Illinois Journal of Mathematics
Volume 55, Number 1, Spring 2011, Pages 397–415
S 0019-2082

# HYPERSURFACES WITH CONSTANT SECTIONAL CURVATURE OF $\mathbb{S}^n \times \mathbb{R}$ AND $\mathbb{H}^n \times \mathbb{R}$

FERNANDO MANFIO AND RUY TOJEIRO

Abstract. We classify the hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with constant sectional curvature and dimension $n \geq 3$.

## 1. Introduction

The submanifold geometry of the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ has been extensively studied in the last years. Here $\mathbb{S}^n$ and $\mathbb{H}^n$ denote the sphere and hyperbolic space of dimension $n$, respectively. Emphasis has been given on minimal and constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, starting with the work in [1] and [15], among others. See [11] for an up-to-date list of references on this topic.

Surfaces of constant *Gaussian* curvature of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ were investigated in [2] and [3], with special attention to their global properties (see also [12] for a local study in $\mathbb{H}^2 \times \mathbb{R}$). In particular, nonexistence of *complete* surfaces of constant Gaussian curvature $c$ in $\mathbb{S}^2 \times \mathbb{R}$ (respectively, $\mathbb{H}^2 \times \mathbb{R}$) was established for $c < -1$ and $0 < c < 1$ (respectively, $c < -1$). It was also shown that a complete surface of constant Gaussian curvature $c > 1$ in $\mathbb{S}^2 \times \mathbb{R}$ (respectively, $c > 0$ in $\mathbb{H}^2 \times \mathbb{R}$) must be a rotation surface. Moreover, the profile curves of such surfaces have been explicitly determined.

Our aim in this paper is to classify all hypersurfaces with constant sectional curvature and dimension $n \geq 3$ of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. It turns out that for $n \geq 4$ a hypersurface of constant sectional curvature $c$ in $\mathbb{S}^n \times \mathbb{R}$ (respectively, $\mathbb{H}^n \times \mathbb{R}$) only exists, even locally, if $c \geq 1$ (respectively, $c \geq -1$), and for any such values of $c$ it must be an open subset of a complete rotation hypersurface. In the case $n = 3$, exactly one class of nonrotational hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with constant sectional curvature arises. Each hypersurface in this class in $\mathbb{S}^3 \times \mathbb{R}$ (respectively, $\mathbb{H}^3 \times \mathbb{R}$) has constant sectional curvature

Received September 11, 2009; received in final form February 22, 2010.
2010 *Mathematics Subject Classification.* 53B25.

$c \in (0,1)$ (respectively, $c \in (-1,0)$), and is constructed in an explicit way by means of a family of parallel flat surfaces in $\mathbb{S}^3$ (respectively, $\mathbb{H}^3$). An interesting property of such a hypersurface is that its unit normal vector field makes a constant angle with the unit vector field spanning the factor $\mathbb{R}$. All surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ with this property were classified in [8] and [9], where they were called *constant angle surfaces*. Here we give a simple proof of a generalization of this result to constant angle hypersurfaces of arbitrary dimension of both $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

## 2. Preliminaries

Let $\mathbb{Q}^n_\varepsilon$ denote either the sphere $\mathbb{S}^n$ or hyperbolic space $\mathbb{H}^n$, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively. In order to study hypersurfaces $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, our approach is to regard $f$ as an isometric immersion into $\mathbb{E}^{n+2}$, where $\mathbb{E}^{n+2}$ denotes either Euclidean space or Lorentzian space of dimension $(n+2)$, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively. More precisely, let $(x_1, \ldots, x_{n+2})$ be the standard coordinates on $\mathbb{E}^{n+2}$ with respect to which the flat metric is written as

$$ds^2 = \varepsilon dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2.$$

Regard $\mathbb{E}^{n+1}$ as

$$\mathbb{E}^{n+1} = \{(x_1, \ldots, x_{n+2}) \in \mathbb{E}^{n+2} : x_{n+2} = 0\}$$

and

$$\mathbb{Q}^n_\varepsilon = \{(x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n+1} : \varepsilon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = \varepsilon\} \quad (x_1 > 0 \text{ if } \varepsilon = -1).$$

Then we consider the inclusion

$$i : \mathbb{Q}^n_\varepsilon \times \mathbb{R} \to \mathbb{E}^{n+1} \times \mathbb{R} = \mathbb{E}^{n+2}$$

and study the composition $i \circ f$, which we also denote by $f$.

Given a hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, let $N$ denote a unit normal vector field to $f$ and let $\frac{\partial}{\partial t}$ be a unit vector field tangent to the second factor. Then, a vector field $T$ and a smooth function $\nu$ on $M^n$ are defined by

$$\frac{\partial}{\partial t} = f_* T + \nu N.$$

Notice that $T$ is the gradient of the height function $h = \langle f, \frac{\partial}{\partial t} \rangle$.

Two trivial classes of hypersurfaces of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ arise if either $\nu$ or $T$ vanishes identically:

PROPOSITION 2.1. *Let $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a hypersurface.*

(i) *If $T$ vanishes identically, then $f(M^n)$ is an open subset of a slice $\mathbb{Q}^n_\varepsilon \times \{t\}$.*
(ii) *If $\nu$ vanishes identically, then $f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{Q}^n_\varepsilon$.*

Let $\nabla$ and $R$ be the Levi–Civita connection and the curvature tensor of $M^n$, respectively, and let $A$ be the shape operator of $f$ with respect to $N$. Then the Gauss and Codazzi equations are

$$R(X,Y)Z = (AX \wedge AY)Z + \varepsilon\big((X \wedge Y)Z - \langle Y,T\rangle(X \wedge T)Z + \langle X,T\rangle(Y \wedge T)Z\big), \tag{2.1}$$

and

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = \varepsilon\nu(X \wedge Y)T, \tag{2.2}$$

respectively, where $X,Y,Z \in TM$. Moreover, the fact that $\frac{\partial}{\partial t}$ is parallel in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ yields for all $X \in TM$ that

$$\nabla_X T = \nu AX,$$

and

$$X(\nu) = -\langle AX, T\rangle. \tag{2.3}$$

## 3. A basic lemma

Our main goal in this section is to prove the following lemma.

LEMMA 3.1. *Let $f : M^n_c \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a hypersurface of dimension $n \geq 3$ and constant sectional curvature $c \neq 0$. Assume that $T \neq 0$ at $x \in M^n_c$. Then $T$ is a principal direction at $x$.*

Lemma 3.1 will follow by putting together Lemma 3.2 and Proposition 3.3 below:

LEMMA 3.2. *Let $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a hypersurface. Suppose that $T \neq 0$ at $x \in M^n$. Then $f$ has flat normal bundle at $x$ as an isometric immersion into $\mathbb{E}^{n+2}$ if and only if $T$ is a principal direction at $x$.*

PROPOSITION 3.3. *Any isometric immersion $g : M^n_c \to \mathbb{E}^{n+2}$ of a Riemannian manifold with dimension $n \geq 3$ and constant sectional curvature $c \neq 0$ has flat normal bundle.*

Lemma 3.2 was first proved in [7] for $n = 2$ and $\varepsilon = 1$. A proof of the general case can be found in [16]. For the proof of Proposition 3.3, we make use of standard facts from [13] on the theory of flat bilinear forms. Recall that a symmetric bilinear form $\beta : V \times V \to W$, where $V$ and $W$ are finite-dimensional vector spaces, is said to be *flat* with respect to an inner product $\langle\cdot,\cdot\rangle : W \times W \to \mathbb{R}$ if

$$\big\langle \beta(X,Y), \beta(Z,T)\big\rangle - \big\langle \beta(X,T), \beta(Z,Y)\big\rangle = 0$$

for all $X,Y,Z,T \in V$. Clearly, the standard example of a flat bilinear form is the second fundamental form of an isometric immersion between space forms with the same constant sectional curvature.

