THE DEPTH FORMULA FOR MODULES WITH REDUCIBLE COMPLEXITY

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ABSTRACT. We prove that the depth formula holds for Tor-independent modules in certain cases over a Cohen–Macaulay local ring, provided one of the modules has reducible complexity.

1. Introduction

Two finitely generated modules M and N over a local ring A satisfy the *depth formula* if

$$\text{(1)} \qquad \operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} A + \operatorname{depth}(M \otimes_A N).$$

This formula is not at all true in general; an obvious counterexample appears by taking modules of depth zero over a ring of positive depth. The natural question is then: for which pairs of modules does the formula hold? The first systematic treatment of this question was done by Auslander in [Aus], where he considered the case when one of the modules involved has finite projective dimension. In this situation, let q be the largest integer such that $\operatorname{Tor}_q^A(M, N)$ is nonzero. Auslander proved that if either $\operatorname{depth} \operatorname{Tor}_q^A(M, N) \leq 1$ or $q = 0$, then the formula

$$\text{(2)} \qquad \operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} A + \operatorname{depth} \operatorname{Tor}_q^A(M, N) - q$$

holds. The case $q = 0$ is the depth formula.

Auslander's result indicated that in order to decide which pairs of modules satisfy the depth formula, one should concentrate on Tor-*independent* pairs, that is, modules M and N satisfying $\operatorname{Tor}_n^A(M, N) = 0$ for $n > 0$. In [HuW], Huneke and Wiegand showed that the depth formula holds for such modules over complete intersections. This (and Auslander's result) was later generalized in [ArY] by Araya and Yoshino, who considered the case when one of the modules involved has finite complete intersection dimension. If

$\operatorname{Tor}_n^A(M, N) = 0$ for $n \gg 0$, let q be the largest integer such that $\operatorname{Tor}_q^A(M, N)$ is nonzero. In this situation, Araya and Yoshino proved Auslander's original result (2) above if either depth $\operatorname{Tor}_q^A(M, N) \le 1$ or $q = 0$.

The aim of this paper is to investigate the depth formula (1) for Tor-independent modules over a local Cohen–Macaulay ring, provided one of the modules has reducible complexity. In particular, we show that the formula (1) holds for Tor-independent modules over a Cohen–Macaulay ring if one module has reducible complexity and is not maximal Cohen–Macaulay, or if *both* modules have reducible complexity. Moreover, we prove that the depth formula holds if one of the modules involved has reducible complexity, and the other has finite Gorenstein dimension.

In the final section, we show that there exist modules having reducible complexity of any finite complexity, but not finite complete intersection dimension. Knowing that such modules exist is a critical point of the investigation. Modules of infinite complete intersection dimension are in a precise sense far from resembling those modules considered in the original explorations of the formulas (1) and (2). Thus, we show that the depth formula holds in a context that is fundamentally departed from previous considerations. We know of no example of finitely generated Tor-independent modules that do not satisfy the depth formula (1), nor are we aware of a counterexample to Auslander's formula (2) when $q < \infty$, and depth $\operatorname{Tor}_q^R(M, N) \le 1$ or $q = 0$.

2. Reducible complexity

Throughout the rest of this paper, we assume that all modules encountered are finitely generated. In this section, we fix a local (meaning commutative Noetherian local) ring $(A, \mathfrak{m}, k)$. Under these assumptions, every A-module M admits a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

which is unique up to isomorphism. The rank of the free A-module F_n is the nth *Betti number* of M; we denote it by $\beta_n^A(M)$. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M \stackrel{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n^A(M) \le an^{t-1} \text{ for all } n \gg 0\}.$$

In other words, the complexity of a module is the polynomial rate of growth of its Betti sequence. It follows from the definition that $\operatorname{cx} M = 0$ precisely when M has finite projective dimension, and that $\operatorname{cx} M = 1$ if and only if the Betti sequence of M is bounded. An arbitrary local ring may have many modules with infinite complexity; by a theorem of Gulliksen (cf. [Gul]), the local rings over which all modules have finite complexity are precisely the complete intersections.

