Illinois Journal of Mathematics
Volume 55, Number 2, Summer 2011, Pages 495–508
S 0019-2082

INTEGRATION OF VECTOR-VALUED FUNCTIONS WITH RESPECT TO VECTOR MEASURES DEFINED ON δ-RINGS

N. D. CHAKRABORTY AND SANTWANA BASU

ABSTRACT. This paper extends the theory of scalar-valued integrable functions with respect to vector measures defined on δ-rings to the case of vector-valued tensor integrable functions with respect to vector measures defined on δ-rings. This paper also generalizes some results of G. F. Stefánsson for tensor integration theory of vector-valued functions with respect to vector measures defined on σ-algebras.

1. Introduction

The main purpose of our paper is to develop an integration theory of vector valued functions with respect to vector measures defined on δ-rings. The theory of integration of scalar valued functions with respect to vector measures defined on δ-rings was introduced in 1972 by D. R. Lewis in [13]. In 1989, Masani and Niemi [16], [17] continued the study of integration theory developed by Lewis [13]. In [5], O. Delgado further developed this theory and analysed the subtle differences between the L_1-spaces of vector measures defined on δ-rings and defined on σ-algebras. In fact, she showed that the space $L_1(\nu)$ of a vector measure ν defined on a δ-ring is an order continuous Banach lattice which may not have a weak order unit. Since a countably additive vector measure defined on a δ-ring may not be strongly additive, she studied the effect of strong additivity of ν on $L_1(\nu)$ and connected the analytic properties of ν with the lattice properties of $L_1(\nu)$.

Vector measures defined on σ-algebras have become a very important tool for the study of operators $T: Z \to Y$ between Banach function spaces. In fact, the optimal domain of T can be described as the space $L_1(\nu)$ of integrable functions with respect to the vector measure ν canonically associated to T by $\nu(A) = T(\chi_A)$ (see [4, p. 133] and [18, Chapters 3 and 4]).

Received October 6, 2009; received in final form March 15, 2010.
2010 *Mathematics Subject Classification.* Primary 46G10, 28B05. Secondary 46B99.

The theory of tensor integration of vector-valued functions $f : \Omega \to X$ with respect to a countably additive vector measure $\nu : \Sigma \to Y$ defined on a σ-algebra Σ, where (Ω, Σ) is a measurable space and X and Y are real Banach spaces, was systematically studied by G. F. Stefánsson in [22]. It has been shown in [22, Theorem 4, p. 932] that the space $L_1(\nu, X, Y)$ of all tensor integrable functions is a Banach space with respect to the norm

$$N(f) = \sup\left\{ \int_\Omega \|f\| \, d\left|y^*\nu\right| : y^* \in B_{Y^*} \right\}.$$

In [2], we studied some general properties of the Banach space $L_1(\nu, X, Y)$, such as order continuity, separability, weak sequential compactness and weak compactly generated property.

In [3], we also studied the space of p-tensor integrable functions and related Banach space properties, which extends the theory of integration developed in [11], [21].

In this connection, we would like to mention that I. Dobrakov developed a theory of integration for vector-valued functions with respect to operator-valued measures defined on δ-rings in a series of papers initiated by his fundamental papers in 1970 [7], [8].

In 2004, Dobrakov and Panchapagesan [10] provided detailed proofs of many results of [7] and [8] and discussed some of the distinguishing features of this theory including the stronger version of the Pettis measurability criteria.

On the other hand, B. Jefferies and S. Okada studied the theory of tensor integration of vector-valued functions with respect to vector-valued measures defined on a σ-algebra [12, Definition 1.5, p. 521]. Their definition of tensor integration is weaker than the definition as given by Stefansson in [22, Definition 1, p. 927]. However, they succeeded in developing a relationship between the tensor integrable functions and Dobrakov integrable functions [12, Theorem 3.5, and Corollary 3.6].

In [19], R. Pallu de la Barriére also studied a theory of integration based on the notion of semivariation and developed the theories of bilinear integration and tensor integration and proved Dominated Convergence Theorem and Convergence Theorem of Vitali type from this notion.

In [20], J. Rodríguez studied the integration theory of vector-valued functions with respect to operator-valued measures and extended the theories of Birkhoff and Mcshane integrals and connected these integrals with the S^*-integrals as developed by Dobrakov in [9] (see [20, Theorem 3.7, p. 817 and Theorem 3.8, p. 823]).

In this paper, we construct a theory of tensor integration of vector-valued functions with respect to vector measures defined on δ-rings keeping in mind the papers of Stefánsson [22] and Delgado [5]. In this setting, we prove the Dominated Convergence Theorem for tensor integrable functions and also give an alternative proof of the completeness of $L_1(\nu, X, Y)$ when ν is defined on a δ-ring (see [22, Theorem 3, p. 931 and Theorem 4, p. 932]).

2. Notations, definitions and preliminaries

Throughout this paper, X and Y are two real Banach spaces with topological duals X^* and Y^*, respectively. B_X (respectively B_{X^*}) denotes the closed unit ball of X (respectively, X^*). $X \check{\otimes} Y$ is the injective tensor product of X and Y (see [6, Chapter VIII]) and $L(X,Y)$ denotes the set of all bounded linear transformations from X to Y.