Denote by $N(\beta) \subset V$ the *nullity subspace* of $\beta$, given by

$$N(\beta) = \{X \in V : \beta(X,Y) = 0 \text{ for all } Y \in V\},$$

and by $S(\beta) \subset W$ its *image subspace*

$$S(\beta) = \text{span}\{\beta(X,Y) : X, Y \in V\}.$$

The next result is a basic fact on flat bilinear forms (cf. Corollary 1 and Corollary 2 in [13]):

THEOREM 3.4. [13] *Let $\beta : V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle \cdot, \cdot \rangle$ on $W$. Assume that $\langle \cdot, \cdot \rangle$ is either positive-definite or Lorentzian and, in the latter case, suppose that $S(\beta)$ is a nongenerate subspace of $W$, that is, $S(\beta) \cap S(\beta)^\perp = \{0\}$. Then*

$$\dim N(\beta) \geq \dim V - \dim S(\beta).$$

Another fact we will need in order to handle the case $n = 3$ in Proposition 3.3 is the following consequence of Theorem 2 in [13].

THEOREM 3.5. [13] *Let $\beta : V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle \cdot, \cdot \rangle$ on $W$. Assume that* $\dim V = \dim W$, *that* $N(\beta) = \{0\}$ *and that $\langle \cdot, \cdot \rangle$ is either positive-definite or Lorentzian. Moreover, in the latter case suppose that there exists a vector $e \in W$ such that $\langle \beta(\cdot,\cdot), e \rangle$ is positive definite. Then there exists a diagonalyzing basis $\{e_1, \ldots, e_n\}$ for $\beta$, that is, $\beta(e_i, e_j) = 0$ for $1 \leq i \neq j \leq n$.*

*Proof of Proposition* 3.3. First, recall that $\mathbb{R}^{n+2}$ admits an umbilical inclusion $i$ into both hyperbolic space $\mathbb{H}^{n+3}_c$ and the Lorentzian sphere $\mathbb{S}^{n+2,1}_c$ of constant sectional curvature $c$, according as $c < 0$ or $c > 0$, respectively, that is, its second fundamental form $\alpha$ is

$$\alpha(X,Y) = \sqrt{|c|}\langle X, Y \rangle \eta,$$

where $\eta$ is one of the two normal vectors such that $\langle \eta, \eta \rangle = -\operatorname{sgn}(c)$, and $\operatorname{sgn}(c) = c/|c|$. Similarly, Lorentzian space $\mathbb{L}^{n+2}$ admits umbilical inclusions into $\mathbb{H}^{n+2,1}_c$ or $\mathbb{S}^{n+1,2}_c$, according as $c < 0$ or $c > 0$, respectively.

Then, the second fundamental form $\alpha_\phi = g^*\alpha + i_*\alpha_g$ of $\phi = i \circ g$ at every $x \in M^n_c$ is a flat bilinear form with respect to the inner product $\langle \cdot, \cdot \rangle$ on its three-dimensional normal space. The inner product $\langle \cdot, \cdot \rangle$ is positive-definite if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$, Lorentzian if either $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$ or if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$, and has index two if $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$. In the latter case, $\alpha_\phi$ is also flat with respect to the Lorentzian inner product $-\langle \cdot, \cdot \rangle$. Moreover, since

$$\langle \alpha_\phi(\cdot,\cdot), \eta \rangle = \langle g^*\alpha(\cdot,\cdot), \eta \rangle = -\operatorname{sgn}(c)\sqrt{|c|}\langle \cdot, \cdot \rangle,$$

it follows that $N(\alpha_\phi) = \{0\}$. Let us consider the two possible cases:

(i) $S(\alpha_\phi)$ *is nondegenerate*: in this case, Theorem 3.4 gives

$$\dim S(\alpha_\phi) \geq n - \dim N(\alpha_\phi) = n.$$

Since $\dim S(\alpha_\phi) \leq 3$, this implies that $n = 3 = \dim S(\alpha_\phi)$. The bilinear form $\langle \alpha_\phi(\cdot,\cdot), -\operatorname{sgn}(c)\eta\rangle$ being positive definite, it follows from Theorem 3.5 that there exists a basis $\{e_1, e_2, e_3\}$ of $T_xM_c^3$ such that $\alpha_\phi(e_i, e_j) = 0$ for $i \neq j$. In particular, we have

$$0 = \big\langle \alpha_\phi(e_i, e_j), \eta\big\rangle = -\operatorname{sgn}(c)\sqrt{|c|}\langle e_i, e_j\rangle \quad \text{for } i \neq j,$$

that is, $\{e_1, e_2, e_3\}$ is an orthogonal basis. Since $\{e_1, e_2, e_3\}$ also diagonalizes $\alpha_g$, we conclude that $g$ has flat normal bundle.

(ii) $S(\alpha_\phi)$ *is degenerate*: in this case, there exists a nonzero vector $\rho \in S(\alpha_\phi) \cap S(\alpha_\phi)^\perp$. Writing $\rho = \eta + i_*\zeta$, with $\zeta$ a unit normal vector to $g$, we obtain from $0 = \langle \alpha_\phi(X, Y), \rho\rangle$ for all $X, Y \in T_xM_c^n$ that

$$\big\langle \alpha_g(X, Y), \zeta\big\rangle = \operatorname{sgn}(c)\sqrt{|c|}\langle X, Y\rangle$$

for all $X, Y \in T_xM_c^n$, that is, $g$ has an umbilical normal direction. Since $g$ has codimension two, the Ricci equation implies that its normal bundle is flat. □

The flat case $c = 0$ can also be handled by means of Theorem 3.4.