In [Be1], the concept of modules with reducible complexity was introduced. These are modules which in some sense generalize modules of finite complete

intersection dimension (see [AGP]), in particular modules over complete intersections. Before we state the definition, we recall the following. Let M and N be A-modules, and consider an element $\eta \in \mathrm{Ext}_A^t(M, N)$. By choosing a map $f_\eta : \Omega_A^t(M) \to N$ representing η, we obtain a commutative pushout diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_A^t(M) & \longrightarrow & F_{t-1} & \longrightarrow & \Omega_A^{t-1}(M) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_\eta} & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & K_\eta & \longrightarrow & \Omega_A^{t-1}(M) & \longrightarrow & 0
\end{array}
$$

with exact rows. The module K_η is independent, up to isomorphism, of the map f_η chosen as a representative for η. We now recall the definition of modules with reducible complexity. Given A-modules X and Y, we denote the graded A-module $\bigoplus_{i=0}^\infty \mathrm{Ext}_A^i(X, Y)$ by $\mathrm{Ext}_A^*(X, Y)$.

DEFINITION. The full subcategory of A-modules consisting of the modules having *reducible complexity* is defined inductively as follows:

(i) Every A-module of finite projective dimension has reducible complexity.
(ii) An A-module M of finite positive complexity has reducible complexity if there exists a homogeneous element $\eta \in \mathrm{Ext}_A^*(M, M)$, of positive degree, such that $\mathrm{cx}\, K_\eta < \mathrm{cx}\, M$ and K_η has reducible complexity.

Thus, an A-module M of finite positive complexity c, say, has reducible complexity if and only if the following hold: there exist nonnegative integers $n_1, \ldots, n_t$, with $t \le c$, and exact sequences (with $K_0 = M$)

$$
\eta_1 : \qquad 0 \longrightarrow K_0 \longrightarrow K_1 \longrightarrow \Omega_A^{n_1}(K_0) \longrightarrow 0
$$

$$
\vdots \qquad\qquad \vdots \qquad\qquad \vdots
$$

$$
\eta_t : \qquad 0 \longrightarrow K_{t-1} \longrightarrow K_t \longrightarrow \Omega_A^{n_t}(K_{t-1}) \longrightarrow 0
$$

in which $\mathrm{cx}\, M > \mathrm{cx}\, K_1 > \cdots > \mathrm{cx}\, K_t = 0$. We say that these sequences $\eta_1, \ldots, \eta_t$ *reduce the complexity* of M. As shown in [Be1], every module of finite complete intersection dimension has reducible complexity. In particular, if A is a complete intersection, then every A-module has this property.

REMARK 1. In the original definition in [Be1], the extra requirement $\mathrm{depth}\, M = \mathrm{depth}\, K_1 = \cdots = \mathrm{depth}\, K_t$ was included. However, as we will only be working over Cohen–Macaulay rings, this requirement is redundant. Namely, when A is Cohen–Macaulay and M is any A-module, then the depth of any syzygy of M is at least the depth of M. Consequently, in a short exact sequence

$$
0 \to M \to K \to \Omega_A^n(M) \to 0
$$

the depth of M automatically equals that of K.

3. The depth formula

Let A be a local ring, and let M be an A-module with reducible complexity. If the complexity of M is positive, then by definition there exist a number t and short exact sequences

$$\eta_i : 0 \to K_{i-1} \to K_i \to \Omega_A^{|\eta_i|-1}(K_{i-1}) \to 0$$

for $1 \leq i \leq t$ reducing the complexity of M. We define the *upper reducing degree* of M, denoted $\operatorname{reddeg}^* M$, to be the supremum of the minimal degree of the cohomological elements η_i, the supremum taken over all such sequences reducing the complexity of M:

$$\operatorname{reddeg}^* M$$
$$\stackrel{\text{def}}{=} \sup\{\min\{|\eta_1|, \ldots, |\eta_t|\} \mid \eta_1, \ldots, \eta_t \text{ reduces the complexity of } M\}.$$

If the complexity of M is zero, that is, if M has finite projective dimension, then we define $\operatorname{reddeg}^* M = \infty$. Note that the inequality $\operatorname{reddeg}^* M \geq 1$ always holds.

We now prove our first result, namely the depth formula in the situation when the tensor product of the two modules involved has depth zero. In this result, we also include a generalized version of half of [ArY, Theorem 2.5].

THEOREM 3.1. *Let A be a Cohen–Macaulay local ring, and let M and N be nonzero A-modules such that M has reducible complexity. Suppose that $\operatorname{Tor}_n^A(M, N) = 0$ for $n \gg 0$, and let q be the largest integer such that $\operatorname{Tor}_q^A(M, N)$ is nonzero. Furthermore, suppose that one of the following holds:*

(i) $\operatorname{depth} \operatorname{Tor}_q^A(M, N) = 0$,
(ii) $q \geq 1$, $\operatorname{depth} \operatorname{Tor}_q^A(M, N) \leq 1$ *and* $\operatorname{reddeg}^* M \geq 2$.
Then the formula

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth} \operatorname{Tor}_q^A(M, N) - q$$

holds.