Unless otherwise stated, we always assume that τ is a δ-ring of subsets of a non-empty set Ω and $C(\tau)$ is the σ-algebra of sets locally in τ, that is, $A \in C(\tau)$ if and only if $A \cap B \in \tau$ for all $B \in \tau$ and $\nu : \tau \to Y$ a countably additive vector measure (or simply, a vector measure). Let $\mathcal{M}(X)$ denote the space of all X-valued measurable functions on $(\Omega, C(\tau))$. If $X = \mathbb{R}$, we simply denote $\mathcal{M}(\mathbb{R})$ by $\mathcal{M}$. $\mathcal{M}^*$ denotes the set of all extended real-valued measurable functions on $C(\tau)$. The space of all X-valued τ-simple functions is denoted by $S(\tau, X)$. If $X = \mathbb{R}$, $S(\tau, \mathbb{R})$ is simply denoted by $S(\tau)$.

The semivariation of ν is the set function defined on $C(\tau)$ by

$$\|\nu\|(A) = \sup\{ |y^*\nu|(A) : y^* \in B_{Y^*} \},$$

where $|y^*\nu|$ is the variation of the scalar measure $y^*\nu$ defined on $C(\tau)$.

$\|\nu\|$ is finite on τ and a set $B \in C(\tau)$ is called ν-null if $\|\nu\|(B) = 0$.

A property holds ν-almost everywhere (ν-a.e.) if it holds except on a ν-null set [5, p. 433].

We denote by $w - L_1(\nu)$ the space of all equivalence classes of functions in $\mathcal{M}$ which are integrable with respect to $y^*\nu$ for all $y^* \in Y^*$. The space $w - L_1(\nu)$ is a Banach lattice with respect to the norm

$$\|f\|_\nu = \sup\left\{ \int_\Omega |f|\, d|y^*\nu| : y^* \in B_{Y^*} \right\}, \quad f \in w - L_1(\nu)$$

and the usual order structure.

A function f in $\mathcal{M}$ is said to be ν-integrable if

(1) f is $y^*\nu$ integrable for each $y^* \in Y^*$ and
(2) for each $A \in C(\tau)$, there is a vector in Y denoted by $\int_A f d\nu$ such that

$$y^*\left(\int_A f\, d\nu \right) = \int_A f\, dy^*\nu \quad \text{for all } y^* \in Y^*.$$

Let $L_1(\nu)$ denote the space of all equivalence classes of $\mathcal{M}$-measurable ν-integrable functions, equipped with the norm

$$\|f\|_\nu = \sup\left\{ \int_\Omega |f|\, d|y^*\nu| : y^* \in B_{Y^*} \right\}, \quad f \in L_1(\nu).$$

It is well known that $L_1(\nu)$ is an order continuous Banach lattice which, in general, may not have a weak order unit and so it may not be a Banach function space, in sharp contrast with the case when ν is defined on a σ-algebra (see [17, Theorem 4.7(c), p. 141] and [5, p. 435 and p. 438]).

If $\phi = \sum_{i=1}^{n} a_i \chi_{A_i} \in S(\tau)$, then $\phi \in L_1(\nu)$ with

$$\int_A \phi\, d\nu = \sum_{i=1}^{n} a_i \nu(A_i \cap A) \quad \text{for } A \in C(\tau).$$

It is also well known that $S(\tau)$ is dense in $L_1(\nu)$ (see [13, Theorem 3.5, p. 297] and [17, Theorem 4.7, p. 141]).

The vector measure $\nu : \tau \to Y$ is said to be strongly additive if $\nu(A_n) \to 0$ for every pair wise disjoint sequence $\{A_n\}$ in $\tau.\nu$ is called σ-finite if there exists a sequence $\{A_n\} \subset \tau$ and a ν-null set $N \in C(\tau)$ such that $\Omega = (\bigcup A_n) \cup N$. It is obvious that any vector measure defined on a σ-algebra is strongly additive and σ-finite.

Strongly additive vector measures are σ-finite [1, Lemma 1.1, p. 158]. The converse does not hold, in general (see [5, Example 2.1, p. 435, p. 437]).

3. Definition of the integral and main properties

Let $\phi = \sum_{i=1}^{n} x_i \chi_{A_i}$, be an X-valued τ-simple function and let $A \in C(\tau)$. We define $\int_A \phi\, d\nu$ by the equation

$$\int_A \phi\, d\nu = \sum_{i=1}^{n} x_i \otimes \nu(A_i \cap A).$$

Then it follows by [22, p. 927] that $\int_A \phi\, d\nu$ can be viewed as an element of $X \check{\otimes} Y$ and

$$\left\| \int_A \phi\, d\nu \right\| \leq \sup \left\{ \int_A \|\phi\|\, d|y^*\nu| : y^* \in B_{Y^*} \right\}.$$

Definition 3.1. Let $f \in \mathcal{M}(X)$ and $\nu : \tau \to Y$ be a vector measure. The function f is said to be weakly Bochner integrable with respect to ν if f is Bochner integrable with respect to $y^*\nu$ for each $y^* \in Y^*$, that is, if for each $y^* \in Y^*$, there exists a sequence of X-valued τ-simple functions $\{\psi_n\}$ such that $\lim_n \int_\Omega \|f - \psi_n\|\, d|y^*\nu| = 0$.

If f is weakly Bochner integrable, then it follows that $\int_\Omega \|f\|\, d|y^*\nu| < \infty$, for each $y^* \in Y^*$, that is, $\|f\| \in w - L_1(\nu)$.

The space of all weakly Bochner integrable functions is denoted by $w - L_1(\nu, X)$.