Lemma 3.6. *Let $f : M_0^n \to \mathbb{Q}_\varepsilon^n \times \mathbb{R}$ be a flat hypersurface of dimension $n \geq 3$. Assume that $T \neq 0$ at $x \in M_0^n$.*

(i) *If $\varepsilon = 1$, then $n = 3$ and $\nu$ vanishes at $x$.*
(ii) *If $\varepsilon = -1$, then either $\nu$ vanishes at $x$ or $A_N = A_\xi$ for one of the two possible choices of a unit normal vector $N$ to $f$ and the outward pointing unit normal vector $\xi$ to $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$ at $x$.*

*Proof.* Regard $f$ as an isometric immersion into $\mathbb{E}^{n+2}$. Then, its second fundamental form $\alpha$ is a flat bilinear map by the Gauss equation. On the other hand, it is easily seen that the shape operator of $f$ with respect to $\xi$ is given by

$$A_\xi T = -\nu^2 T \quad \text{and} \quad A_\xi X = -X \quad \text{for } X \in \{T\}^\perp.$$

If either $\varepsilon = 1$ or $\varepsilon = -1$ and $S(\alpha)$ is a nondegenerate subspace of the (Lorentzian) two-dimensional normal space of $f$ in $\mathbb{E}^{n+2}$ at $x$, then it follows from Theorem 3.4 that

$$2 \geq \dim S(\alpha) \geq n - \dim N(\alpha) \geq n - \dim\ker A_\xi.$$

Since $\dim\ker A_\xi \leq 1$, and $\dim\ker A_\xi = 1$ only if $\nu = 0$ at $x$, we obtain that $n = 3$ and $\nu = 0$ at $x$.

Now assume that $S(\alpha)$ is degenerate. Then $S(\alpha)$ is spanned by the light-like vector $N - \xi$ for one of the two unit normal vectors $N$ to $f$ in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ at $x$. But the fact that $N - \xi \in S(\alpha)^\perp$ just means that $A_N = A_\xi$. □

## 4. Rotation hypersurfaces

Rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ have been defined and their principal curvatures computed in [6], as an extension of the work in [4] on rotation hypersurfaces of space forms.

With notations as in Section 2, let $P^3$ be a three-dimensional subspace of $\mathbb{E}^{n+2}$ containing the $\frac{\partial}{\partial x_1}$ and the $\frac{\partial}{\partial x_{n+2}}$ directions. Then $(\mathbb{Q}^n_\varepsilon \times \mathbb{R}) \cap P^3 = \mathbb{Q}^1_\varepsilon \times \mathbb{R}$. Denote by $\mathcal{I}$ the group of isometries of $\mathbb{E}^{n+2}$ that fix pointwise a two-dimensional subspace $P^2 \subset P^3$ also containing the $\frac{\partial}{\partial x_{n+2}}$-direction. Consider a curve $\alpha$ in $\mathbb{Q}^1_\varepsilon \times \mathbb{R} \subset P^3$ that lies in one of the two half-spaces of $P^3$ determined by $P^2$.

Definition 4.1. *A* rotation hypersurface in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ with profile curve $\alpha$ and axis $P^2$ *is the orbit of $\alpha$ under the action of $\mathcal{I}$.*

We will always assume that $P^3$ is spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$. In the case $\varepsilon = 1$, we also assume that $P^2$ is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$, and that the curve $\alpha$ is parametrized by arc length as

$$\alpha(s) = \big(\sin\big(k(s)\big), 0, \dots, 0, \cos\big(k(s)\big), h(s)\big),$$

where $s$ runs over an interval $I$ where $\cos(k(s)) \geq 0$, so that $\alpha(I)$ is contained in a closed half-space determined by $P^2$. Here $k, h : I \to \mathbb{R}$ are smooth functions satisfying

$$k'(s)^2 + h'(s)^2 = 1 \quad \text{for all } s \in I. \tag{4.1}$$

In this case, the rotation hypersurface in $\mathbb{S}^n \times \mathbb{R}$ with profile curve $\alpha$ and axis $P^2$ can be parametrized by

$$f(s,t) = \big(\sin\big(k(s)\big), \cos\big(k(s)\big)\varphi_1(t), \dots, \cos\big(k(s)\big)\varphi_n(t), h(s)\big), \tag{4.2}$$

where $t = (t_1, \dots, t_{n-1})$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ parametrizes $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. The metric induced by $f$ is

$$d\sigma^2 = ds^2 + \cos^2\big(k(s)\big)dt^2, \tag{4.3}$$

where $dt^2$ is the standard metric of $\mathbb{S}^{n-1}$.

For $\varepsilon = -1$, one has three distinct possibilities, according as the two-plane $P^2$ is Lorentzian, Riemannian or degenerate, respectively. We call $f$, accordingly, a rotation hypersurface of *spherical*, *hyperbolic* or *parabolic* type, because the orbits of $\mathcal{I}$ are spheres, hyperbolic spaces or horospheres, respectively. In the first case, we can assume that $P^2$ is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$ and that the curve $\alpha$ is parametrized by

$$\alpha(s) = \big(\cosh\big(k(s)\big), 0, \dots, 0, \sinh\big(k(s)\big), h(s)\big). \tag{4.4}$$

Then $f$ can be parametrized by

$$f(s,t) = \big(\cosh\big(k(s)\big), \sinh\big(k(s)\big)\varphi_1(t), \dots, \sinh\big(k(s)\big)\varphi_n(t), h(s)\big). \tag{4.5}$$

The induced metric is

$$d\sigma^2 = ds^2 + \sinh^2\big(k(s)\big)dt^2, \tag{4.6}$$

where $dt^2$ is the standard metric of $\mathbb{S}^{n-1}$.

In the second case, assuming that $P^2$ is spanned by $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$, the curve $\alpha$ can also be parametrized as in (4.4), and a parametrization of $f$ is

$$f(s,t) = \big(\cosh\big(k(s)\big)\varphi_1(t), \dots, \cosh\big(k(s)\big)\varphi_n(t), \sinh\big(k(s)\big), h(s)\big), \tag{4.7}$$

where $t = (t_1, \dots, t_{n-1})$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ parametrizes $\mathbb{H}^{n-1} \subset \mathbb{L}^n$. The induced metric is

$$d\sigma^2 = ds^2 + \cosh^2\big(k(s)\big)dt^2, \tag{4.8}$$

where $dt^2$ is the standard metric of $\mathbb{H}^{n-1}$.

Finally, when $P^2$ is degenerate, we choose a pseudo-orthonormal basis

$$e_1 = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}}\right), \quad e_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}}\right), \quad e_j = \frac{\partial}{\partial x_j}$$

for $j \in \{2, \dots, n, n+2\}$, and assume that $P^2$ is spanned by $e_{n+1}$ and $e_{n+2}$. Notice that $\langle e_1, e_1\rangle = 0 = \langle e_{n+1}, e_{n+1}\rangle$ and $\langle e_1, e_{n+1}\rangle = 1$. Then we can parametrize $\alpha$ by

$$\alpha(s) = \left(k(s), 0, \dots, 0, -\frac{1}{2k(s)}, h(s)\right),$$

with

$$k(s) > 0 \quad \text{and} \quad (\ln k)'^2(s) + h'(s)^2 = 1, \tag{4.9}$$

and a parametrization of $f$ is

$$\begin{aligned} &f(s, t_2, \dots, t_n) \\ &= \left(k(s), k(s)t_2, \dots, k(s)t_n, -\frac{1}{2k(s)} - \frac{k(s)}{2}\sum_{i=2}^{n} t_i^2, h(s)\right). \end{aligned} \tag{4.10}$$

The induced metric is

$$d\sigma^2 = ds^2 + k^2(s)dt^2, \tag{4.11}$$

where $dt^2$ is the standard metric of $\mathbb{R}^{n-1}$.

