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THE WEAK LEFSCHETZ PROPERTY, MONOMIAL IDEALS, AND LOZENGES

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ABSTRACT. We study the weak Lefschetz property and the Hilbert function of level Artinian monomial almost complete intersections in three variables. Several such families are shown to have the weak Lefschetz property if the characteristic of the base field is zero or greater than the maximal degree of any minimal generator of the ideal. Two of the families have an interesting relation to tilings of hexagons by lozenges. This lends further evidence to a conjecture by Migliore, Miró-Roig, and the second author. Finally, using our results about the weak Lefschetz property, we show that the Hilbert function of each level Artinian monomial almost complete intersection in three variables is peaked strictly unimodal.

1. Introduction

Let A be a standard graded Artinian algebra over a field K. Then A has the *weak Lefschetz property* if there is a linear form $\ell \in A$ such that, for all integers d, the multiplication map

$$\times\ell : [A]_d \to [A]_{d+1}$$

has maximal rank, that is, it is surjective or injective. In this case, the linear form ℓ is called a *Lefschetz element* of A.

This property is of interest mainly because it constrains the Hilbert function as shown in [7], which in turn has interesting consequences (see, e.g., [10] for a spectacular application). Furthermore, it is a difficult task to classify which rings do (and do not) have the weak Lefschetz property. For example, in [7] it was shown that all height three complete intersections over a field of

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characteristic zero have the weak Lefschetz property, but this is still unknown if we consider height four complete intersections.

In this note, we further explore level Artinian monomial almost complete intersections in three variables, as discussed in [2], [3], and more extensively in [9]. Even in this restricted setting, it is still unclassified which rings have the weak Lefschetz property. However, in [9] a conjectural solution is put forth, restated here along with known partial results in Section 2.

Several parts of this conjecture have been established in [9]. Here we resolve some of the open cases, thus lending further evidence to the conjecture. In Section 3, we consider three rather straightforward cases, where three of the four parameters are equal. In Section 4, we consider the two cases where a parameter is extremal.

The key to these results is the computation of a certain determinant which was shown to play a crucial role in [9]. Interestingly, the computation of the determinant in the two extremal cases reveals a connection to combinatorial objects, namely to tilings of hexagons by lozenges.

While the conjecture in [9] is for algebras over fields of characteristic zero only, our computation of the determinants allows us also to establish the weak Lefschetz property also over fields of sufficiently large characteristic. In fact, we give an effective lower bound on the characteristic in each case. However, in Remark 4.8 we notice that in general the maximal degree of the minimal generators gives no indication of a such bound on the characteristic.

Last, in Section 5 we show, using also our results from Section 4, that *every* level Artinian monomial almost complete intersection R/I has a *peaked strictly unimodal* Hilbert function; that is, if h is the Hilbert function of R/I, then

$$h(0) < \cdots < h(s) = \cdots = h(s+t-1) > h(s+t) > \cdots > h(e),$$

where $s, \ldots, s+t-1$ are the peak degrees and e is the socle degree of R/I. This result in turn gives a partial answer to Question 8.2(1) from [9]. It shows that for these algebras the knowledge of the Hilbert function does not allow one to decide whether the algebra has the weak Lefschetz property or not.

2. A conjecture

Throughout this note, we assume K is an arbitrary field unless otherwise specified.

We consider level Artinian monomial almost complete intersections in $R = K[x, y, z]$. These are precisely the ideals of the form

$$I = \left(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma\right), \tag{2.1}$$

where $0 < t$ and, after a change of variables, $0 \le \alpha \le \beta \le \gamma$, as shown in Section 6 of [9].

Given known results and extensive computations, the authors of [9] made the following conjecture.

Conjecture 2.1. *Let K be an algebraically closed field of characteristic zero and let $I \subset R = K[x,y,z]$ be a level Artinian monomial almost complete intersection, that is,*

$$I = \left(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^{\alpha}y^{\beta}z^{\gamma}\right),$$

where $0 < t$ and $0 \le \alpha \le \beta \le \gamma$. Then:

(i) *R/I has the weak Lefschetz property if any of the following conditions hold:*
  (a) $\alpha = 0$,
  (b) *$\alpha+\beta+\gamma$ is not divisible by* 3,
  (c) $\gamma > 2(\alpha+\beta)$, *or*
  (d) $t < \frac{1}{3}(\alpha+\beta+\gamma)$.

(ii) *R/I does not have the weak Lefschetz property if (α,β,γ,t) is either* $(2,9,13,9)$ *or* $(3,7,14,9)$.

(iii) *Assuming the parameters fail all conditions in* (i) *and are not as in* (ii), *then R/I does not have the weak Lefschetz property if and only if t is even and any of the following conditions hold:*
  (a) *α is even, $\alpha = \beta$, and* $\gamma - \alpha \equiv 3 \pmod 6$;
  (b) *α is odd, $\alpha = \beta$, and* $\gamma - \alpha \equiv 0 \pmod 6$; *or*
  (c) *α is odd, $\beta = \gamma$, and* $\gamma - \alpha \equiv 0 \pmod 3$.

Notice that the conditions in part (iii) of Conjecture 2.1 can be restated in a more compact form.

Conjecture 2.2. *Under the assumptions as in part* (iii) *of Conjecture* 2.1, *then R/I does not have the weak Lefschetz property if and only if t is even, $\alpha+\beta+\gamma$ is odd, and either $\alpha = \beta$ or $\beta = \gamma$.*

In order to begin working on this conjecture, the authors in [9] established a particular matrix in Theorem 7.2 and the corresponding Corollary 7.3, whose determinant completely determines if the ring R/I has the weak Lefschetz property.