Definition 3.2. $f \in \mathcal{M}(X)$ is said to be $\check{\otimes}$-integrable with respect to $\nu : \tau \to Y$ if there exists a sequence of X-valued τ-simple functions $\{\phi_n\}$ such that $\lim_n \sup_{\|y^*\| \leq 1} \int_\Omega \|f - \phi_n\|\, d|y^*\nu| = 0$.

In this case, the sequence $\{\int_A \phi_n \, d\nu\}$ is a Cauchy sequence in $X \check{\otimes} Y$ for each $A \in C(\tau)$. The limit $\int_A f \, d\nu = \lim_n \int_A \phi_n \, d\nu$, is called the $\check{\otimes}$-integral of f over A with respect to ν (see [22, Definition 1, p. 927]).

The space of all $\check{\otimes}$-integrable functions with respect to ν is denoted by $L_1(\nu, X, Y)$.

It follows from Definitions 3.1 and 3.2 that $L_1(\nu, X, Y) \subset w - L_1(\nu, X)$.

Define $N(f) = \sup\{\int_\Omega \|f\| \, d|y^*\nu| : y^* \in B_{Y^*}\}, f \in w - L_1(\nu, X)$.

Then it is easy to see that both $L_1(\nu, X, Y)$ and $w - L_1(\nu, X)$ are normed linear spaces with respect to $N(\cdot)$.

If $f \in L_1(\nu, X, Y)$, we write $\mu_f(A) = \int_A f \, d\nu \in X \check{\otimes} Y$ for each $A \in C(\tau)$. Then we can view $\mu_f(A)$ as an element of $L(Y^*, X)$ and $\mu_f(A)(y^*) = \int_A f \, dy^*\nu$ is an X-valued countably additive measure for each $y^* \in Y^*$ and so $x^*(\int_A f \, dy^*\nu) = \int_A x^* f \, dy^*\nu$ is a countably additive scalar-valued measure for each $x^* \in X^*$ and $y^* \in Y^*$.

Thus, we have $(x^* \otimes y^*)(\int_A f \, d\nu) = \int_A x^* f \, dy^*\nu$.

Consequently, following the arguments as in [22, Theorem 2, p. 929] we have the following theorem.

THEOREM 3.3. *Let $f \in \mathcal{M}(X)$ and let $\nu : \tau \to Y$ be a vector measure. If f is $\check{\otimes}$-integrable, then we have*

(a) $\mu_f(\cdot)$ *is a countably additive vector measure on* $C(\tau)$.
(b) $\|\mu_f\|(A) = \sup\{\int_A |x^* f| \, d|y^*\nu| : x^* \in B_{X^*}, y^* \in B_{Y^*}\}$.
(c) $\lim_{\|\nu(A)\| \to 0} \|\mu_f\|(A) = 0$, *if ν is strongly additive.*

THEOREM 3.4. *Let $f \in \mathcal{M}(X)$. Then f is $\check{\otimes}$-integrable with respect to the vector measure $\nu : \tau \to Y$ if and only if $\|f\|$ is ν-integrable.*

Proof. We first note that $L_1(\nu, X, Y) \subset w - L_1(\nu, X)$. So if f is $\check{\otimes}$-integrable, then $\|f\| \in w - L_1(\nu)$. Let $\{\phi_n\}$ be a sequence of X-valued τ-simple functions such that $\lim_n N(f - \phi_n) = 0$.

Then $|\|f(\omega)\| - \|\phi_n(\omega)\|| \le \|f(\omega) - \phi_n(\omega)\|$ and therefore $\|\|f\| - \|\phi_n\|\|_\nu \le N(f - \phi_n)$.

This implies that $\{\|\phi_n\|\}$ converges to $\|f\|$ in $w - L_1(\nu)$. Since each $\|\phi_n\| \in L_1(\nu)$ and $L_1(\nu)$ is a closed subspace of $w - L_1(\nu)$, it follows that $\|f\| \in L_1(\nu)$ and this means that $\|f\|$ is ν-integrable.

Conversely, let $\|f\|$ be ν-integrable. Let $\varepsilon > 0$. So an application of [13, Lemma 3.4, p. 297] allows us to choose an $A \in \tau$ such that

$$\sup_{\|y^*\| \le 1} \int_{\Omega \setminus A} \|f\| \, d|y^*\nu| < \varepsilon/2. \tag{1}$$

Define a countably additive Y-valued measure $\nu_A : C(\tau) \to Y$ by

$$\nu_A(B) = \nu(A \cap B) \quad \text{for all } B \in C(\tau).$$

Since $f \in \mathcal{M}(X)$, by [22, Corollary B, p. 927], there exists a sequence $\{f_n\}$ of countably valued functions in $\mathcal{M}(X)$ such that $\|f - f_n\| < \frac{1}{n}$ $\|\nu_A\|$-a.e.

Then $\|f\chi_A - f_n\chi_A\| \le \frac{1}{n}\ \|\nu_A\|$-a.e. and so, $\|f_n\chi_A\| \le \|f\chi_A\| + \frac{1}{n}\ \|\nu_A\|$-a.e. From this, it follows that $\|f_n\chi_A\|$ is ν_A-integrable for each n and consequently

$$\lim_{\|\nu_A\|(E)\to 0} N(f_n\chi_A\chi_E) = 0 \quad \text{for } E \in C(\tau). \tag{2}$$

Let $f_n = \sum_{k=1}^{\infty} x_{n,k}\chi_{A_{n,k}}$, where $A_{n,i}\cap A_{n,j} = \phi$ if $i \ne j$ and $A_{n,k} \in C(\tau)$ and $x_{n,k} \in X$. So $f_n\chi_A = \sum_{k=1}^{\infty} x_{n,k}\chi_{A_{n,k}\cap A}$. Since $A_{n,k} \in C(\tau)$ and $A \in \tau$ we have $A_{n,k}\cap A \in \tau$.