Proof. Part (i) is just [Be1, Theorem 3.4(i)], so we only need to prove (ii). We do this by induction on the complexity of M, where the case $\operatorname{cx} M = 0$ follows from Auslander's original result [Aus, Theorem 1.2]. Suppose therefore the complexity of M is nonzero. Since $\operatorname{reddeg}^* M \geq 2$, there exists an exact sequence

$$0 \to M \to K \to \Omega_A^n(M) \to 0$$

with $n \geq 1$, in which the complexity of K is at most $\operatorname{cx} M - 1$ and $\operatorname{reddeg}^* K \geq 2$. From this sequence, we see that $\operatorname{Tor}_q^A(K, N)$ is isomorphic to $\operatorname{Tor}_q^A(M, N)$,

and that $\operatorname{Tor}_i^A(K, N) = 0$ for $i \geq q + 1$. The formula therefore holds with K replacing M, but since $\operatorname{depth} K = \operatorname{depth} M$ we are done. $\qquad\square$

As mentioned, the proof of this result generalizes the first half of [ArY, Theorem 2.5]. Namely, if A is a local ring (not necessarily Cohen–Macaulay) and M is a module of finite complete intersection dimension, then M has reducible complexity (including the depth condition of Remark 1) by [Be1, Proposition 2.2(i)], and $\operatorname{reddeg}^* M = \infty$ by [Be2, Lemma 2.1(ii)].

Next, we show that the depth formula is valid for Tor-independent modules over a local Cohen–Macaulay ring in the following situation: one of the modules has reducible complexity, and the other has finite Gorenstein dimension. Recall therefore that if A is a local ring, then a module M has *Gorenstein dimension zero* if the following hold: the module is reflexive (i.e., the canonical homomorphism $M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, A), A)$ is bijective), and

$$\operatorname{Ext}_A^n(M, A) = 0 = \operatorname{Ext}_A^n\big(\operatorname{Hom}_A(M, A), A\big)$$

for all $n > 0$. The *Gorenstein dimension* of M is defined to be the infimum of all nonnegative integers n, such that there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

in which all the G_i have Gorenstein dimension zero. By [AuB, Theorem 4.13], if M has finite Gorenstein dimension, then it equals $\operatorname{depth} A - \operatorname{depth} M$. Moreover, by [AuB, Theorem 4.20], a local ring is Gorenstein precisely when *every* module has finite Gorenstein dimension.

PROPOSITION 3.2. *Let A be a local Cohen–Macaulay ring, and M and N be nonzero Tor-independent A-modules. Assume that M is maximal Cohen–Macaulay and has reducible complexity, and that N has finite Gorenstein dimension. Then if $\operatorname{depth}(M \otimes_A N)$ is nonzero, so is $\operatorname{depth} N$.*

Proof. By [CFH, Lemma 2.17], there exists an exact sequence

$$0 \to N \to I \to X \to 0$$

in which the projective dimension of I is finite and X has Gorenstein dimension zero. Then $\operatorname{Tor}_n^A(M, I)$ and $\operatorname{Tor}_n^A(M, X)$ both vanish for $n \gg 0$, and since M is maximal Cohen–Macaulay it follows from [Be1, Theorem 3.3] that $\operatorname{Tor}_n^A(M, I) = 0 = \operatorname{Tor}_n^A(M, X)$ for $n \geq 1$. Hence, the pairs $(M, N), (M, I)$ and (M, X) are all Tor-independent.

Suppose $\operatorname{depth} N = 0$. Then the depth of I is also zero. Tensoring the exact sequence with M yields the exact sequence

$$0 \to M \otimes_A N \to M \otimes_A I \to M \otimes_A X \to 0.$$

By Auslander's original result, the depth formula holds for the pair (M, I). Moreover, by [Be1, Theorem 3.4(iii)], the formula also holds for the pair

(M, X), hence

$$\operatorname{depth} M + \operatorname{depth} I = \dim A + \operatorname{depth}(M \otimes_A I)$$

and

$$\operatorname{depth} M + \operatorname{depth} X = \dim A + \operatorname{depth}(M \otimes_A X).$$

The first of these formulas implies that the depth of $M \otimes_A I$ is zero. The second formula, together with the fact that X is maximal Cohen–Macaulay, implies that $M \otimes_A X$ is maximal Cohen–Macaulay. Therefore, $\operatorname{depth}(M \otimes_A N) = 0$ by the depth lemma. $\square$

We can now prove that the depth formula holds when one module has reducible complexity, and the other has finite Gorenstein dimension.