Theorem 2.3. *Let K be an arbitrary field and let I be as in* (2.1) *with the additional assumptions as in Conjecture* 2.1, *part* (iii). *Consider the square* *integer matrix M of size $t+\frac{1}{3}(\alpha+\beta-2\gamma)$:*

$$M = \begin{bmatrix} \binom{\gamma}{\frac{1}{3}(\alpha+\beta+\gamma)} & \binom{\gamma}{\frac{1}{3}(\alpha+\beta+\gamma)-1} & \cdots & \binom{\gamma}{\gamma-t+1} \\ \binom{\gamma}{\frac{1}{3}(\alpha+\beta+\gamma)+1} & \binom{\gamma}{\frac{1}{3}(\alpha+\beta+\gamma)} & \cdots & \binom{\gamma}{\gamma-t+2} \\ & & \vdots & \\ \binom{\gamma}{t-1} & \binom{\gamma}{t-2} & \cdots & \binom{\gamma}{\frac{1}{3}(2\gamma-\alpha-\beta)} \\ \binom{\gamma+t}{t+\beta-1} & \binom{\gamma+t}{t+\beta-2} & \cdots & \binom{\gamma+t}{\frac{1}{3}(2(\beta+\gamma)-\alpha)} \\ \binom{\gamma+t}{t+\beta-2} & \binom{\gamma+t}{t+\beta-3} & \cdots & \binom{\gamma+t}{\frac{1}{3}(2(\beta+\gamma)-\alpha)-1} \\ & & \vdots & \\ \binom{\gamma+t}{t+\frac{1}{3}(\beta+\gamma-2\alpha)} & \binom{\gamma+t}{t-1+\frac{1}{3}(\beta+\gamma-2\alpha)} & \cdots & \binom{\gamma+t}{\gamma-\alpha+1} \end{bmatrix}.$$

Then $\det M \equiv 0 \pmod{\operatorname{char} K}$ *if and only if R/I fails to have the weak Lefschetz property.*

Notice that the matrix M has two distinct portions: a top part which has $t-\frac{1}{3}(\alpha+\beta+\gamma)$ rows and a bottom part which has $\frac{1}{3}(2(\alpha+\beta)-\gamma)$ rows. This will be especially useful in Section 4.

A large portion of Conjecture 2.1 has been proven; the results are summarised as follows.

Remark 2.4. Part (i) of Conjecture 2.1 is true by Corollary 6.3, Lemma 6.6, and Lemma 6.7 in [9]. Part (ii) is true by direct computation (e.g., using a computer algebra system such as [5] or [6]). Furthermore, the sufficiency of part (iii) holds by Corollary 7.4 in [9]. Hence, only the necessity of part (iii) remains to be shown.

3. Some straightforward cases

We establish necessary and sufficient numerical conditions for the weak Lefschetz property to hold in three families, all of which have the property $\alpha=\beta=\gamma$.

Proposition 3.1. *Suppose $\alpha=\beta=\gamma=1$ and $t\geq 1$. Let M be the matrix defined in Theorem 2.3. Then*

$$\det M = \begin{cases} 0, & \text{if } t \text{ is even;} \\ 2, & \text{if } t \text{ is odd.} \end{cases}$$

Proof. Notice, $M \in \mathbb{Z}^{t\times t}$ is the matrix

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ \binom{t+1}{t} & \binom{t+1}{t-1} & \binom{t+1}{t-2} & \cdots & \binom{t+1}{3} & \binom{t+1}{2} & \binom{t+1}{1} \end{bmatrix},$$

hence we need only $t-1$ elimination steps along the bottom row to make this matrix upper triangular.

For the first step, subtract $\binom{t+1}{t}$ copies of the first row from the last row, so the first entry in the last row becomes 0 and the second entry becomes $\binom{t+1}{t-1} - \binom{t+1}{t}$. For the second step, subtract $\binom{t+1}{t-1} - \binom{t+1}{t}$ copies of the second row from the (new) last row, so the second entry becomes 0 and the third entry becomes $\binom{t+1}{t-2} - \binom{t+1}{t-1} + \binom{t+1}{t}$. Continuing in this way, we see that after i elimination steps, the first i entries of the last row are 0 and the $(i+1)$st entry of the last row is $\binom{t+1}{t-i+1} - \binom{t+1}{t-i+2} + \cdots + (-1)^i\binom{t+1}{t}$.

Thus, $t-1$ elimination steps of this form yield the $t\times t$ matrix

$$\tilde{M} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \ell \end{bmatrix},$$

where $\ell = \binom{t+1}{1} - \binom{t+1}{2} + \cdots + (-1)^{t-1}\binom{t+1}{t}$. Using the identity

$$\sum_{i=0}^{t+1} (-1)^i \binom{t+1}{i} = 0,$$

we see that $\det M = \ell = 1 + (-1)^{t+1}$. □

Now the following is immediate using Theorem 2.3.

COROLLARY 3.2. *Suppose $I = (x^{t+1}, y^{t+1}, z^{t+1}, xyz)$ where $t \geq 1$. Then the algebra R/I has the weak Lefschetz property if and only if t is odd and the characteristic of K is not 2.*

PROPOSITION 3.3. *Suppose $\alpha = \beta = \gamma = 2$ and $t \geq 2$. Let M be the matrix defined in Theorem 2.3. Then*

$$\det M = \begin{cases} -t^2(t+3), & \text{if } t \text{ is even;} \\ (t+2)^2(t-1), & \text{if } t \text{ is odd.} \end{cases}$$

Proof. Notice, $M \in \mathbb{Z}^{t\times t}$ is given by

$$M = \begin{bmatrix} 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ \binom{t+2}{t+1} & \binom{t+2}{t} & \binom{t+2}{t-1} & \binom{t+2}{t-2} & \cdots & \binom{t+2}{5} & \binom{t+2}{4} & \binom{t+2}{3} & \binom{t+2}{2} \\ \binom{t+2}{t} & \binom{t+2}{t-1} & \binom{t+2}{t-2} & \binom{t+2}{t-3} & \cdots & \binom{t+2}{4} & \binom{t+2}{3} & \binom{t+2}{2} & \binom{t+2}{1} \end{bmatrix}.$$

We will eliminate the first $t-2$ entries in the last two rows, independently, starting with the penultimate row.