Let $f_n\chi_A = h_n$. Then h_n is a countably valued function based on τ and we have

$$\lim_{\|\nu_A\|(E)\to 0} N(h_n\chi_E) = 0, \quad \text{by (2)}. \tag{3}$$

For each n, it follows from equation (3) that we can choose p_n large enough so that

$$\sup_{\|y^*\|\le 1} \int_{\bigcup_{k>p_n} A_{n,k}\cap A} \|h_n\|\, d\big|y^*\nu_A\big| < \frac{\|\nu_A\|(\Omega)}{n}. \tag{4}$$

If we let $\phi_n = \sum_{k\le p_n} x_{n,k}\chi_{A_{n,k}\cap A}$, then ϕ_n is an X-valued τ-simple function and since $f_n\chi_A$ vanishes off A, ϕ_n also vanishes off A and

$$N(f\chi_A - \phi_n) \le N(f\chi_A - h_n) + N(h_n - \phi_n) < \frac{2\|\nu_A\|(\Omega)}{n},$$

by (3) and (4).

This implies that $N(f\chi_A - \phi_n) \to 0$, as $n \to \infty$ and so we have

$$N(f\chi_A - \phi_{n_0}) < \varepsilon/2 \quad \text{for some positive integer } n_0. \tag{5}$$

Now $\phi_{n_0}(\omega) = 0$ for $\omega \in \Omega\setminus A$ and so $\|f(\omega) - \phi_{n_0}(\omega)\| = \|f(\omega)\|$ on $\Omega\setminus A$ and $\nu = \nu_A$ on A. Therefore,

$$\begin{aligned}
&\sup_{\|y^*\|\le 1} \int_{\Omega} \big\|f(\omega) - \phi_{n_0}(\omega)\big\|\, d\big|y^*\nu\big| \\
&\quad = \sup_{\|y^*\|\le 1} \int_{\Omega\setminus A} \big\|f(\omega)\big\|\, d\big|y^*\nu\big| + \sup_{\|y^*\|\le 1} \int_{A} \big\|f(\omega) - \phi_{n_0}(\omega)\big\|\, d\big|y^*\nu_A\big| \\
&\quad = \sup_{\|y^*\|\le 1} \int_{\Omega\setminus A} \big\|f(\omega)\big\|\, d\big|y^*\nu\big| + N(f\chi_A - \phi_{n_0}) \\
&\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \text{by (1) and (5)}.
\end{aligned}$$

Thus, $\sup_{\|y^*\|\le 1}\int_{\Omega} \|f - \phi_{n_0}\|\, d|y^*\nu| < \varepsilon$, where ϕ_{n_0} is an X-valued τ-simple function and this implies that f is $\check{\otimes}$-integrable with respect to ν. □

Corollary 1. *If $f \in \mathcal{M}(X)$ is bounded, then f is $\check{\otimes}$-integrable with respect to the vector measure $\nu : \tau \to Y$, if ν is strongly additive.*

The proof follows from Corollary 3.2(b) of [5, p. 438].

COROLLARY 2. *Let both f and $g \in \mathcal{M}(X)$. If g is $\check{\otimes}$-integrable with respect to the vector measure ν and $\|f\| \le \|g\|$ $\|\nu\|$-a.e., then f is $\check{\otimes}$-integrable with respect to the vector measure ν.*

The proof follows from Theorem 3.4 and Theorem 4.10 of [17].

THEOREM 3.5 (Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of $\check{\otimes}$-integrable functions which converges $\|\nu\|$-a.e. to a function f. If there exists a $\check{\otimes}$-integrable function g such that $\|f_n\| \le \|g\| \|\nu\|$-a.e. then f is $\check{\otimes}$-integrable and*

$$\lim_n \int_B f_n \, d\nu = \int_B f \, d\nu, \quad \textit{uniformly with respect to } B \in C(\tau).$$

Proof. Since $\|f_n\| \le \|g\| \|\nu\|$-a.e., $\|f\| \le \|g\| \|\nu\|$-a.e. Since g is $\check{\otimes}$-integrable, it follows by the above theorem that $\|g\|$ is ν-integrable and so by [17, Theorem 4.10, p. 145] that $\|f\|$ is ν-integrable and hence, by the above theorem, f is $\check{\otimes}$-integrable.

Again, since each f_n is $\check{\otimes}$-integrable with respect to ν, it follows that each $f_n \in L_1(|y^*\nu|, X)$ for $y^* \in Y^*$ and since both $\|f_n\|$ and $\|g\|$ are $y^*\nu$-integrable, an application of dominated convergence theorem in $L_1(|y^*\nu|, X)$ shows that $f \in L_1(|y^*\nu|, X)$ and this implies that f is $y^*\nu$-integrable and moreover

$$\int_E f \, dy^*\nu = \lim_n \int_E f_n dy^*\nu \quad \text{for all } E \in C(\tau).$$

This implies that

$$\int_E x^* f \, dy^*\nu = \lim_n \int_E x^* f_n \, dy^*\nu.$$

Following the arguments as in the proof of [22, Theorem 3, p. 931], it suffices to show that the sequence $\{\int_B f_n \, d\nu\}$ is Cauchy uniformly with respect to $B \in C(\tau)$.