THEOREM 3.3 (Depth formula—Gorenstein case 1). *Let A be a local Cohen–Macaulay ring, and M and N be nonzero* Tor-*independent A-modules. If M has reducible complexity and N has finite Gorenstein dimension, then*

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth}(M \otimes_A N).$$

Proof. We prove this result by induction on the depth of the tensor product. If $\operatorname{depth}(M \otimes_A N) = 0$, then the formula holds by Theorem 3.1, so assume that $\operatorname{depth}(M \otimes_A N)$ is positive. If M has finite projective dimension, then the formula holds by Auslander's original result, hence we assume that the complexity of M is positive.

Suppose the depth of N is zero. Choose short exact sequences (with $K_0 = M$)

$$0 \longrightarrow K_0 \longrightarrow K_1 \longrightarrow \Omega_A^{n_1}(K_0) \longrightarrow 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \longrightarrow K_{t-1} \longrightarrow K_t \longrightarrow \Omega_A^{n_t}(K_{t-1}) \longrightarrow 0$$

reducing the complexity of M, and note that the pair (K_i, N) is Tor-independent for all i. Since the projective dimension of K_t is finite, the depth formula holds for K_t and N, that is,

$$\operatorname{depth} K_t + \operatorname{depth} N = \dim A + \operatorname{depth}(K_t \otimes_A N).$$

Since $\operatorname{depth} N = 0$, we see that K_t, and hence also M, is maximal Cohen–Macaulay. But this contradicts Proposition 3.2, hence the depth of N must be positive.

Choose an element $x \in A$ which is regular on both N and $M \otimes_A N$. Tensoring the exact sequence

$$0 \to N \xrightarrow{\cdot x} N \to N/xN \to 0$$

with M, we get the exact sequence

$$0 \to \operatorname{Tor}_1^A(M, N/xN) \to M \otimes_A N \xrightarrow{\cdot x} M \otimes_A N \to M \otimes_A N/xN \to 0.$$

We also see that $\operatorname{Tor}_n^A(M, N/xN) = 0$ for $n \geq 2$. However, the element x is regular on $M \otimes_A N$, hence $\operatorname{Tor}_1^A(M, N/xN) = 0$ and $(M \otimes_A N)/x(M \otimes_A N) \simeq M \otimes_A N/xN$. The modules M and N/xN are therefore Tor-independent, and $\operatorname{depth}(M \otimes_A N/xN) = \operatorname{depth}(M \otimes_A N) - 1$. By induction, the depth formula holds for M and N/xN, giving

$$\begin{aligned}
\operatorname{depth} M + \operatorname{depth} N &= \operatorname{depth} M + \operatorname{depth} N/xN + 1 \\
&= \dim A + \operatorname{depth}(M \otimes_A N/xN) + 1 \\
&= \dim A + \operatorname{depth}(M \otimes_A N).
\end{aligned}$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

COROLLARY 3.4 (Depth formula—Gorenstein case 2). *Let A be a Gorenstein local ring, and M and N be nonzero* Tor-*independent A-modules. If M has reducible complexity, then*

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth}(M \otimes_A N).$$

REMARK. In work in progress by Lars Winther Christensen and the second author (cf. [ChJ]), the depth formula is proved for modules M and N over a local ring A under the following assumptions: the module M has finite Gorenstein dimension, and the Tate homology group $\widehat{\operatorname{Tor}}_n^A(M, N)$ vanishes for all $n \in \mathbb{Z}$.

What can we say if the ring is not necessarily Gorenstein, or, more general, when we do not assume that one of the modules has finite Gorenstein dimension? The following result shows that if the ring is Cohen–Macaulay and the module having reducible complexity is not maximal Cohen–Macaulay, then the depth formula holds.

THEOREM 3.5 (Depth formula—Cohen–Macaulay case 1). *Let A be a Cohen–Macaulay local ring, and let M and N be nonzero* Tor-*independent A-modules. If M has reducible complexity and is not maximal Cohen–Macaulay, then*

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth}(M \otimes_A N).$$

Proof. We prove this result by induction on the complexity of M. As before, if M has finite projective dimension, then the depth formula follows from Auslander's original result. We therefore assume that the complexity of M is positive.