For the first step, subtract $\binom{t+2}{t+1}$ copies of the first row from the penultimate row, so the first entry becomes 0, the second entry becomes $\binom{t+2}{t} - 2\binom{t+2}{t+1}$, and the third entry becomes $\binom{t+2}{t-1} - \binom{t+2}{t+1}$. For the second step, subtract $\binom{t+2}{t} - 2\binom{t+2}{t+1}$ copies of the second row from the (new) penultimate row, so the second entry becomes 0, the third entry becomes

$$\begin{aligned} &\binom{t+2}{t-1} - \binom{t+2}{t+1} - 2\left[\binom{t+2}{t} - 2\binom{t+2}{t+1}\right] \\ &= \binom{t+2}{t-1} - 2\binom{t+2}{t} + 3\binom{t+2}{t+1}, \end{aligned}$$

and the fourth entry becomes

$$\binom{t+2}{t-2} - \left[\binom{t+2}{t} - 2\binom{t+2}{t+1}\right] = \binom{t+2}{t-2} - \binom{t+2}{t} + 2\binom{t+2}{t+1}.$$

Continuing in this way, after i elimination steps, the first i entries of the penultimate row are 0, the $(i+1)$st entry is

$$\binom{t+2}{t-i+1} - 2\binom{t+2}{t-i+2} + 3\binom{t+2}{t-i+3} + \cdots + (-1)^{i+1}(i+1)\binom{t+2}{t+1}$$

and the $(i+2)$nd entry is

$$\binom{t+2}{t-i} - \binom{t+2}{t-i+2} + 2\binom{t+2}{t-i+3} + \cdots + (-1)^{i} i\binom{t+2}{t+1}.$$

Noticing that the penultimate and ultimate rows only differ by the lower index of the binomial coefficient, then we also see that after i elimination steps of the last row, analogous to the elimination of the penultimate row given above, the first i entries of the ultimate row are 0, the $(i+1)$st entry is

$$\binom{t+2}{t-i} - 2\binom{t+2}{t-i+1} + 3\binom{t+2}{t-i+2} + \cdots + (-1)^{i}(i+1)\binom{t+2}{t}$$

and the $(i+2)$nd entry is

$$\binom{t+2}{t-i-1} - \binom{t+2}{t-i+1} + 2\binom{t+2}{t-i+2} + \cdots + (-1)^{i} i\binom{t+2}{t}.$$

Thus, after $t-2$ elimination steps of both the penultimate and ultimate rows of M, we get the $t \times t$ matrix

$$\tilde{M} = \begin{bmatrix} 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p & q \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & r & s \end{bmatrix},$$

where

$$p = \binom{t+2}{3} - 2\binom{t+2}{4} + 3\binom{t+2}{5} + \cdots + (-1)^{t-2}(t-1)\binom{t+2}{t+1},$$
$$q = \binom{t+2}{2} - \binom{t+2}{4} + 2\binom{t+2}{5} + \cdots + (-1)^{t-2}(t-2)\binom{t+2}{t+1},$$
$$r = \binom{t+2}{2} - 2\binom{t+2}{3} + 3\binom{t+2}{4} + \cdots + (-1)^{t-2}(t-1)\binom{t+2}{t}, \quad \text{and}$$
$$s = \binom{t+2}{1} - \binom{t+2}{3} + 2\binom{t+2}{4} + \cdots + (-1)^{t-2}(t-2)\binom{t+2}{t}.$$

Notice that for $n \geq 2$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j = n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1} = -n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j} = 0.$$

Thus, $\sum_{j=0}^{n}(-1)^n\binom{n}{j}f(j) = 0$ for $n \geq 2$ and any linear polynomial f, and we then see that

$$\begin{aligned} p &= (-2)\binom{t+2}{0} - (-1)\binom{t+2}{1} + (-1)^t t\binom{t+2}{t} \\ &= (-1)^t t + t, \\ q &= (-3)\binom{t+2}{0} - (-2)\binom{t+2}{1} + (-1)^t(t-1)\binom{t+2}{t} \\ &= (-1)^t(t-1) + 2t + 1, \\ r &= 1\binom{t+2}{0} + (-1)^t t\binom{t+2}{t+1} + (-1)^{t+1}(t+1)\binom{t+2}{t+2} \\ &= (-1)^t\left(t^2+t-1\right) + 1, \quad \text{and} \\ s &= 2\binom{t+2}{0} + (-1)^t(t-1)\binom{t+2}{t+1} + (-1)^{t+1}t\binom{t+2}{t+2} \\ &= (-1)^t\left(t^2-2\right) + 2. \end{aligned}$$

Hence, as $\det M = \det \tilde{M} = ps - qr$, the claim follows. $\square$

Corollary 3.4. *Suppose* $I = (x^{t+2}, y^{t+2}, z^{t+2}, x^2y^2z^2)$ *where* $t \geq 2$. *Then the algebra* R/I *has the weak Lefschetz property if the characteristic of* K *is zero or greater than* $t+3$.

This reduction generalises nicely.

Remark 3.5. In the general case when $1 \leq \alpha = \beta = \gamma \leq t$, then the associated matrix M defined in Theorem 2.3 can be reduced to a diagonal matrix with entries 1 on the diagonal except for the bottom-right $\alpha \times \alpha$ matrix. Hence, finding $\det M$ can be reduced to finding the determinant of an $\alpha \times \alpha$ matrix.

Using this technique, we have been able to verify that when $\alpha = \beta = \gamma = 3$ and $t \geq 3$, then

$$\det M = \begin{cases} 0, & \text{if } t \text{ is even;} \\ -\frac{1}{4}(t-1)^2(t+1)(t+2)(t+4)^2, & \text{if } t \text{ is odd.} \end{cases}$$

Thus, if $I = (x^{t+3}, y^{t+3}, z^{t+3}, x^3y^3z^3)$ where $t \geq 3$, then the algebra R/I fails to have the weak Lefschetz property if t is even. Further, R/I has the weak Lefschetz property if t is odd and either the characteristic of K is zero or greater than $t+4$.