Let $\varepsilon > 0$. Since each f_n and g are $\check{\otimes}$-integrable and $\|f_n\| \le \|g\|$ $\|\nu\|$-a.e., it follows by Lemma 3.4 of [13] that there is an $A \in \tau$ such that

$$\sup_{\|y^*\| \le 1} \int_{\Omega \setminus A} \|g\| \, d\big|y^*\nu\big| < \varepsilon/3 \quad \text{and so}$$

$$\sup_{\|y^*\| \le 1} \int_{\Omega \setminus A} \|f_n\| \, d\big|y^*\nu\big| < \varepsilon/3 \quad \text{for all } n.$$

Therefore,

$$\sup_{\|y^*\| \le 1} \int_{B \setminus A} \|f_n\| \, d\big|y^*\nu\big| < \varepsilon/3 \quad \text{for } B \in C(\tau) \text{ and for all } n.$$

Now by an easy calculation, we have

$$\left\|\int_B f_n\,d\nu - \int_{B\cap A} f_n\,d\nu\right\| = \left\|\int_{B\setminus A} f_n\,d\nu\right\| \le \sup_{\|y^*\|\le 1}\int_{B\setminus A}\|f_n\|\,d|y^*\nu| \le \sup_{\|y^*\|\le 1}\int_{\Omega\setminus A}\|f_n\|\,d|y^*\nu| < \varepsilon/3 \quad \text{for all } n \text{ and } B\in C(\tau). \tag{6}$$

Define $F: C(\tau)\to Y$ by

$$F(B)=\nu_A(B)=\nu(A\cap B)\quad \text{for all } B\in C(\tau).$$

Then F is a countably additive Y-valued vector measure on $C(\tau)$ and since f and each f_n are $\check{\otimes}$-integrable with respect to ν, they are also $\check{\otimes}$-integrable with respect to F and consequently by Theorem 3 of [22] we have,

$$\left\|\int_B f_n\,dF - \int_B f_m\,dF\right\| < \varepsilon/3 \quad \text{for } m,n>n_0, \text{ for some positive integer } n_0 \text{ and } B\in C(\tau). \tag{7}$$

Now

$$\begin{aligned}
&\left\|\int_B f_n\,d\nu - \int_B f_m\,d\nu\right\| \\
&\quad\le \left\|\int_B f_n\,d\nu - \int_{B\cap A} f_n\,d\nu\right\| + \left\|\int_{B\cap A} f_n\,d\nu - \int_{B\cap A} f_m\,d\nu\right\| \\
&\qquad + \left\|\int_{B\cap A} f_m\,d\nu - \int_B f_m\,d\nu\right\| \\
&\quad= \left\|\int_B f_n\,d\nu - \int_{B\cap A} f_n\,d\nu\right\| + \left\|\int_B f_n\,dF - \int_B f_m\,dF\right\| \\
&\qquad + \left\|\int_{B\cap A} f_m\,d\nu - \int_B f_m\,d\nu\right\| < \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon,
\end{aligned}$$

by (6) and (7), for $B\in C(\tau)$ and for all $m,n>n_0$.

This implies that $\{\int_B f_n\,d\nu\}$ is a Cauchy sequence in $X\check{\otimes}Y$ and since $X\check{\otimes}Y$ is complete, there exists an element $\mathcal{U}_B\in X\check{\otimes}Y$ such that $\int_B f_n\,d\nu\to\mathcal{U}_B$ in $X\check{\otimes}Y$ and so $\int_B f_n\,d\nu\to\mathcal{U}_B$ weakly in $X\check{\otimes}Y$.

Also, since f is $\check{\otimes}$-integrable we have that

$$\begin{aligned}
(x^*\otimes y^*)\left(\int_B f_n\,d\nu\right) &= \int_B x^* f_n\,dy^*\nu \to \int_B x^* f\,dy^*\nu \\
&= (x^*\otimes y^*)\left(\int_B f\,d\nu\right)
\end{aligned}$$

for all $x^* \in X^*$ and $y^* \in Y^*$. So, by Lemma 1.1 of [14], it follows that $\int_B f_n \, d\nu \to \int_B f \, d\nu$ weakly in $X \check{\otimes} Y$. Thus, we have $\int_B f \, d\nu = \mathcal{U}_B$ and consequently $\lim_n \int_B f_n \, d\nu = \int_B f \, d\nu$ for each $B \in C(\tau)$ and the theorem is proved. □

THEOREM 3.6. *$w - L_1(\nu, X)$ is a Banach space with respect to the norm*

$$N(f) = \sup\left\{ \int_\Omega \|f\| \, d\left|y^*\nu\right| : y^* \in B_{Y^*} \right\}, \quad f \in w - L_1(\nu, X).$$

Proof. It is easy to see that $w - L_1(\nu, X)$ is a normed linear space with respect to the norm $N(\cdot)$. So we have only to show that it is complete with respect to the norm $N(\cdot)$.