REMARK 4.2. Our definition of a rotation hypersurface in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ was taken from [6], and naturally extends the one given in [4] for space forms. For $\varepsilon = -1$, it differs from that used in [2], where only rotation surfaces of spherical type were considered.

We are now in a position to classify rotation hypersurfaces of $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ with constant sectional curvature $c$ and dimension $n \geq 3$. We state separately the cases $\varepsilon = 1$ and $\varepsilon = -1$.

THEOREM 4.3. *Let $f : M_c^n \to \mathbb{S}^n \times \mathbb{R}$ be a rotation hypersurface with constant sectional curvature $c$ and dimension $n \geq 3$. Then $c \geq 1$. Moreover,*

(i) *if $c = 1$ then $f(M_c^n)$ is an open subset of a slice $\mathbb{S}^n \times \{t\}$.*
(ii) *If $c > 1$ then $f(M_c^n)$ is an open subset of a complete hypersurface that can be parametrized by (4.2), with*

$$k(s) = \arccos\left(\frac{1}{\sqrt{c}}\sin(\sqrt{c}s)\right) \tag{4.12}$$

*and*

$$h(s) = -\sqrt{\frac{c-1}{c}}\ln\left(\frac{\cos(\sqrt{c}s)+\sqrt{c-\sin^2(\sqrt{c}s)}}{1+\sqrt{c}}\right), \quad s \in [0, \pi/\sqrt{c}]. \tag{4.13}$$

THEOREM 4.4. *Let $f : M_c^n \to \mathbb{H}^n \times \mathbb{R}$ be a rotation hypersurface with constant sectional curvature $c$ and dimension $n \geq 3$. Then $c \geq -1$. Moreover,*

(i) *if $c = -1$ then $f(M^n)$ is an open subset of a slice $\mathbb{H}^n \times \{t\}$.*
(ii) *If $c \in (-1, 0)$ then one of the following possibilities holds:*
   (a) *$f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (4.5), with*

$$k(s) = \operatorname{arcsinh}\left(\frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}s)\right) \tag{4.14}$$

*and*

$$h(s) = \sqrt{\frac{c+1}{-c}}\ln\left(\frac{\cosh(\sqrt{-c}s)+\sqrt{-c+\sinh^2(\sqrt{-c}s)}}{1+\sqrt{-c}}\right). \tag{4.15}$$

   (b) *$f(M^n)$ is an open subset of a complete hypersurface of hyperbolical type that can be parametrized by (4.7), with*

$$k(s) = \operatorname{arccosh}\frac{1}{\sqrt{-c}}\cosh(\sqrt{-c}s) \tag{4.16}$$

*and*

$$h(s) = \sqrt{\frac{c+1}{-c}}\ln\big(\sinh(\sqrt{-c}s)+\sqrt{c+\cosh^2(\sqrt{-c}s)}\big). \tag{4.17}$$

   (c) *$f(M^n)$ is an open subset of a complete hypersurface of parabolical type that can be parametrized by (4.10), with*

$$k(s) = \exp\sqrt{-c}s \tag{4.18}$$

*and*

$$h(s) = \sqrt{1+c}s. \tag{4.19}$$

(iii) *If $c = 0$, then one of the following possibilities holds:*

(a) *$f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (4.5), with*

$$k(s) = \operatorname{arcsinh}(s) \tag{4.20}$$

*and*

$$h(s) = -1 + \sqrt{1+s^2}. \tag{4.21}$$

(b) *$f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a horosphere of $\mathbb{H}^n$.*

(iv) *If $c > 0$, then $f(M^n)$ is an open subset of a complete hypersurface of spherical type that can be parametrized by (4.5), with*

$$k(s) = \operatorname{arcsinh}\left(\frac{1}{\sqrt{c}}\sin(\sqrt{c}s)\right) \tag{4.22}$$

*and*

$$h(s) = -\sqrt{\frac{c+1}{c}}\arctan\left(\frac{\cos(\sqrt{c}s)}{\sqrt{c+\sin^2(\sqrt{c}s)}}\right). \tag{4.23}$$

REMARK 4.5. The hypersurfaces in Theorems 4.3 and 4.4 also occur in dimension $n = 2$. In particular, those in parts (ii)(b) and (ii)(c) of Theorem 4.4 provide examples of complete surfaces of constant Gaussian curvature $c \in (-1, 0)$ in $\mathbb{H}^2 \times \mathbb{R}$ that do not appear in [2].

For the proofs of Theorems 4.3 and 4.4, we make use of the following fact.

PROPOSITION 4.6. *Assume that the warped product $I \times_\rho \mathbb{Q}^n_\delta$, $n \geq 2$, $\delta \in \{-1, 0, 1\}$, has constant sectional curvature $c$.*

(i) *If $c > 0$, then $\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{c}}\sin(\sqrt{c}s + \theta_0), \theta_0 \in \mathbb{R}$.*

(ii) *If $c = 0$, then one of the following possibilities holds:*
   (a) *$\delta = 1$ and $\rho(s) = \pm s + s_0, s_0 \in \mathbb{R}$.*
   (b) *$\delta = 0$ and $\rho(s) = A \in \mathbb{R}$.*

(iii) *If $c < 0$, then one of the following possibilities holds:*
   (a) *$\delta = -1$ and $\rho(s) = \frac{1}{\sqrt{-c}}\cosh(\sqrt{-c}s + \theta_0), \theta_0 \in \mathbb{R}$.*
   (b) *$\delta = 0$ and $\rho(s) = \exp(\pm\sqrt{-c}s + s_0), s_0 \in \mathbb{R}$.*
   (c) *$\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}s + \theta_0), \theta_0 \in \mathbb{R}$.*

*Proof.* In a warped product $I \times_\rho \mathbb{Q}^n_\delta$, $n \geq 2$, the sectional curvature along a plane tangent to $\mathbb{Q}^n_\delta$ is $(\delta - (\rho')^2)/\rho^2$, whereas the sectional curvature along a plane spanned by unit vectors $\partial/\partial s$ and $X$ tangent to $I$ and $\mathbb{Q}^n_\delta$, respectively, is $-\rho''/\rho$. Therefore, $I \times_\rho \mathbb{Q}^n_\delta$ has constant sectional curvature $c$ if and only if

$$\left(\rho'\right)^2 + c\rho^2 = \delta. \tag{4.24}$$

Notice that $-\rho''/\rho = c$, or equivalently,

$$\rho'' + c\rho = 0, \tag{4.25}$$

follows by differentiating (4.24). If $c>0$, we obtain from (4.24) that $\delta=1$. Moreover, by (4.25) we have that

$$\rho(s)=A\cos\sqrt{c}s+B\sin\sqrt{c}s$$

for some $A,B\in\mathbb{R}$, which gives $(\rho')^2+c\rho^2=c(A^2+B^2)$. From (4.24) we get $c(A^2+B^2)=1$, hence we may write

$$A=\frac{1}{\sqrt{c}}\sin\theta_0 \quad \text{and} \quad B=\frac{1}{\sqrt{c}}\cos\theta_0$$