It is important to notice how the results in this section verify parts of Conjecture 2.2:

Remark 3.6. For this remark, assume K is a field of characteristic zero.

In Corollary 3.2 and Remark 3.5, we have $\alpha+\beta+\gamma$ is odd, $\alpha = \beta = \gamma$, and R/I has the weak Lefschetz property if and only if t is odd. This confirms Conjecture 2.2 for their respective cases.

Further still, in Corollary 3.4 we have that $\alpha+\beta+\gamma$ is even and R/I always has the weak Lefschetz property. This also confirms Conjecture 2.2 for the case $\alpha = \beta = \gamma = 2$.

4. Two extremal cases

In this section, we consider two extremal cases for the parameters in Conjecture 2.2 where the weak Lefschetz property can be shown to hold. We do this by computing the determinants of the associated matrices from Theorem 2.3.

A nice concept that will allow a drastic simplification in the following determinants is the *hyperfactorial*.

Notation 4.1. Let $n \geq 0$ be an integer. Then define the *hyperfactorial* of n to be

$$\mathcal{H}(n) = \prod_{i=0}^{n-1} i!$$

where it is important to notice that the product goes to $n-1$ and $\mathcal{H}(0) = 1$.

We need the following formula.

LEMMA 4.2. *Let $T \geq B \geq 0$ be integers and let N be an $n \times n$ matrix with entry (i, j) given by*

$$N_{(i,j)} = \binom{T}{B+i-j} \quad (1 \leq i, j \leq n).$$

Then

$$\det N = \frac{\mathcal{H}(n)\mathcal{H}(B)\mathcal{H}(T-B)\mathcal{H}(T+n)}{\mathcal{H}(B+n)\mathcal{H}(T-B+n)\mathcal{H}(T)}.$$

Proof. This follows by an application of Lemma 3 in [8] as described there on page 8. We have written the result more conveniently, in particular, making use of the hyperfactorial form. □

We consider the case of R/I as in Conjecture 2.2 where γ is maximal with respect to given α and β, that is, $\gamma = 2(\alpha + \beta)$. Notice here, that the parameters α, β, γ, satisfy the conditions of Theorem 2.3.

THEOREM 4.3. *Let $1 \leq \alpha \leq \beta, \gamma = 2(\alpha+\beta)$, and let $t \geq \frac{1}{3}(\alpha+\beta+\gamma) = \alpha+\beta$. Set $n = t - (\alpha + \beta)$. Then the matrix M from Theorem 2.3 is a $n \times n$ matrix which has entry (i, j) given by*

$$M_{(i,j)} = \binom{\gamma}{\alpha+\beta+i-j} \quad (1 \leq i, j \leq n),$$

and determinant

$$\det M = \frac{\mathcal{H}(n)\mathcal{H}^2(\alpha+\beta)\mathcal{H}(\gamma+n)}{\mathcal{H}(\gamma)\mathcal{H}^2(t)}.$$

Proof. First, notice that since $\gamma = 2(\alpha + \beta)$ the bottom part of M from Theorem 2.3 has zero rows, so only the top part remains. This gives precisely the matrix defined above.

Setting $B = \frac{1}{2}\gamma = \alpha + \beta$ and $T = \gamma = 2(\alpha + \beta)$, then applying Lemma 4.2 provides

$$\begin{aligned}\det M &= \frac{\mathcal{H}(n)\mathcal{H}(\alpha+\beta)\mathcal{H}(\gamma-(\alpha+\beta))\mathcal{H}(\gamma+n)}{\mathcal{H}(\alpha+\beta+n)\mathcal{H}(\gamma-(\alpha+\beta)+n)\mathcal{H}(\gamma)} \\ &= \frac{\mathcal{H}(n)\mathcal{H}^2(\alpha+\beta)\mathcal{H}(\gamma+n)}{\mathcal{H}(\gamma)\mathcal{H}^2(t)},\end{aligned}$$

where we use that $\gamma - (\alpha + \beta) = \alpha + \beta$ and $\alpha + \beta + n = t$. □

As noted before, the parameters satisfy the conditions of Theorem 2.3.

COROLLARY 4.4. *Let $1 \leq \alpha \leq \beta, \gamma = 2(\alpha + \beta)$, and let $t \geq \frac{1}{3}(\alpha + \beta + \gamma) = \alpha + \beta$. Consider the ideal given by*

$$I = (x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma) \subset R = K[x, y, z].$$

Then R/I has the weak Lefschetz property if $\operatorname{char} K = 0$ *or* $\operatorname{char} K \geq t + \alpha + \beta$.

Proof. Given the closed form of the determinant in Theorem 4.3, it is clear that determinant is never 0. Further still, we see that $\det M$ is not divisible by any prime greater than or equal to $t+\alpha+\beta$. $\square$

We now consider the case of R/I as in Conjecture 2.2 where t is minimal with respect to given α, β, and γ, that is, $t=\frac{1}{3}(\alpha+\beta+\gamma)$. If we assume that $1 \le \alpha \le \beta \le \gamma \le 2(\alpha+\beta)$ and $\alpha+\beta+\gamma$ is divisible by 3, then the parameters satisfy the conditions of Theorem 2.3.