Choose a short exact sequence

$$0 \to M \to K \to \Omega_A^t(M) \to 0$$

in $\operatorname{Ext}_A^t(M, M)$, with $\operatorname{cx} K < \operatorname{cx} M$ and $t \geq 0$. Since M and K are Tor-independent and $\operatorname{depth} K = \operatorname{depth} M$, the depth formula holds for these mod-

ules by induction, i.e.

(†) $$\operatorname{depth} K + \operatorname{depth} N = \dim A + \operatorname{depth}(K \otimes_A N).$$

Therefore, we need only to show that $\operatorname{depth}(K \otimes_A N) = \operatorname{depth}(M \otimes_A N)$.

If $t = 0$, then by tensoring the above exact sequence with N, we obtain the exact sequence

$$0 \to M \otimes_A N \to K \otimes_A N \to M \otimes_A N \to 0.$$

In this situation, the equality $\operatorname{depth}(K \otimes_A N) = \operatorname{depth}(M \otimes_A N)$ follows from the depth lemma, and we are done. What remains is therefore the case $t \geq 1$. Moreover, by considering the short exact sequence

(††) $$0 \to M \otimes_A N \to K \otimes_A N \to \Omega_A^t(M) \otimes_A N \to 0,$$

we see that if the depth of $M \otimes_A N$ is zero, then so is the depth of $K \otimes_A N$. In this case we are done, hence we may assume that the depth of $M \otimes_A N$ is positive.

Suppose $\operatorname{depth}(K \otimes_A N) > \operatorname{depth}(M \otimes_A N)$. Then $\operatorname{depth}(\Omega_A^t(M) \otimes_A N) = \operatorname{depth}(M \otimes_A N) - 1$ by the depth lemma. Now for each $i \geq 1$, let

$$0 \to \Omega_A^{i+1}(M) \to A^{\beta_i} \to \Omega_A^i(M) \to 0$$

be a projective cover of $\Omega_A^i(M)$, and note that this sequence stays exact when we tensor with N. Let s be the largest integer in $\{0, \ldots, t-1\}$ such that in the exact sequence

(†††) $$0 \to \Omega_A^{s+1}(M) \otimes_A N \to N^{\beta_s} \to \Omega_A^s(M) \otimes_A N \to 0$$

the inequality $\operatorname{depth}(\Omega_A^{s+1}(M) \otimes_A N) < \operatorname{depth}(\Omega_A^s(M) \otimes_A N)$ holds. From the depth lemma applied to this sequence, we see that

$$\begin{aligned}
\operatorname{depth} N &= \operatorname{depth}\big(\Omega_A^{s+1}(M) \otimes_A N\big) \\
&\leq \operatorname{depth}\big(\Omega_A^t(M) \otimes_A N\big) \\
&= \operatorname{depth}(M \otimes_A N) - 1 \\
&< \operatorname{depth}(K \otimes_A N) - 1.
\end{aligned}$$

But then from (†), we obtain the contradiction $\dim A < \operatorname{depth} K - 1$, and consequently the inequality $\operatorname{depth}(K \otimes_A N) > \operatorname{depth}(M \otimes_A N)$ cannot hold.

Next, suppose that $\operatorname{depth}(K \otimes_A N) < \operatorname{depth}(M \otimes_A N)$. Applying the depth lemma to (††), we see that $\operatorname{depth}(K \otimes_A N) = \operatorname{depth}(\Omega_A^t(M) \otimes_A N)$. Again, let s be the largest integer in $\{0, \ldots, t-1\}$ such that $\operatorname{depth}(\Omega_A^{s+1}(M) \otimes_A N) < \operatorname{depth}(\Omega_A^s(M) \otimes_A N)$. Then the depth lemma applied to (†††) gives

$$\begin{aligned}
\operatorname{depth} N &= \operatorname{depth}\big(\Omega_A^{s+1}(M) \otimes_A N\big) \\
&\leq \operatorname{depth}\big(\Omega_A^t(M) \otimes_A N\big) \\
&= \operatorname{depth}(K \otimes_A N).
\end{aligned}$$

From ($\dagger$), it now follows that K, and hence also M, is maximal Cohen–Macaulay, a contradiction. This shows that the depth of $K \otimes_A N$ equals that of $M \otimes_A N$. $\square$

Next, we show that if both the Tor-independent modules have reducible complexity, then the depth formula holds without the assumption that M is not maximal Cohen–Macaulay.