for some $\theta_0\in\mathbb{R}$. It follows that

$$\rho(s)=\frac{1}{\sqrt{c}}\sin(\sqrt{c}s+\theta_0).$$

The remaining cases are similar. □

*Proof of Theorems 4.3 and 4.4.* First, we determine the possible values of $c$ for a rotation hypersurface $f: M_c^n\to\mathbb{Q}_\varepsilon^n\times\mathbb{R}$ with constant sectional curvature $c$ and dimension $n\geq 3$. If $T$ vanishes on an open subset, then $c=\varepsilon$ by Proposition 2.1. Otherwise, we can assume that $T$ is nowhere vanishing. Then $f$ has two principal curvatures $\lambda$ and $\mu\neq 0$, the first one with $T$ as principal direction (cf. [6]). Let $\{T,X_1,\ldots,X_{n-1}\}$ be an orthogonal basis of eigenvectors of $A$ at $x$, with

$$AT=\lambda T \quad \text{and} \quad AX_i=\mu X_i, \quad 1\leq i\leq n-1.$$

From the Gauss equation (2.1) of $f$ for $X=X_i$ and $Y=Z=X_j$, $i\neq j$, we get

$$c-\varepsilon=\mu^2,$$

and hence $c>\varepsilon$. This proves the first assertions in Theorems 4.3 and 4.4.

Now assume that $\varepsilon=1$. Then $f$ can be parametrized by (4.2), with $k(s)$ and $h(s)$ satisfying (4.1), and the metric induced by $f$ is given by (4.3). Since $c\geq 1$, by Proposition 4.6 we must have

$$\cos\big(k(s)\big)=\frac{1}{\sqrt{c}}\sin(\sqrt{c}s+\theta_0)$$

for some $\theta_0\in\mathbb{R}$. Replacing $s$ by $s-\theta_0/\sqrt{c}$, we can assume that $\theta_0=0$. If $c=1$, then $f$ just parametrizes an open subset of a slice $\mathbb{S}^n\times\{t\}$. If $c>1$, we obtain that $k(s)$ and $h(s)$ are given by (4.12) and (4.13), respectively. The corresponding profile curve is exactly that of the complete surface of constant sectional curvature $c$ in $\mathbb{S}^2\times\mathbb{R}$ determined in [2], and their argument also applies to show the completeness of $f$ in any dimension $n\geq 3$.

From now on, we deal with the case $\varepsilon=-1$. Assume first that $f$ is of spherical type. Then $f$ can be parametrized by (4.5), with $k(s)$ and $h(s)$ satisfying

(4.1), and the metric induced by $f$ is given by (4.6). By Proposition 4.6, the warping function $\sinh(k(s))$ must be equal to

$$\frac{1}{\sqrt{c}}\sin(\sqrt{c}s+\theta_0), \qquad \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}s+\theta_0), \quad \theta_0\in\mathbb{R}, \quad \text{or}$$
$$\pm s+s_0, \quad s_0\in\mathbb{R},$$

according as $c>0$, $c<0$ or $c=0$, respectively. After suitably replacing the parameter $s$, we can assume that $\theta_0=0$ in the first two cases, and that $\sinh(k(s))=s$ in the last one. Each possibility gives rise to the expressions (4.14), (4.22) and (4.20) for $k(s)$, and (4.15), (4.23) and (4.21) for $h(s)$, respectively. The corresponding profile curves are exactly those of the complete rotation surfaces with constant sectional curvature of spherical type determined in [2], and the completeness of the corresponding hypersurfaces can be seen in the same way as in [2].

Now suppose that $f$ is of hyperbolical type. Then, it can be parametrized by (4.7), with $k(s)$ and $h(s)$ satisfying (4.1), and the induced metric is (4.8). Since $c\geq -1$, by Proposition 4.6 we must have $c\in[-1,0)$ and

$$\cosh\big(k(s)\big)=\frac{1}{\sqrt{-c}}\cosh(\sqrt{-c}s+\theta_0), \quad \theta_0\in\mathbb{R}.$$

As before, we can assume that $\theta_0=0$. If $c=-1$, then $f(M^n)$ is an open subset of a slice $\mathbb{H}^n\times\{t\}$. Otherwise, $k$ and $h$ are given by (4.16) and (4.17), respectively.

Finally, suppose that $f$ is of parabolical type. Then, it can be parametrized by (4.10), with $k(s)$ and $h(s)$ satisfying (4.9), and the induced metric is (4.11). By Proposition 4.6, we must have $c\leq 0$ and

$$k(s)=A\in\mathbb{R} \quad \text{or} \quad k(s)=\exp(\pm\sqrt{-c}s+s_0), \quad s_0\in\mathbb{R},$$

according as $c=0$ or $c<0$, respectively. In the first case, $f$ just parametrizes an open subset of a Riemannian product $M^{n-1}\times\mathbb{R}$, where $M^{n-1}$ is a horosphere of $\mathbb{H}^n$. In the second case, we can assume that $k$ is given by (4.18), and then $h$ is as in (4.19). Completeness of the hypersurfaces in this and the preceding case is straightforward. □

## 5. Constant angle hypersurfaces

Let $g: M^{n-1}\to\mathbb{Q}^n_\varepsilon$ be a hypersurface and let $g_s: M^{n-1}\to\mathbb{Q}^n_\varepsilon$ be the family of parallel hypersurfaces to $g$, that is,

$$g_s(x)=C_\varepsilon(s)g(x)+S_\varepsilon(s)N(x),$$

where $N$ is a unit normal vector field to $g$,

$$C_\varepsilon(s)=\begin{cases}\cos s, & \text{if } \varepsilon=1,\\ \cosh s, & \text{if } \varepsilon=-1\end{cases} \quad \text{and} \quad S_\varepsilon(s)=\begin{cases}\sin s, & \text{if } \varepsilon=1,\\ \sinh s, & \text{if } \varepsilon=-1.\end{cases}$$

For $\varepsilon = 1$, write the principal curvatures of $g$ as

$$\lambda_i = \cot\theta_i, \quad 0 < \theta_i < \pi, 1 \le i \le m,$$

where the $\theta_i$ form an increasing sequence. For $X$ in the eigenspace of the shape operator $A_N$ of $g$ corresponding to the principal curvature $\lambda_i$, $1 \le i \le m$, we have

$$g_{s*}X = g_*(\cos sX - \sin sA_N X) = (\cos s - \sin s\cot\theta_i)X = \frac{\sin(\theta_i - s)}{\sin\theta_i}X.$$

Thus, $g_s$ is an immersion at $x$ if and only if $s \ne \theta_i(x) (\mathrm{mod}\, \pi)$ for $1 \le i \le m$.

For $\varepsilon = -1$, write the principal curvatures of $g$ with absolute value greater than 1 as

$$\lambda_i = \coth\theta_i, \quad \theta_i \ne 0, 1 \le i \le m.$$

As in the preceding case, for $X$ in the eigenspace of the shape operator $A_N$ corresponding to the principal curvature $\lambda_i$, $1 \le i \le m$, we have

$$g_{s*}X = \frac{\sinh(\theta_i - s)}{\sinh\theta_i}X.$$

Thus, $g_s$ is an immersion at $x$ if and only if $s \ne \theta_i(x)$ for any $1 \le i \le m$.