Let $\{f_k\}$ be a sequence in $w - L_1(\nu, X)$ such that $\sum_{k=1}^\infty N(f_k) < \infty$. So we have to show that there exists an $f \in w - L_1(\nu, X)$ such that

$$N\left(f - \sum_{k=1}^n f_k\right) \to 0, \quad \text{as } n \to \infty.$$

Let $\omega \in \Omega$ and let

$$\phi(\omega) = \sum_{k=1}^\infty \left\|f_k(\omega)\right\| \in [0, \infty],$$
$$\Omega_\infty = \left\{\omega \in \Omega \text{ and } \phi(\omega) = \infty\right\}.$$

Since each $f_k \in \mathcal{M}(X)$, it follows that $\phi \in \mathcal{M}^*$ and $\Omega_\infty \in C(\tau)$.

We contend that $\|\nu\|(\Omega_\infty) = 0$ and $\|\phi\chi_{\Omega_\infty}\|_\nu = 0$.

Fix $y^* \in B_{Y^*}$. Then

$$\int_\Omega \phi \, d\left|y^*\nu\right| = \int_\Omega \left[\lim_n \sum_{k=1}^n \|f_k\|\right] d\left|y^*\nu\right| = \lim_n \sum_{k=1}^n \int_\Omega \|f_k\| \, d\left|y^*\nu\right|$$
$$\leq \lim_n \sum_{k=1}^n N(f_k) = \sum_{k=1}^\infty N(f_k) < \infty.$$

Hence, $\phi \in L_1(|y^*\nu|)$ and so it follows by a classical result that $|y^*\nu|(\Omega_\infty) = 0$.

Therefore,

$$\int_\Omega \phi\chi_{\Omega_\infty} \, d\left|y^*\nu\right| = \int_{\Omega_\infty} |\phi| \, d|y^*\nu| = 0 \quad \text{for all } y^* \in B_{Y^*}.$$

Taking supremum over $y^* \in B_{Y^*}$ we have, $\|\nu\|(\Omega_\infty) = 0$ and $\|\phi\chi_{\Omega_\infty}\|_\nu = 0$.

Define $f(\cdot)$ on Ω by

$$f(\omega) = \begin{cases} 0, & \text{if } \omega \in \Omega_\infty, \\ \sum_{k=1}^\infty f_k(\omega), & \text{if } \omega \in \Omega \setminus \Omega_\infty. \end{cases}$$

Then an easy calculation shows that $f \in w - L_1(\nu, X)$. Now,

$$\begin{aligned} N\left(f - \sum_{k=1}^{n} f_k\right) &= \sup_{\|y^*\| \le 1} \int_\Omega \left\| f - \sum_{k=1}^{n} f_k \right\| d|y^*\nu| \\ &= \sup_{\|y^*\| \le 1} \int_{\Omega \setminus \Omega_\infty} \left\| \sum_{k=n+1}^{\infty} f_k \right\| d|y^*\nu| \\ &\le \sup_{\|y^*\| \le 1} \int_\Omega \left\| \sum_{k=n+1}^{\infty} f_k \right\| d|y^*\nu| = N\left(\sum_{k=n+1}^{\infty} f_k\right) \\ &\le \sum_{k=n+1}^{\infty} N(f_k) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that $\sum_{k=1}^{n} f_k$ converges to f in $w - L_1(\nu, X)$ and so $w - L_1(\nu, X)$ is a Banach space with respect to the norm $N(\cdot)$ and the proof is complete. □

Note that $L_1(\nu, X, Y) \subset w - L_1(\nu, X)$. Since by definition, $S(\tau, X)$ is dense in $L_1(\nu, X, Y)$ with respect to the norm $N(\cdot)$, it follows that $L_1(\nu, X, Y)$ is a closed subspace of $w - L_1(\nu, X)$ with respect to the norm $N(\cdot)$ and consequently we have that

THEOREM 3.7. *$L_1(\nu, X, Y)$ is a Banach space with respect to the norm $N(\cdot)$.*

PROPOSITION 3.8. *Let $f \in \mathcal{M}(X)$. A necessary condition that f belongs to $L_1(\nu, X, Y)$ is that*

(a) *there exists a sequence $\{\phi_n\} \subset S(\tau, X)$ such that $\{\phi_n\}$ converges to f $\|\nu\|$-a.e.*

(b) *$\{\int_A \phi_n \, d\nu\}$ converges in the norm of $X \check{\otimes} Y$ for all $A \in C(\tau)$.*

Proof. Let $f \in L_1(\nu, X, Y)$. Then, by definition, there exists a sequence of X-valued τ-simple functions $\{\psi_n\}$ such that $N(f - \psi_n) \to 0$, as $n \to \infty$. So, there exists a subsequence $\{\phi_n\}$ of $\{\psi_n\}$ such that $\{\phi_n\}$ converges to f $\|\nu\|$-a.e.

Now, an easy calculation shows that

$$\begin{aligned} &\left\| \int_A f \, d\nu - \int_A \phi_n \, d\nu \right\|_{X \check{\otimes} Y} \\ &\quad = \sup\left\{ \left| (x^* \otimes y^*) \left(\int_A f \, d\nu - \int_A \phi_n \, d\nu \right) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\} \\ &\quad = \sup\left\{ \left| \int_A x^*(f - \phi_n) \, dy^*\nu \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\} \\ &\quad \le N(f - \phi_n) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

So, $\{\int_A \phi_n \, d\nu\}$ converges to $\int_A f \, d\nu$ in the norm of $X \check{\otimes} Y$ for all $A \in C(\tau)$. □

REMARK. Whether the conditions (a) and (b) are sufficient for the above proposition is not known, even when the vector measure ν is defined on a σ-algebra.