THEOREM 4.5. *Let $1 \le \alpha \le \beta \le \gamma \le 2(\alpha+\beta)$ such that $\alpha+\beta+\gamma$ is divisible by 3 and let $t=\frac{1}{3}(\alpha+\beta+\gamma)$. Set $n=\frac{1}{3}(2(\alpha+\beta)-\gamma)$. Then the matrix M from Theorem 2.3 is a $n \times n$ matrix which has entry (i,j) given by*

$$M_{(i,j)} = \binom{\gamma+t}{\beta+t+1-i-j} \quad (1 \le i,j \le n),$$

and determinant

$$\det M = (-1)^{\binom{n}{2}} \frac{\mathcal{H}(2t-\gamma)\mathcal{H}(2t-\beta)\mathcal{H}(2t-\alpha)\mathcal{H}(\alpha+\beta+\gamma)}{\mathcal{H}(\alpha+t)\mathcal{H}(\beta+t)\mathcal{H}(\gamma+t)}.$$

Proof. First, notice that since $t=\frac{1}{3}(\alpha+\beta+\gamma)$ then the top part of M from Theorem 2.3 has zero rows, so only the bottom part remains. This gives precisely the matrix defined above.

In order to compute the determinant, we must first "flip" the matrix upside down. This can be done in $\binom{n}{2}$ operations (it can be done with fewer operations, but this does not matter here) yielding the matrix $\bar{M}$ such that $\det M = (-1)^{\binom{n}{2}} \det \bar{M}$. More importantly, the matrix $\bar{M}$ has, for $1 \le i,j \le n$, entry (i,j) given by

$$\bar{M}_{(i,j)} = \binom{\gamma+t}{\beta+t-n+i-j}.$$

Setting $B=\beta+t-n$ and $T=\gamma+t$, then applying Lemma 4.2 provides

$$\begin{aligned}\det \bar{M} &= \frac{\mathcal{H}(n)\mathcal{H}(\beta+t-n)\mathcal{H}(\gamma-\beta+n)\mathcal{H}(n+\gamma+t)}{\mathcal{H}(\beta+t)\mathcal{H}(\gamma-\beta+2n)\mathcal{H}(\gamma+t)} \\ &= \frac{\mathcal{H}(2t-\gamma)\mathcal{H}(2t-\alpha)\mathcal{H}(2t-\beta)\mathcal{H}(\alpha+\beta+\gamma)}{\mathcal{H}(\beta+t)\mathcal{H}(\alpha+t)\mathcal{H}(\gamma+t)},\end{aligned}$$

where we use that $\gamma+n=2t$ and $\alpha+\beta=n+t$. $\square$

As noted before, the parameters satisfy the conditions of Theorem 2.3.

COROLLARY 4.6. *Let $1 \le \alpha \le \beta \le \gamma \le 2(\alpha+\beta)$ such that $\alpha+\beta+\gamma$ is divisible by 3 and let $t=\frac{1}{3}(\alpha+\beta+\gamma)$. Consider the ideal given by*

$$I = \left(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^{\alpha}y^{\beta}z^{\gamma}\right) \subset R = K[x,y,z].$$

Then R/I has the weak Lefschetz property if $\operatorname{char} K = 0$ *or* $\operatorname{char} K \ge \alpha+\beta+\gamma$.

Proof. Given the closed form of the determinant in Theorem 4.5, it is clear that the determinant is never 0. Further still, we see that $\det M$ is not divisible by any prime greater than or equal to $\alpha+\beta+\gamma$ because $\alpha+\beta+\gamma-1$ is the maximum of the multiplicands in the numerator of the determinant. □

It is important to notice how the two results in this section verify parts of Conjecture 2.2.

REMARK 4.7. Consider the case presented in Theorem 4.3. Notice that $\beta<\gamma$ and if $\alpha=\beta$ then $\alpha+\beta+\gamma=6\alpha$ is even, and this verifies Conjecture 2.2 in the case of γ being maximal.

Consider now the case presented in Theorem 4.5. Notice that $t=\frac{1}{3}(\alpha+\beta+\gamma)$ is even if and only if $\alpha+\beta+\gamma$ is even. Hence, t cannot be even at the same time as $\alpha+\beta+\gamma$ is odd, and this verifies Conjecture 2.2 for the case of t being minimal.

REMARK 4.8. We notice that in the cases of γ being maximal and t being minimal, the characteristics of K where R/I can possibly fail to have the weak Lefschetz property are bounded above by the maximum of the degrees of the generators of I. However, in other cases described in Conjecture 2.2, this is not true.

For example, consider the case $(\alpha,\beta,\gamma,t)=(2,9,13,12)$ where the maximum degree of a generator of I is 25. In this case,

$$\begin{aligned}\det M &= -4108937448492761153197500 \\ &= -2\cdot 3^2\cdot 5^3\cdot 11^4\cdot 13^5\cdot 19\cdot 23^3\cdot 29\cdot 5011.\end{aligned}$$

Hence, when $\operatorname{char} K=5011$ (or any other prime divisor of $\det M$) the algebra R/I fails to have the weak Lefschetz property.

There is a natural explanation why the determinants in Theorems 4.3 and 4.5 are non-trivial. The determinants compute the number of certain combinatorial objects. More specifically, let a,b,c be positive integers and consider a hexagon with side lengths a,b,c,a,b,c with angles 120°; a hexagon as described is called an *(a,b,c)-hexagon.* A *lozenge* is a rhombus of unit side-length with angles 60° and 120°.

The number of lozenge tilings is familiar (see Equation (1.1) in [4]).

PROPOSITION 4.9. *Let $a,b,c\in\mathbb{N}$. Then the number of lozenge tilings of an (a,b,c)-hexagon) is*

$$\frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a+b+c)}{\mathcal{H}(a+b)\mathcal{H}(a+c)\mathcal{H}(b+c)}.$$

Notice that if we set $a=n, b=B$, and $c=T-B$ then the determinant found in Lemma 4.2 counts the number of lozenge tilings of an (a,b,c)-hexagon, i.e. an $(n,B,T-B)$-hexagon. This connection is noted in both [4] and [8]. See Figure 1 for an example tiling of a hexagon.

