ up to homotopy equivalence.

We record for convenience that

$$\tilde{H}^*\left(\mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}\right) = \begin{cases} \mathbb{Z} & \text{if } * = k+1-\left(-1\right)^k, \\ \mathbb{Z}/2 & \text{if } * = k + \frac{3+(-1)^k}{2}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{H}^*\left(\mathbb{R}P^4/\mathbb{R}P^2\right) = \begin{cases} \mathbb{Z}/2 & \text{if } * = 4, \\ 0 & \text{otherwise.} \end{cases}$$

As noted earlier, although our results agree with the published version of [1] when $n=2$, they disagree when $n=3$. For reference, the following is an explicit listing of the groups in this case.

$$H^j\left(Y[3]\right) = \begin{cases} \mathbb{Z} & \text{if } j=0, \\ 0 & \text{if } j=1, \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } j=2, \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } j=3, \\ \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 & \text{if } j=4, \\ \mathbb{Z} & \text{if } j=5, \\ 0 & \text{if } j>5. \end{cases}$$

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