However, the conditions (a) and (b) are sufficient for the case of integration theory of scalar valued functions with respect to a vector measure ν defined on a δ-ring (see [5, Proposition 2.3, p. 436]), where the proof of the sufficiency part depends on the fact that $\|f\|_\nu = \|\mu_f\|(\Omega)$ for $f \in L_1(\nu)$, which is not true for $f \in L_1(\nu, X, Y)$.

It has been shown in [5, Corollary 3.2, p. 438] the following.

THEOREM A. *Let Y be a Banach space, τ a δ-ring of subsets of a non-empty set Ω and $\nu : \tau \to Y$ a vector measure.*

(a) *If ν is strongly additive, then $L_1(\nu)$ coincides with $L_1(\widehat{\nu})$ where $\widehat{\nu} : C(\tau) \to Y$ is a vector measure which extends ν.*
(b) *The vector measure ν is strongly additive if and only if $\chi_\Omega \in L_1(\nu)$.*
(c) *If ν is strongly additive then $L_1(\nu)$ is an order continuous Banach function space with respect to $(\Omega, C(\tau), \lambda)$ where $\lambda = |x_0^*\nu|$, $x_0^* \in B_{X^*}$, is the Rybakov control measure for ν.*

We are now in a position to extend Theorem A to $L_1(\nu, X, Y)$.

THEOREM 3.9. *Let X be an order continuous Banach lattice and Y a Banach space. Let $\nu : \tau \to Y$ be a vector measure. If ν is strongly additive, then $L_1(\nu, X, Y)$ coincides with $L_1(\widehat{\nu}, X, Y)$, where $\widehat{\nu} : C(\tau) \to Y$ is a vector measure which extends ν and $L_1(\nu, X, Y)$ is an order continuous Banach lattice with weak order unit.*

Proof. We first show that $S(\tau, X)$ is dense in $L_1(\widehat{\nu}, X, Y)$. Since ν is strongly additive, we have $\Omega = (\bigcup_{n=1}^\infty A_n) \cap N$ where $\{A_n\} \subset \tau$ and $N \in C(\tau)$ is a ν-null set and $(\bigcup_{n=1}^\infty A_n) \cap N = \phi$. So $\Omega \setminus N = \bigcup_{n=1}^\infty A_n$. Let $B_n = \bigcup_{k=1}^n A_k$.

Then $\{B_n\}$ is an increasing sequence in τ with $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n$.

Let $f \in L_1(\widehat{\nu}, X, Y)$ and $\varepsilon > 0$. Since X is order complete and

$$|f|\chi_{B_1} \le |f|\chi_{B_2} \le \cdots \le |f|\chi_{\Omega\setminus N} \quad \text{and} \quad \lim_n |f|\chi_{B_n} = |f|\chi_{\Omega\setminus N},$$

it follows that $\{|f|\chi_{B_n}\}$ is an order bounded increasing sequence in $L_1(\widehat{\nu}, X, Y)$. Since $L_1(\widehat{\nu}, X, Y)$ is an order continuous Banach lattice [2, Theorem 1, p. 5], we have by [15, Proposition 1.a.8, p. 7] that there exists some $B_k \in \tau$ such that

$$\sup_{\|y^*\|\le 1} \int_\Omega \big\||f|\chi_{\Omega\setminus N} - |f|\chi_{B_k}\big\|\, d\big|y^*\widehat{\nu}\big| < \varepsilon/2.$$

This implies that

$$\sup_{\|y^*\|\le 1} \int_\Omega \|f\chi_{\Omega\setminus N} - f\chi_{B_k}\|\, d\big|y^*\widehat{\nu}\big| < \varepsilon/2$$

and so in notational form, we have

$$N(f\chi_{\Omega\setminus N} - f\chi_{B_k}) < \varepsilon/2. \tag{8}$$

Since $S(C(\tau),X)$ is dense in $L_1(\widehat{\nu},X,Y)$ [22, Definition 1, p. 927], there exists $g \in S(C(\tau),X)$ such that

$$N(f-g) < \varepsilon/2.$$

This implies that

$$N(f\chi_{B_k} - g\chi_{B_k}) < \varepsilon/2. \tag{9}$$

Let

$$g = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad x_i \in X, E_i \in C(\tau).$$

Then

$$g\chi_{B_k} = \sum_{i=1}^{n} x_i \chi_{E_i \cap B_k}, \quad E_i \cap B_k \in \tau \quad \text{and so} \quad g\chi_{B_k} \in S(\tau, X).$$

Now

$$\begin{aligned} N(f\chi_{\Omega\setminus N} - g\chi_{B_k}) &\le N(f\chi_{\Omega\setminus N} - f\chi_{B_k}) + N(f\chi_{B_k} - g\chi_{B_k}) \\ &< \varepsilon, \quad \text{by (8) and (9)} \quad \text{and so} \quad N(f - g\chi_{B_k}) < \varepsilon. \end{aligned}$$

Thus, $S(\tau,X)$ is dense in $L_1(\widehat{\nu},X,Y)$.

We next show that $L_1(\nu,X,Y)$ coincides with $L_1(\widehat{\nu},X,Y)$. By Definition 3.2, $S(\tau,X)$ is dense in $L_1(\nu,X,Y)$ with respect to the norm $N(\cdot)$ and $L_1(\nu,X,Y) \subset L_1(\widehat{\nu},X,Y)$.