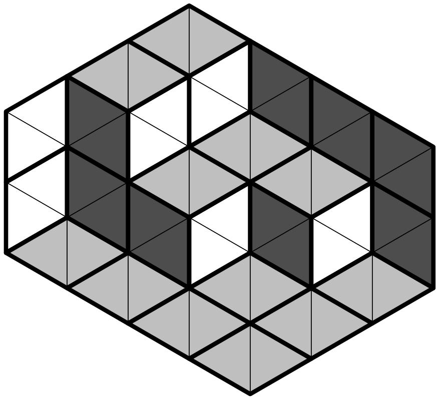

FIGURE 1. A $(2,4,3)$-hexagon tiled by lozenges.

In particular, the matrix M associated to the case when γ is maximal (resp., t is minimal) in Theorem 4.3 (resp., Theorem 4.5) has determinant whose modulus counts the number of lozenge tilings of $(\alpha+\beta, t-\alpha-\beta, \alpha+\beta)$-hexagons (resp., $(2t-\alpha, 2t-\beta, 2t-\gamma)$-hexagons).

This observation raises some natural questions:

(i) Can the connection above be extended in some way to all matrices M appearing in Theorem 2.3?
(ii) More generally, can *any* such combinatorial connection be found?
(iii) Is there some natural property of the level Artinian monomial almost complete intersections which directly associates to the tilings of hexagons by lozenges?

5. The Hilbert function is peaked strictly unimodal

The Hilbert function is strongly tied to many properties of graded algebras. If a graded algebra A has the weak Lefschetz property, then its Hilbert function is unimodal as shown in [7]. Ahn and Shin strengthened this result for level graded algebras.

PROPOSITION 5.1 ([1], Theorem 3.6). *Let A be a level artinian standard graded K-algebra with the weak Lefschetz property. Then the Hilbert function of A is peaked strictly unimodal.*

We have seen that some level artinian monomial almost complete intersections in three variables fail to have the weak Lefschetz property. Nevertheless, we show in Lemma 5.3 that their Hilbert functions are always peaked strictly unimodal regardless whether the quotient has the weak Lefschetz property or not.

First, we recall the form of a free resolution of level artinian monomial almost complete intersections in three variables ([9], Proposition 6.1).

PROPOSITION 5.2. *Let I be as in (2.1) and let $\sigma = \alpha + \beta + \gamma$. Then R/I has a free resolution of the form*

$$0 \to R^3(-\sigma - 2t) \to \begin{array}{c} R^3(-\sigma - t) \\ \oplus \\ R(-\alpha - \beta - 2t) \\ \oplus \\ R(-\alpha - \gamma - 2t) \\ \oplus \\ R(-\beta - \gamma - 2t) \end{array} \to \begin{array}{c} R(-\sigma) \\ \oplus \\ R(-\alpha - t) \\ \oplus \\ R(-\beta - t) \\ \oplus \\ R(-\gamma - t) \end{array} \to R \to R/I \to 0. \tag{5.1}$$

Furthermore, if $\alpha > 0$ then this resolution is minimal.

We are ready to prove the following key result.

LEMMA 5.3. *Let $1 \le \alpha \le \beta \le \gamma < 2(\alpha+\beta), t > \frac{1}{3}(\alpha+\beta+\gamma)$, and let $\alpha+\beta+\gamma$ be divisible by 3. Consider the ideal*

$$I = \left(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma\right) \subset R = K[x, y, z].$$

Then the Hilbert function of R/I is peaked strictly unimodal with exactly two peaks in degrees $s = \frac{2}{3}(\alpha + \beta + \gamma) + t - 2$ and $s + 1$.

Proof. Let h be the Hilbert function of R/I, that is, $h(d) = \dim_K [R/I]_d$ for $d \in \mathbb{Z}$.

By Proposition 5.2 the socle-degree of R/I is $e = \alpha + \beta + \gamma + 2t - 3$ and the Cohen–Macaulay type of R/I is 3. This implies $h(e) = 3$ and $h(d) = 0$ for $d > e$.

Further, setting $s = \frac{2}{3}(\alpha + \beta + \gamma) + t - 2$, then, by Lemma 7.1 in [9], $h(s) = h(s+1)$. Moreover, since $2(\alpha + \beta) > \gamma$ and as $\alpha + \beta + \gamma$ is divisible by 3, then

$$2(\alpha + \beta) - \gamma \ge 3. \tag{5.2}$$

Now the proof is carried out in two steps.

Step 1: Strict increase for $0 \le d \le s$. First, notice that since $d \le s$, then $\alpha + \beta + \gamma + t > s \ge d$ and further $\alpha + \beta + 2t > \frac{1}{3}(4\alpha + 4\beta + \gamma) + t > s \ge d$. This implies that the ultimate and penultimate free modules in the Resolution (5.1) yield no contribution to the Hilbert function in degree d. Hence, if $d \le s$, we have that $h(d)$ is given by

$$\binom{2+d}{2} - \binom{2+d-\alpha-\beta-\gamma}{2} \\ - \left[\binom{2+d-\alpha-t}{2} + \binom{2+d-\beta-t}{2} + \binom{2+d-\gamma-t}{2}\right]$$

and thus $h(d+1)-h(d)$ is

$$\binom{2+d}{1}-\binom{2+d-\alpha-\beta-\gamma}{1} - \left[\binom{2+d-\alpha-t}{1}+\binom{2+d-\beta-t}{1}+\binom{2+d-\gamma-t}{1}\right].$$

For $0 \le d < s$, when considering $h(d+1)-h(d)$, there are eight possible cases where the different binomial terms are non-zero in $h(d+1)-h(d)$. Furthermore, these eight cases are broken into two families: when $d+1 < \alpha+\beta+\gamma$ and when $\alpha+\beta+\gamma \le d+1$.

Assume $d+1 < \alpha+\beta+\gamma$.