In the case $\varepsilon = 1$, set

$$U := \big\{(x,s) \in M^{n-1} \times \mathbb{R} : s \in \big(\theta_m(x) - \pi, \theta_1(x)\big)\big\}.$$

For $\varepsilon = -1$, let $\theta_+$ (respectively, $\theta_-$) be the least (respectively, greater) of the $\theta_i$ that is greater than 1 (respectively, less than $-1$), and set

$$U := \big\{(x,s) \in M^{n-1} \times \mathbb{R} : s \in \big(\theta_-(x), \theta_+(x)\big)\big\}.$$

In both cases, if $V \subset M^{n-1}$ is an open subset and $I$ is an open interval containing 0 such that $V \times I \subset U$, then $g_s$ is an immersion on $V$ for every $s \in I$, with

$$N_s(x) = -\varepsilon S_\varepsilon(s)g(x) + C_\varepsilon(s)N(x) \tag{5.1}$$

as a unit normal vector at $x$.

Now define

$$f : M^n := V \times I \to \mathbb{Q}^n_\varepsilon \times \mathbb{R} \subset \mathbb{E}^{n+2}$$

by

$$f(x,s) = g_s(x) + Bs\frac{\partial}{\partial t}, \quad B > 0. \tag{5.2}$$

Then

$$f_*X = g_{s*}X \quad \text{for any } X \in TV,$$

and

$$f_*\frac{\partial}{\partial s} = N_s + B\frac{\partial}{\partial t}.$$

Since $g_s$ is an immersion on $V$ for every $s \in I$, it follows that $f$ is an immersion on $M^n$ with

$$\eta(x,s) = -\frac{B}{a} N_s(x) + \frac{1}{a}\frac{\partial}{\partial t}, \quad a = \sqrt{1+B^2}, \tag{5.3}$$

as a unit normal vector field. Thus, $f$ has the property that

$$\left\langle \eta, \frac{\partial}{\partial t} \right\rangle = \frac{1}{a}$$

is constant on $M^n$. Following [8], $f$ was called in [16] a *constant angle hypersurface*. Constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been classified in [8] and [9], respectively. The next result was obtained in [16] as a consequence of a more general theorem. For the sake of completeness, we provide here a simple and direct proof.

THEOREM 5.1. *Any constant angle hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is either an open subset of a slice $\mathbb{Q}^n_\varepsilon \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{Q}^n_\varepsilon$, or it is locally given by the preceding construction.*

*Proof.* Let $\eta$ be a unit normal vector field to $f$. By assumption, $\nu = \langle \eta, \partial/\partial t\rangle$ is constant on $M^n$, which we can assume to belong to $[0,1]$. Since $\|T\|^2 + \nu^2 = 1$, the vector field $T$ has also constant length. By Proposition 2.1, the cases $\nu = 1$ and $\nu = 0$ correspond to the first two possibilities in the statement, respectively. From now on, we assume that $\nu \in (0,1)$, hence $T$ is a vector field whose length is also a constant in $(0,1)$. Since $T$ is a gradient vector field, its integral curves are (not unit-speed) geodesics in $M^n$. The fact that $T$ is a gradient also implies that the orthogonal distribution $\{T\}^\perp$ is integrable. Thus, there exists locally a diffeomorphism $\psi : M^{n-1} \times I \to M^n$, where $I$ is an open interval containing 0, such that $\psi(x,\cdot) : I \to M^n$ are integral curves of $T$ and $\psi(\cdot,s) : M^{n-1} \to M^n$ are integral manifolds of $\{T\}^\perp$. Set $F = f \circ \psi$, with $f$ being regarded as an isometric immersion into $\mathbb{E}^{n+2}$. Then

$$X\left\langle F, \frac{\partial}{\partial t}\right\rangle = \left\langle f_*\psi_* X, \frac{\partial}{\partial t}\right\rangle = \langle \psi_* X, T\rangle = 0$$

for any $X \in TM^{n-1}$. Thus $\langle F(x,s), \frac{\partial}{\partial t}\rangle = \rho(s)$ for some smooth function $\rho$ on $I$.

On the other hand, since $\nu$ is constant, it follows from (2.3) that

$$0 = d\nu(X) = -\langle AX, T\rangle \quad \text{for all } X \in TM^n,$$

hence $AT = 0$. Thus $F(x,\cdot) : I \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ are geodesics in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, where $F = f \circ \psi$. Therefore, the projections $\Pi_1 \circ F(x,\cdot) : I \to \mathbb{Q}^n_\varepsilon$ and $\Pi_2 \circ F(x,\cdot) : I \to \mathbb{R}$ are geodesics of $\mathbb{Q}^n_\varepsilon$ and $\mathbb{R}$, respectively.

That $\Pi_2 \circ F(x,\cdot) : I \to \mathbb{R}$ are geodesics in $\mathbb{R}$ just means that $\rho(s) = Bs$, for some constant $B > 0$, after possibly a translation in the parameter $s$ and changing $s$ by $-s$. Now define $g : M^{n-1} \to \mathbb{Q}^n_\varepsilon$ by

$$g(x) = \Pi_1 \circ F(x, 0).$$

Rescaling the parameter $s$ so that the geodesics $\Pi_1 \circ F(x,\cdot) : I \to \mathbb{Q}^n_\varepsilon$ have unit speed, the fact that they are normal to $g$ at $g(x)$ for any $x \in M^{n-1}$ just says that

$$\Pi_1 \circ F(x, s) = g_s(x),$$

where $g_s$ denotes the parallel hypersurface to $g$ at a distance $s$. □

REMARK 5.2. The proof of Theorem 5.1 also applies to hypersurfaces of $\mathbb{R}^{n+1}$ whose unit normal vector field makes a constant angle with a fixed direction $\partial/\partial t$. Namely, writing $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, with the second factor being spanned by $\partial/\partial t$, it shows that any such hypersurface is either an open subset of an affine subspace $\mathbb{R}^n \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{R}^n$, or it is locally given by (5.2), where $g_s$ is the family of parallel hypersurfaces to some hypersurface $g$ in the first factor $\mathbb{R}^n$, namely, $g_s(x) = g(x) + sN(x)$ for a unit vector field $N$ to $g$. A proof of this fact for surfaces in $\mathbb{R}^3$ was given in [14].

## 6. Nonrotational examples in dimension three

Here we use the construction of the previous section to produce a family of nonrotational hypersurfaces of $\mathbb{S}^3 \times \mathbb{R}$ (respectively, $\mathbb{H}^3 \times \mathbb{R}$) with constant sectional curvature $c$ for any $c \in (0, 1)$ (respectively, $c \in (-1, 0)$).