So $S(\tau,X) \subset L_1(\nu,X,Y) \subset L_1(\widehat{\nu},X,Y)$.

Since $S(\tau,X)$ is dense in $L_1(\widehat{\nu},X,Y)$ with respect to the norm $N(\cdot)$ and since $L_1(\nu,X,Y)$ is complete with respect to $\widehat{\nu}$ restricted to τ, it follows by an easy calculation that $L_1(\nu,X,Y) = L_1(\widehat{\nu},X,Y)$. Since by [2, Theorem 1, p. 5], $L_1(\widehat{\nu},X,Y)$ is an order continuous Banach lattice with weak order unit, it follows that $L_1(\nu,X,Y)$ is an order continuous Banach lattice with weak order unit. $\square$

Acknowledgments. The authors wish to express their heartfelt thanks to the anonymous referee for his thorough and meticulous reading of the original version of their manuscript. They gratefully acknowledge the valuable comments and suggestions of the referee, which helped them to improve the presentation of the paper to a great extent. He brought to the notice of the authors the works of Dobrakov, Jefferies and Okada, Pallu De la Barriére, Panchapagesan and Rodríguez, which have been very useful in preparing the revised version. He also pointed out some minor mistakes which have also been corrected.

The authors also wish to thank Dr. A. A. Shaikh, former Head of the Department of Mathematics, University of Burdwan, for his constant encouragement during the preparation of this paper.

References

[1] J. K. Brooks and N. Dinculeanu, *Strong additivity, absolute continuity and compactness in spaces of measures*, J. Math. Anal. Appl. **45** (1974), 156–175. MR 0346505

[2] N. D. Chakraborty and S. Basu, *On some properties of the space of tensor integrable functions*, Anal. Math. **33** (2007), 1–16. MR 2312937

[3] N. D. Chakraborty and S. Basu, *Spaces of p-tensor integrable functions and related Banach space properties*, Real Anal. Exchange **34** (2008/2009), 87–104. MR 2527124

[4] G. P. Curbera and W. J. Ricker, *Vector measures, integration and applications*, Positivity, Trends Math., Birkhäuser, Basel, 2007, pp. 127–160. MR 2382217

[5] O. Delgado, *L^1-spaces of vector measures defined on δ-rings*, Arch. Math. **84** (2005), 432–443. MR 2139546

[6] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, vol. 15, *Amer. Math. Soc.*, Providence, RI, 1977. MR 0453964

[7] I. Dobrakov, *On integration in Banach spaces*, I, Czechoslovak Math. J. **20** (1970), 511–536. MR 0365138

[8] I. Dobrakov, *On integration in Banach spaces*, II, Czechoslovak Math. J. **20** (1970), 680–695. MR 0365139

[9] I. Dobrakov, *On integration in Banach spaces*, VII, Czechoslovak Math. J. **38** (1988), 434–449. MR 0950297

[10] I. Dobrakov and T. V. Panchapagesan, *A generalized Pettis measurability criterion and integration of vector functions*, Studia Math. **164** (2004), 205–229. MR 2079949

[11] A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E. A. Sánchez-Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10** (2006), 1–16. MR 2223581

[12] B. Jefferies and S. Okada, *Bilinear integration in tensor products*, Rocky Mountain J. Math. **28** (1998), 517–545. MR 1651584

[13] D. R. Lewis, *On integrability and summability in vector spaces*, Illinois J. Math. **16** (1972), 294–307. MR 0291409

[14] D. R. Lewis, *Conditional weak compactness in certain inductive tensor products*, Math. Ann. **201** (1973), 201–209. MR 0326417

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin, 1979. MR 0540367

[16] P. R. Masani and H. Niemi, *The integration theory of Banach space valued measures and the Tonelli–Fubini theorems. I. Scalar-valued measures on δ-rings*, Adv. Math. **73** (1989), 204–241. MR 0987275

[17] P. R. Masani and H. Niemi, *The integration theory of Banach space valued measures and the Tonelli–Fubini theorems. II. Pettis integration*, Adv. Math. **75** (1989), 121–167. MR 1002206

[18] S. Okada, W. J. Ricker and E. A. Sánchez Pérez, *Optimal domain and integral extension of operators acting in function spaces*, Operator Theory: Advances and Applications, vol. 180, Birkhäuser Verlag, Basel, 2008. MR 2418751

[19] R. Pallu de La Barriére, I*ntegration of vector functions with respect to vector measures*, Studia Univ. Babes-Bolyai Math. **43** (1998), 55–93. MR 1855339

[20] J. Rodríguez, *On integration of vector functions with respect to vector measures*, Czechoslovak Math. J. **56** (2006), 805–825. MR 2261655

[21] E. A. Sánchez-Pérez, C*ompactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue–Bochner spaces*, Illinois J. Math. **45** (2001), 907–923. MR 1879243

[22] G. F. Stefánsson, *Integration in vector spaces*, Illinois J. Math. **45** (2001), 925–938. MR 1879244

N. D. Chakraborty, Department of Mathematics, University of Burdwan, Burdwan-713104, West Bengal, India

E-mail address: cms_ndc@yahoo.co.in

Santwana Basu, Department of Mathematics, University of Burdwan, Burdwan-713104, West Bengal, India

E-mail address: basu_snw@yahoo.co.in