(i) If $d+1 < \alpha+t$, then

$$\begin{aligned} h(d+1)-h(d) &= 2+d \\ &\ge 2. \end{aligned}$$

(ii) If $\alpha+t \le d+1 < \beta+t$, then

$$\begin{aligned} h(d+1)-h(d) &= 2+d-(2+d-\alpha-t) \\ &= \alpha+t \\ &\ge 3 \quad (\text{as } t \ge 2 \text{ and } \alpha \ge 1). \end{aligned}$$

(iii) If $\beta+t \le d+1 < \gamma+t$, then

$$\begin{aligned} h(d+1)-h(d) &= 2+d-\big[(2+d-\alpha-t)+(2+d-\beta-t)\big] \\ &= \alpha+\beta+2t-(2+d) \\ &\ge \alpha+\beta+t-\gamma \quad (\text{since } \gamma+t > d+1) \\ &\ge \frac{2}{3}\big(2(\alpha+\beta)-\gamma\big)+1 \quad \left(\text{as } t > \frac{1}{3}(\alpha+\beta+\gamma)\right) \\ &\ge 3 \quad \big(\text{by Inequality (5.2)}\big). \end{aligned}$$

(iv) If $\gamma+t \le d+1$, then

$$\begin{aligned} h(d+1)-h(d) &= 2+d \\ &\qquad -\big[(2+d-\alpha-t)+(2+d-\beta-t)+(2+d-\gamma-t)\big] \\ &= \alpha+\beta+\gamma+3t-2(2+d) \\ &\ge \alpha+\beta+\gamma+3t-2(s+1) \quad \left(\text{as } t > \frac{1}{3}(\alpha+\beta+\gamma)\right) \\ &= t-\frac{1}{3}(\alpha+\beta+\gamma)+2 \quad \left(\text{since } s = \frac{2}{3}(\alpha+\beta+\gamma)+2t-2\right) \\ &\ge 3 \quad \left(\text{again by } t > \frac{1}{3}(\alpha+\beta+\gamma)\right). \end{aligned}$$

Assume $\alpha+\beta+\gamma \le d+1$.

(i) If $d+1 < \alpha + t$, then

$$\begin{aligned} h(d+1) - h(d) &= 2 + d - (2 + d - \alpha - \beta - \gamma) \\ &= \alpha + \beta + \gamma \\ &\geq 3 \quad (\text{as } \gamma \geq \beta \geq \alpha \geq 1). \end{aligned}$$

(ii) If $\alpha + t \leq d+1 < \beta + t$, then

$$\begin{aligned} h(d+1) - h(d) &= 2 + d - \big[(2 + d - \alpha - \beta - \gamma) + (2 + d - \alpha - t)\big] \\ &= 2\alpha + \beta + \gamma + t - (2 + d) \\ &\geq 2\alpha + \gamma \quad (\text{as } \gamma + t \geq \beta + t > d + 1, \text{ since } \gamma \geq \beta) \\ &\geq 3 \quad (\text{since } \gamma \geq \alpha \geq 1). \end{aligned}$$

(iii) If $\beta + t \leq d+1 < \gamma + t$, then

$$\begin{aligned} h(d+1) - h(d) &= 2 + d \\ &\quad - \big[(2 + d - \alpha - \beta - \gamma) + (2 + d - \alpha - t) + (2 + d - \beta - t)\big] \\ &= 2\alpha + 2\beta + \gamma + 2t - 2(2 + d) \\ &\geq 2\alpha + 2\beta - \gamma \quad (\text{as } \gamma + t > d + 1) \\ &\geq 3 \quad \big(\text{by Inequality (5.2)}\big), \end{aligned}$$

where the second inequality uses Inequality (5.2).

(iv) If $\gamma + t \leq d+1$, then

$$\begin{aligned} h(d+1) - h(d) &= 2 + d - \big[(2 + d - \alpha - \beta - \gamma) + (2 + d - \alpha - t)\big] \\ &\quad - \big[(2 + d - \beta - t) + (2 + d - \gamma - t)\big] \\ &= 2(\alpha + \beta + \gamma) + 3t - 3(2 + d) \\ &\geq 2(\alpha + \beta + \gamma) + 3t - 3(s + 2) + 3 \quad (\text{as } s > d) \\ &= 3 \quad \left(\text{since } s = \frac{2}{3}(\alpha + \beta + \gamma) + t - 2\right). \end{aligned}$$

Thus, we have that $h(d+1) - h(d) > 0$ for all $0 \leq d < s$ implying that the Hilbert function is strictly increasing from degree 0 to degree s.

Step 2: Strict decrease for $s+1 \leq d \leq e$. Let k be the Hilbert function of the K-dual of R/I, that is of $(R/I)^\vee$. Then $k(d) = \dim_K [(R/I)^\vee]_d$, so $h(d) = k(-d)$ for all $d \in \mathbb{Z}$.

Since $d \geq s+1$, then $\alpha+\beta+\gamma \leq s+1 \leq d$ and, using Inequality (5.2), $\alpha+t \leq \beta+t \leq \gamma+t \leq s+1 \leq d$. This implies that the ultimate and penultimate free modules in the resolution of $(R/I)^\vee$ (which is dual to the Resolution (5.1)) yield no contribution to the Hilbert function of $(R/I)^\vee$ in degree $-d$. Hence, if $s+1 \le d$, we have that $k(-d)$ is

$$3\binom{-d-1+\alpha+\beta+\gamma+2t}{2} - 3\binom{-d-1+\alpha+\beta+\gamma+t}{2} - \left[\binom{-d-1+\beta+\gamma+2t}{2} + \binom{-d-1+\alpha+\gamma+2t}{2} + \binom{-d-1+\alpha+\beta+2t}{2}\right]$$

and thus $k(-d)-k(-d-1)$ is

$$3\binom{-d-2+\alpha+\beta+\gamma+2t}{1} - 3\binom{-d-2+\alpha+\beta+\gamma+t}{1} - \left[\binom{-d-2+\beta+\gamma+2t}{1} + \binom{-d-2+\alpha+\gamma+2t}{1} + \binom{-d-2+\alpha+\beta+2t}{1}\right].$$

For $s+1 \le d < e$, when considering $h(d)-h(d+1) = k(-d)-k(-d-1)$, there are eight possible cases where the different binomial terms are non-zero in $k(-d)-k(-d-1)$. Furthermore, these eight cases are broken into two families: when $d+1 \le \alpha+\beta+\gamma+t-2$ and when $\alpha+\beta+\gamma+t-2 < d+1$.