THEOREM 3.6 (Depth formula—Cohen–Macaulay case 2). *Let A be a Cohen–Macaulay local ring, and let M and N be nonzero* Tor*-independent A-modules. If both M and N have reducible complexity, then*

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth}(M \otimes_A N).$$

Proof. If one of the modules is not maximal Cohen–Macaulay, the result follows from Theorem 3.5. If not, then the result follows from [Be1, Theorem 3.4(iii)]. $\square$

What happens over a Cohen–Macaulay ring if we only require that one of the modules has reducible complexity? We end this section with the following result, showing that, in this situation, if the depth of the tensor product is nonzero, then so is the depth of the module having reducible complexity.

PROPOSITION 3.7. *Let A be a Cohen–Macaulay local ring, and let M and N be nonzero* Tor*-independent A-modules such that M has reducible complexity. Then if* $\operatorname{depth}(M \otimes_A N)$ *is nonzero, so is* $\operatorname{depth} M$.

Proof. If M is maximal Cohen–Macaulay, then the result trivially holds. If not, then the depth formula holds by Theorem 3.5, that is,

$$\operatorname{depth} M + \operatorname{depth} N = \dim A + \operatorname{depth}(M \otimes_A N).$$

Thus, if the depth of $(M \otimes_A N)$ is nonzero, then so is $\operatorname{depth} M$. $\square$

4. Modules with reducible complexity and inifinite complete intersection dimension

We shall shortly give examples showing that there exist modules having reducible complexity of any finite complexity, but not finite complete intersection dimension. In order to do this, we opt to work with complexes in the derived category $D(A)$ of A-modules. This is a triangulated category, the suspension functor Σ being the left shift of a complex together with a sign change in the differential. Now let

$$C : \cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

be a complex in $D(A)$. Then C is *bounded below* if $C_n = 0$ for $n \ll 0$, and *bounded above* if $C_n = 0$ for $n \gg 0$. The complex is *bounded* if it is both bounded below and bounded above. The *homology* of C, denoted $\operatorname{H}(C)$, is the complex with $\operatorname{H}(C)_n = \operatorname{H}_n(C)$, and with trivial differentials. When $\operatorname{H}(C)$ is

bounded and degreewise finitely generated, then C is said to be *homologically finite*. We denote the full subcategory of homologically finite complexes by $D^{\mathrm{hf}}(A)$.

When C is homologically finite, it has a minimal free resolution (cf. [Rob]). Thus, there exists a quasi-isomorphism $F \simeq C$, where F is a bounded below complex

$$\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \cdots$$

of finitely generated free A-modules, and where $\operatorname{Im} d_n \subseteq \mathfrak{m} F_{n-1}$. The minimal free resolution is unique up to isomorphism, and so for each integer n the rank of the free module F_n is a well defined invariant of C. Thus, we may define Betti numbers and complexity for homologically finite complexes, and also the concept of reducible complexity. A complex $C \in D^{\mathrm{hf}}(A)$ is said to have finite project dimension if it is quasi-isomorphic to a perfect complex.

DEFINITION. The full subcategory of complexes in $D^{\mathrm{hf}}(A)$ having *reducible complexity* is defined inductively as follows:

(i) Every homologically finite complex of finite projective dimension has reducible complexity.

(ii) A homologically finite complex C of finite positive complexity has reducible complexity if there exists a triangle

$$C \to \Sigma^n C \to K \to \Sigma C$$

with $n > 0$, such that $\operatorname{cx} K < \operatorname{cx} C$ and K has reducible complexity.

The Betti numbers (and hence also the complexity) of an A-module M equal the Betti numbers of M viewed as an element in $D(A)$, that is, as the stalk complex

$$\cdots \to 0 \to 0 \to M \to 0 \to 0 \to \cdots$$

with M concentrated in degree zero. Moreover, the module M has reducible complexity if and only if it has reducible complexity in $D(A)$. To see this, let

$$\eta: 0 \to M \to K \to \Omega_A^{n-1}(M) \to 0$$

be a short exact sequence, and let F be a free resolution of M. Then η corresponds to a map $F \to \Sigma^n F$ in $D(A)$ whose cone is a free resolution of K. Thus, a sequence of short exact sequences of modules (with $K_0 = M$)