Given a hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\varepsilon$ and the family $g_s : M^{n-1} \to \mathbb{Q}^n_\varepsilon$ of parallel hypersurfaces to $g$, an easy computation shows that, whenever $\cot_\varepsilon s := C_\varepsilon(s)/S_\varepsilon(s)$ is not a principal curvature of $g$ at any $x \in M^{n-1}$, the shape operator $A_s$ of $g_s$ with respect to the unit normal vector field $N_s$ given by (5.1) is

$$A_s = (\cot_\varepsilon s I - A)^{-1}(\cot_\varepsilon s A + \varepsilon I). \tag{6.1}$$

Let $g : M^2 \to \mathbb{Q}^3_\varepsilon$ be a surface and let

$$f : M^3 := V \times I \subset M^2 \times \mathbb{R} \to \mathbb{Q}^3_\varepsilon \times \mathbb{R} \subset \mathbb{E}^5$$

be defined as in the previous section in terms of $g$. The normal space of $f$, as a submanifold of $\mathbb{E}^5$, is spanned by the unit normal vector field $\eta$ given by (5.3) and by the unit normal vector field $\xi(x, s) = g_s(x)$, which is normal to $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ at $f(x, s)$. We have

$$a\tilde{\nabla}_X \eta = B g_{s*} A^s X = B f_* A^s X$$

and

$$a\tilde{\nabla}_{\frac{\partial}{\partial s}} \eta = \varepsilon B g_s = \varepsilon B \xi,$$

hence the principal curvatures of $A^f_\eta$ are

$$-\frac{B}{a}k_1^s, \qquad -\frac{B}{a}k_2^s \quad \text{and} \quad 0,$$

where $k_1^s$ and $k_2^s$ are the principal curvatures of $g_s$, the principal curvature 0 corresponding to the principal direction $\partial/\partial s$. On the other hand,

$$\tilde{\nabla}_X \xi = g_{s*}X = f_*X$$

and

$$\tilde{\nabla}_{\frac{\partial}{\partial s}} \xi = N_s = \frac{1}{a^2} f_* \frac{\partial}{\partial s} - \frac{B}{a}\eta.$$

Thus, the principal curvatures of $A^f_\xi$ are $-1/a^2$ and $-1$, the first being simple with $\partial/\partial s$ as principal direction, and the second having multiplicity two with $TV$ as eigenbundle.

Now assume that $M^2 = M^2_0$ is flat. Then, the principal curvatures $k_1$ and $k_2$ of $g$ satisfy $k_1k_2 = -\varepsilon$ everywhere. By (6.1), the principal curvatures of $g_s$ with respect to $N_s$ are

$$k_i^s = \frac{\cot_\varepsilon s k_i + \varepsilon}{\cot_\varepsilon s - k_i}, \quad 1 \le i \le 2,$$

hence $k_1^s k_2^s = -\varepsilon$, that is, $g_s$ is also a flat surface. It follows that the sectional curvature of $M^3$ along $TV$ is

$$\left(-\frac{B}{a}k_1^s\right)\left(-\frac{B}{a}k_2^s\right) + \varepsilon = \frac{\varepsilon}{a^2},$$

which is also the sectional curvature of $M^3$ along any plane spanned by $\partial/\partial s$ and a vector $X \in TV$.

REMARK 6.1. It is easily seen that if the hypersurface $f$ just constructed is regarded as a submanifold of $\mathbb{R}^5$ for $\varepsilon = 1$, then it does not have any umbilical normal direction at any point. Hence, it provides a new example of a constant curvature submanifold of $\mathbb{R}^5$ with codimension two that is free of weak-umbilic points in the sense of [13].

EXAMPLE 6.2. As an explicit example, consider the Clifford torus

$$g : M^2_0 := \mathbb{S}^1(\cos\theta_0) \times \mathbb{S}^1(\sin\theta_0) \to \mathbb{S}^3$$

parametrized by

$$g(t_1, t_2) = (\cos\theta_0 \cos t_1, \cos\theta_0 \sin t_1, \sin\theta_0 \cos t_2, \sin\theta_0 \sin t_2),$$

which has

$$N(t_1, t_2) = (-\sin\theta_0 \cos t_1, -\sin\theta_0 \sin t_1, \cos\theta_0 \cos t_2, \cos\theta_0 \sin t_2)$$

as a unit normal vector field in $\mathbb{S}^3$. Then,

$$f : M^2_0 \times \mathbb{R} \to \mathbb{S}^3$$

given by (5.2) can be reparametrized by

$$f(t_1,t_2,s)=(\cos s\cos t_1,\cos s\sin t_1,\sin s\cos t_2,\sin s\sin t_2,Bs),$$

after replacing $s+\theta_0$ by $s$ and a translation in the $\partial/\partial t$-direction. This hypersurface appears in [5] as an example of a weak-umbilic free doubly-rotation surface with constant sectional curvature having the helix $s\mapsto(\cos s,\sin s,Bs)$ as profile, in the sense of [10].

A similar example can be constructed in $\mathbb{H}^3\times\mathbb{R}$, starting with the flat surface

$$g: M_0^2:=\mathbb{H}^1(\cosh\theta_0)\times\mathbb{S}^1(\sinh\theta_0)\to\mathbb{H}^3$$

parametrized by

$$g(t_1,t_2)=(\cosh\theta_0\cos t_1,\cosh\theta_0\sin t_1,\sinh\theta_0\cos t_2,\sinh\theta_0\sin t_2).$$

In this case, the corresponding constant curvature hypersurface of $\mathbb{H}^3\times\mathbb{R}$ is

$$f(t_1,t_2,s)=(\cosh s\cos t_1,\cosh s\sin t_1,\sinh s\cos t_2,\sinh s\sin t_2,Bs).$$

These examples can be characterized as the only constant curvature hypersurfaces of $\mathbb{Q}_\varepsilon^3\times\mathbb{R}$ with three distinct principal curvatures and 0 as principal curvature in the $T$-direction and whose two remaining principal curvatures are constant along $\{T\}^\perp$.

## 7. The main result

In this section, we prove our main result, namely, we provide a complete classification of all hypersurfaces with constant sectional curvature of $\mathbb{Q}_\varepsilon^n\times\mathbb{R}$, $n\geq 3$. We state separately the cases $\varepsilon=1$ and $\varepsilon=-1$. For $\varepsilon=1$, we have the following theorem.

THEOREM 7.1. *Let $f: M_c^n\to\mathbb{S}^n\times\mathbb{R}$, $n\geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature $c$. Then $c\geq 0$. Moreover,*

- (i) *if $c=0$, then $n=3$ and $f(M_0^3)$ is an open subset of a Riemannian product $M_0^2\times\mathbb{R}$, where $M_0^2$ is a flat surface of $\mathbb{S}^3$.*
- (ii) *If $c\in(0,1)$, then $n=3$ and $f$ is locally given by the construction described in Section 6.*
- (iii) *If $c=1$, then $f(M_1^n)$ is an open subset of a slice $\mathbb{S}^n\times\{t\}$.*
- (iv) *If $c>1$, then $f(M_c^n)$ is an open subset of a rotation hypersurface given by Theorem 4.3*(ii).

The classification of constant curvature hypersurfaces of $\mathbb{H}^n\times\mathbb{R}$ with dimension $n\geq 3$ reads as follows.