Assume $\alpha+\beta+\gamma+t-2 < d+1$.

(i) If $\beta+\gamma+2t-2 < d+1$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t) \\ &\ge 3\big(-(e-1)-2+\alpha+\beta+\gamma+2t\big) \quad \text{(as } e > d) \\ &= 6, \end{aligned}$$

where we use $d+1 \le e = \alpha+\beta+\gamma+2t-3$.

(ii) If $\alpha+\gamma+2t-2 < d+1 \le \beta+\gamma+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t) - (-d-2+\beta+\gamma+2t) \\ &= 3\alpha+2\beta+2\gamma+4t-4-2d \\ &\ge 3\alpha+2 \quad \text{(since } \beta+\gamma+2t-2 \ge d+1) \\ &\ge 5 \quad \text{(as } \alpha \ge 1). \end{aligned}$$

(iii) If $\alpha+\beta+2t-2 < d+1 \le \alpha+\gamma+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t) - (-d-2+\beta+\gamma+2t) \\ &\quad - (-d-2+\alpha+\gamma+2t) \\ &= 2\alpha+2\beta+\gamma+2t-2-d \\ &\ge \alpha+2\beta+1 \quad \text{(since } \alpha+\gamma+2t-2 \ge d+1) \\ &\ge 4 \quad \text{(as } \beta \ge \alpha \ge 1). \end{aligned}$$

(iv) If $s+1 < d+1 \leq \alpha+\beta+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t)-(-d-2+\beta+\gamma+2t) \\ &\quad -(-d-2+\alpha+\gamma+2t)-(-d-2+\alpha+\beta+2t) \\ &= \alpha+\beta+\gamma \\ &\geq 3 \quad (\text{as } \gamma \geq \beta \geq \alpha \geq 1). \end{aligned}$$

Assume $\alpha+\beta+\gamma+t-2 < d+1$.

(i) If $\beta+\gamma+2t-2 < d+1$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t) \\ &\quad -3(-d-2+\alpha+\beta+\gamma+t) \\ &= 3t \\ &\geq 6 \quad (\text{since } t \geq 2). \end{aligned}$$

(ii) If $\alpha+\gamma+2t-2 < d+1 \leq \beta+\gamma+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t)-3(-d-2+\alpha+\beta+\gamma+t) \\ &\quad -(-d-2+\beta+\gamma+2t) \\ &= t+d+2-\beta-\gamma \\ &\geq 3t+\alpha-\beta \quad (\text{since } d+1 > \alpha+\gamma+2t-2) \\ &\geq 3+2\alpha+\gamma \quad \left(\text{as } t > \frac{1}{3}(\alpha+\beta+\gamma)\right) \\ &\geq 6 \quad (\text{since } \gamma \geq \beta \geq \alpha \geq 1), \end{aligned}$$

where the second inequality uses that $t > \frac{1}{3}(\alpha+\beta+\gamma)$.

(iii) If $\alpha+\beta+2t-2 < d+1 \leq \alpha+\gamma+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t)-3(-d-2+\alpha+\beta+\gamma+t) \\ &\quad -(-d-2+\beta+\gamma+2t)-(-d-2+\alpha+\gamma+2t) \\ &= 2d+4-\alpha-\beta-2\gamma-t \\ &\geq \alpha+\beta-2\gamma+3t \quad (\text{as } d+1 > \alpha+\beta+2t-2) \\ &\geq 2\alpha+2\beta-\gamma+3 \quad \left(\text{since } t > \frac{1}{3}(\alpha+\beta+\gamma)\right) \\ &\geq 6 \quad \big(\text{by Inequality (5.2)}\big), \end{aligned}$$

where the second inequality uses that $t > \frac{1}{3}(\alpha+\beta+\gamma)$, and the third inequality uses Inequality (5.2).

(iv) If $s+1 < d+1 \leq \alpha+\beta+2t-2$, then

$$\begin{aligned} k(-d)-k(-d-1) &= 3(-d-2+\alpha+\beta+\gamma+2t)-3(-d-2+\alpha+\beta+\gamma+t) \\ &\quad -(-d-2+\beta+\gamma+2t)-(-d-2+\alpha+\gamma+2t) \\ &\quad -(-d-2+\alpha+\beta+2t) \end{aligned}$$

$$= 3d + 6 - 2(\alpha + \beta + \gamma) - 3t$$
$$\geq 3 \quad \left(\text{as } d + 1 > s + 1 \text{ and } s = \frac{2}{3}(\alpha + \beta + \gamma) + 2t - 2\right).$$

Hence, we have that $h(d) - h(d+1) = k(-d) - k(-d-1) > 0$ for all $s + 1 \leq d < e$ implying that the Hilbert function is strictly decreasing from $s + 1$ to the socle degree e. □

This provides the following result which gives an affirmative answer to (part of) Question 8.2(1) in [9].

THEOREM 5.4. *Let $I \subset R = K[x, y, z]$ be a level Artinian monomial almost complete intersection. Then R/I has a peaked strictly unimodal Hilbert function.*

Proof. In case (i) of Conjecture 2.1, Remark 2.4 guarantees the weak Lefschetz property of R/I, so the claim follows by Proposition 5.1.

Similarly, we conclude in case (iii) of Conjecture 2.1, when γ is maximal or t is minimal by using Corollaries 4.4 and 4.6.

In all the remaining cases of Conjecture 2.1, we conclude by Lemma 5.3. □

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