$$0 \longrightarrow K_0 \longrightarrow K_1 \longrightarrow \Omega_A^{n_1-1}(K_0) \longrightarrow 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$0 \longrightarrow K_{t-1} \longrightarrow K_t \longrightarrow \Omega_A^{n_t-1}(K_{t-1}) \longrightarrow 0$$

reducing the complexity of M, corresponds to a sequence of triangles

$$F(K_0) \longrightarrow \Sigma^{n_1} F(K_0) \longrightarrow F(K_1) \longrightarrow \Sigma F(K_0)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$F(K_{t-1}) \longrightarrow \Sigma^{n_t} F(K_{t-1}) \longrightarrow F(K_t) \longrightarrow \Sigma F(K_{t-1})$$

reducing the complexity of F, with $F(K_i)$ a free resolution of K_i. Conversely, every such sequence of triangles of free resolutions of K_i gives a sequence of short exact sequences reducing the complexity of M.

There is more generally a relation between homologically finite complexes of reducible complexity and modules of reducible complexity. For a complex C in $D^{\mathrm{hf}}(A)$, we define the *supremum* of C to be

$$\sup(C) = \sup\{i | \mathrm{H}_i(C) \neq 0\}.$$

PROPOSITION 4.1. *Let $C \in D^{\mathrm{hf}}(A)$ be a complex with reducible complexity and $n = \sup(C)$. Then the A-module $M = \mathrm{Coker}(C_{n+1} \to C_n)$ has reducible complexity.*

Proof. We may assume that C is a minimal complex of finitely generated free A-modules. Let $F = C_{\geq n}$. Then F is a minimal free resolution of M. Moreover, it is easy to check that F has reducible complexity since C has. Thus, by the discussion above, M has reducible complexity. $\qquad\square$

We say that a complex $C \in D^{\mathrm{hf}}(A)$ has *finite CI-dimension* if there exists a diagram of local ring homomorphisms $A \to R \leftarrow Q$ with $A \to R$ flat and $R \leftarrow Q$ surjective with kernel generated by a regular sequence, such that $R \otimes_A C$ has finite projective dimension as a complex of Q-modules (cf. [S-W]).

There is a connection between finite CI-dimension of a complex and that of a module.

PROPOSITION 4.2. *Let C be in $D^{\mathrm{hf}}(A)$. If $\mathrm{Coker}(C_{n+1} \to C_n)$ has finite CI-dimension for some $n \geq \sup(C)$, then so does C.*

Proof. We may assume that C is a minimal complex of finitely generated free A-modules. The result is then [S-W, Corollary 3.8]. $\qquad\square$

The following is an easy fact whose proof is left as an exercise.

PROPOSITION 4.3. *Let $0 \to Y \to X \to \Sigma^n X \to 0$ be a short exact sequence of complexes in $D^{\mathrm{hf}}(A)$. Then Y has finite CI dimension if X does.*

CONSTRUCTION 4.4. Let k be a field and $A^{(i)}$ be k-algebras for $1 \leq i \leq c$. Furthermore, for each $1 \leq i \leq c$ let $F^{(i)}$ be a complex of finitely generated free

$A^{(i)}$-modules with $F_j^{(i)} = 0$ for $j < 0$, and possessing a surjective chain map $\eta^{(i)} : F^{(i)} \to \Sigma^{n_i} F^{(i)}$ of degree n_i. Then
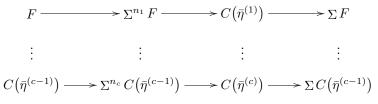
$$F = F^{(1)} \otimes_k \cdots \otimes_k F^{(c)}$$

is a complex of finitely generated free $A = A^{(1)} \otimes_k \cdots \otimes_k A^{(c)}$-modules with $F_j = 0$ for $j < 0$, and each $\eta^{(i)}$ induces a surjective chain map $\bar{\eta}^{(i)} : F \to \Sigma^{n_i} F$. Moreover, the $\bar{\eta}^{(i)}$ commute with one another.

Let $C(\bar{\eta}^{(1)})$ denote the cone of $\bar{\eta}^{(1)}$. Then since $\eta^{(1)}$ and $\eta^{(2)}$ commute with one another, $\bar{\eta}^{(2)}$ induces a surjective chain map $C(\bar{\eta}^{(1)}) \to \Sigma^{n_2} C(\bar{\eta}^{(1)})$. By abuse of notation, we let $C(\bar{\eta}^{(2)})$ denote the cone of this chain map. Inductively we define $C(\bar{\eta}^{(i)})$ to be the cone of the surjective chain map on $C(\bar{\eta}^{(i-1)})$ induced by $\bar{\eta}^{(i)}$.