THEOREM 7.2. *Let $f: M_c^n\to\mathbb{H}^n\times\mathbb{R}$, $n\geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature $c$. Then $c\geq -1$. Moreover,*

(i) *if* $c = -1$, *then* $f(M^n_{-1})$ *is an open subset of a slice* $\mathbb{H}^n \times \{t\}$.
(ii) *If* $c \in (-1, 0)$, *then either* $n = 3$ *and* $f$ *is locally given by the construction described in Section* 6, *or* $f(M^n_0)$ *is an open subset of one of the rotation hypersurfaces given by Theorem* 4.4(ii).
(iii) *If* $c = 0$, *then one of the following possibilities holds*:
  (a) $n = 3$ *and* $f(M^3_0)$ *is an open subset of a Riemannian product* $M^2_0 \times \mathbb{R}$, *where* $M^2_0$ *is a flat surface of* $\mathbb{H}^3$.
  (b) $f(M^n_0)$ *is an open subset of a Riemannian product* $M^{n-1}_0 \times \mathbb{R}$, *where* $M^{n-1}_0$ *is a horosphere of* $\mathbb{H}^n$.
  (c) $f(M^n_0)$ *is an open subset of the spherical rotation hypersurface given by Theorem* 4.4(iii)(a).
(iv) *If* $c > 0$, *then* $f(M^n_c)$ *is an open subset of the spherical rotation hypersurface given by Theorem* 4.4(iv).

*Proof of Theorems* 7.1 *and* 7.2. Assume $c \neq 0$ and that the vector field $T$ does not vanish at $x \in M^n$. Then $T$ is a principal direction of $f$ at $x$ by Lemma 3.1. Let $\{T, X_1, \ldots, X_{n-1}\}$ be an orthogonal basis of eigenvectors of $A_N$ at $x$, with

$$A_N T = \lambda T \quad \text{and} \quad A_N X_i = \lambda_i X_i, \quad 1 \leq i \leq n-1.$$

From the Gauss equation (2.1) of $f$ for $X = X_i$ and $Y = Z = X_j$, $i \neq j$, we get

$$c - \varepsilon = \lambda_i \lambda_j, \quad i \neq j. \tag{7.1}$$

On the other hand, for $X = T$ and $Y = Z = X_i$ the Gauss equation yields

$$c - \varepsilon = \lambda \lambda_i - \varepsilon \|T\|^2. \tag{7.2}$$

Assume first that $c = \varepsilon$. By (7.1), we can assume that $\lambda_i = 0$ for all $2 \leq i \leq n-1$. Then, applying (7.2) for $i \geq 2$ yields a contradiction with $T \neq 0$. We conclude that for $c = \varepsilon$ the vector field $T$ vanishes identically, and this gives part (iii) of Theorem 7.1 and part (i) of Theorem 7.2.

Now suppose that $c \neq \varepsilon$. Then $T$ cannot vanish on any open subset. Thus, we can assume without loss of generality that it is nowhere vanishing. If $n \geq 4$, we obtain from (7.1) that all $\lambda_i's$ coincide for $2 \leq i \leq n-1$. Denote all of them by $\mu$. Then, the Gauss equations now read

$$c - \varepsilon = \mu^2 \tag{7.3}$$

and

$$c - \varepsilon = \lambda \mu - \varepsilon \|T\|^2, \tag{7.4}$$

which can also be written as

$$c = \lambda \mu + \varepsilon \nu^2. \tag{7.5}$$

In particular, it follows from (7.3) that $c > \varepsilon$.

Now, since $T \neq 0$, it follows from (7.3) and (7.4) that $\lambda \neq \mu$. Moreover, since $T$ is a principal direction, we obtain from (2.3) that $\nu$ is constant along the leaves of $\{T\}^{\perp}$, and hence the same holds for $\lambda$ by (7.5) (since $\mu$ has multiplicity greater than one, one can show using the Codazzi equation (2.2) that it is constant along its eigenbundle; cf. the proof of Theorem 1 in [6]). Then, one can use the following result to conclude that $f$ is a rotation hypersurface. It slightly generalizes Theorem 1 in [6], but actually follows from its proof.

Proposition 7.3. *Let $f : M^n \to \mathbb{Q}_\varepsilon^n \times \mathbb{R}$ be a hypersurface with $n \geq 3$ and $T \neq 0$. Assume that $f$ has exactly two principal curvatures $\lambda$ and $\mu$ everywhere, the first one being simple with $T$ as a principal direction. If $\lambda$ is constant along the leaves of the eigenbundle $\{T\}^{\perp}$ of $\mu$, then $f(M^n)$ is an open subset of a rotation hypersurface.*

Thus, the proofs of Theorems 7.1 and 7.2 for $c \neq 0$ and $n \geq 4$ are completed by Theorems 4.3 and 4.4. This also applies to the case $n = 3$ when we have $\lambda_2 = \lambda_3$ everywhere. By (7.1) and (7.2), this is not the case only if $\lambda = 0$. In this situation, equation (7.5) reduces to

$$\varepsilon\nu^2 = c.$$

Hence, $f$ is a constant angle hypersurface. Therefore, by Theorem 5.1 it is locally given by (5.2) for some surface $g : M^2 \to \mathbb{Q}_\varepsilon^3$. Moreover, if we write $\nu = 1/a$, it was shown in Section 6 that the principal curvatures of $f$ are

$$-\frac{B}{a}k_1^s, \qquad -\frac{B}{a}k_2^s \quad \text{and} \quad 0,$$

where $k_1^s$ and $k_2^s$ are the principal curvatures of $g_s$. By the Gauss equation (7.1), we have

$$c - \varepsilon = \left(-\frac{B}{a}k_1^s\right)\left(-\frac{B}{a}k_2^s\right).$$

Replacing $c = \varepsilon/a^2$ and using that $B^2 + 1 = a^2$, it follows that $k_1^s k_2^s = -\varepsilon$, hence $g$ is a flat surface.

Finally, if $c = 0$ then Lemma 3.6 already gives the assertion in Theorem 7.1(i) if $\varepsilon = 1$. For $\varepsilon = -1$, it implies that either $\nu$ vanishes or $f$ has exactly two distinct principal curvatures, one of them simple with $T$ as principal direction. The first possibility corresponds to the two first cases in Theorem 7.2(iii). In the second one, we conclude as before that $f$ is a rotation hypersurface, and the proof is completed by Theorem 4.4. □

Remark 7.4. In [16], a complete classification of all hypersurfaces of $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ that have $T$ as a principal direction was obtained. As a consequence, it was shown that in Proposition 7.3 above the assumption that $\lambda$ is constant along $\{T\}^{\perp}$ is automatically satisfied. Apart from this observation, however, using that classification would apparently not simplify the proofs of Theorems 7.1 and 7.2.

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Fernando Manfio, ICMC–USP, Av. Trab. São-carlense, 400, 13560-970, São Carlos, Brazil

*E-mail address*: manfio@icmc.usp.br

Ruy Tojeiro, Universidade Federal de São Carlos, 13565-905, São Carlos, Brazil

*E-mail address*: tojeiro@dm.ufscar.br