When $\eta_j^{(i)}$ is an isomorphism for $j \geq n_i$, and no such chain map exists of degree less than n_i, we say that $F^{(i)}$ is *periodic of period* n_i.

PROPOSITION 4.5. *With the notation above, assume that A is local. Suppose that each $F^{(i)}$ is periodic of period n_i, with $\eta_i : F^{(i)} \to \Sigma^{n_i} F^{(i)}$ being the surjective endomorphism defining the periodicity of $F^{(i)}$. Then F has reducible complexity and complexity c.*

Proof. By the discussion above, we have a sequence of triangles

$$F \xrightarrow{\hspace{2cm}} \Sigma^{n_1} F \xrightarrow{\hspace{1.5cm}} C(\bar{\eta}^{(1)}) \xrightarrow{\hspace{1.5cm}} \Sigma F$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$C(\bar{\eta}^{(c-1)}) \longrightarrow \Sigma^{n_c} C(\bar{\eta}^{(c-1)}) \longrightarrow C(\bar{\eta}^{(c)}) \longrightarrow \Sigma C(\bar{\eta}^{(c-1)})$$

Since each chain map induced by $\bar{\eta}^{(i)}$, $1 \leq i \leq c$, is onto, the complexity of each $C(\bar{\eta}^{(i)})$ is one less than that of $C(\bar{\eta}^{(i-1)})$. $\square$

Assume that each $F^{(i)}$ is periodic. Define for $0 \leq i \leq c$ the complexes

$$E^{(i)} = F_{<n_1}^{(1)} \otimes_k \cdots \otimes_k F_{<n_i}^{(i)} \otimes_k F^{(i+1)} \otimes_k \cdots \otimes_k F^{(c)}.$$

The chain maps $\eta^{(i)}$ induce short exact sequences

$$(3) \qquad\qquad 0 \to E^{(i)} \to E^{(i-1)} \to \Sigma^{n_i} E^{(i-1)} \to 0.$$

PROPOSITION 4.6. *With the notation above, assume that each $F^{(i)}$ is periodic, $n_i = 1$ for $1 \leq i \leq c - 1$ and $n_c > 2$. Then the complex F has infinite CI-dimension.*

Proof. By applying Proposition 4.3 inductively to the short exact sequences (3), $E^{(c-1)}$ has finite CI-dimension if F does. However, if $n_i = 1$ for $1 \leq i \leq c - 1$ we have

$$E^{(c-1)} = F_0^{(1)} \otimes_k \cdots \otimes_k F_0^{(c-1)} \otimes_k F^{(c)}$$

which is a periodic complex of free A-modules of period $n_c > 2$. It is well known that complexes of finite CI-dimension and complexity one are periodic of period ≤ 2. Thus, $E^{(c-1)}$ has infinite CI-dimension, and therefore so does F.

$\square$

The following corollary is the main point of this section. Its proof follows from the previous results.

COROLLARY 4.7. *Assume that A is local, and that F is defined as in Proposition 4.6, with $n_i = 1$ for $1 \leq i \leq c-1$ and $n_c > 2$. Then the A-module $M = \operatorname{Coker}(F_1 \to F_0)$ has reducible complexity c and infinite CI-dimension.*

REMARK. The hypothesis in 4.5 and 4.7, that A be local, is easy to achieve. Indeed, one could take each k-algebra $A^{(i)}$ to be local and Artinian. Then the same holds for A. Examples of complexes $F^{(i)}$ of complexity one, both periodic of arbitrary period, and aperiodic are well known to exist over local Artinian rings. See, for example, [GP].

In the spirit of Section 3, one would also like to know that there exist modules of reducible complexity c and infinite CI-dimension over rings A of positive dimension. These are easily seen to exist by taking deformations of examples such as above. For instance, if A is local Artinian and M is an A-module of reducible complexity c and infinite CI-dimension, then for indeterminates $x_1, \ldots, x_r$, let A' be the ring $A[x_1, \ldots, x_r]$ suitable localized, and M' the A'-module $M[x_1, \ldots, x_r]$ similarly localized. Then M' has the same properties as M, now over the positive dimensional local ring A'. This same conclusion holds too if one reduces both A' and M' by a regular sequence in the maximal ideal of A'.

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