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MODULAR NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION EQUAL TO THREE

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ABSTRACT. In this paper, we give explicit descriptions of all numerical semigroups, generated by three positive integer numbers, that are the set of solutions of a Diophantine inequality of the form $ax \bmod b \le x$.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integer numbers. A *numerical semigroup* is a subset S of $\mathbb{N}$ such that it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If $A \subseteq \mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \ldots, a_n \in A, \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see [7], [8]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd\{A\} = 1$, where gcd means *greatest common divisor*.

Let S be a numerical semigroup and let X be a subset of S. We say that X is a *system of generators* of S if $S = \langle X \rangle$. In addition, if no proper subset of X generates S, then we say that X is a *minimal system of generators* of S. Every numerical semigroup admits a unique minimal system of generators and, moreover, such system has finitely many elements (see [2], [7], [8]). The cardinal of this system is known as the *embedding dimension* of S and it is denoted by $\mathrm{e}(S)$. On the other hand, if $X = \{n_1 < n_2 < \cdots < n_e\}$ is a minimal system of generators of S, then n_1, n_2 are known as the *multiplicity* and the *ratio* of S, and the first of them is denoted by $\mathrm{m}(S)$. Let us observe that $\mathrm{m}(S)$ is the least positive integer of S.

Let m, n be integers such that $n \neq 0$. We denote by $m \bmod n$ the remainder of the division of m by n. Following the notation of [9], we say that a

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proportionally modular Diophantine inequality is an expression of the form

$$ax \bmod b \leq cx, \tag{1.1}$$

where a, b, c are positive integers. We call a, b, and c the *factor*, the *modulus*, and the *proportion* of the inequality, respectively. Let $\mathrm{S}(a,b,c)$ be the set of integer solutions of (1.1). Then $\mathrm{S}(a,b,c)$ is a numerical semigroup (see [9]) that we call *proportionally modular numerical semigroup* (PM-*semigroup*).

Let $x_1, x_2, \ldots, x_q$ be a sequence of integers. We say that it is arranged in a convex form if one of the following conditions is satisfied,

(1) $x_1 \leq x_2 \leq \cdots \leq x_q$;
(2) $x_1 \geq x_2 \geq \cdots \geq x_q$;
(3) there exists $h \in \{2, \ldots, q-1\}$ such that $x_1 \geq \cdots \geq x_h \leq \cdots \leq x_q$.

As a consequence of [11, Theorem 31] (see its proof and [11, Corollary 18]), we have an easy characterization for PM-semigroups.

Lemma 1.1. *A numerical semigroup S is a* PM-*semigroup if and only if there exists a convex arrangement $n_1, n_2, \ldots, n_e$ of its set of minimal generators that satisfies the following conditions*,

(1) $\gcd\{n_i, n_{i+1}\} = 1$ *for all* $i \in \{1, \ldots, e-1\}$,
(2) $(n_{i-1} + n_{i+1}) \equiv 0 \bmod n_i$ *for all* $i \in \{2, \ldots, e-1\}$.

A *modular Diophantine inequality* (see [10]) is an expression of the form

$$ax \bmod b \leq x, \tag{1.2}$$

that is, it is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is a *modular numerical semigroup* (M-*semigroup*) if it is the set of integer solutions of a modular Diophantine inequality. Therefore, every M-semigroup is a PM-semigroup, but the reciprocal is false. In effect, from [9, Example 26], we have that the numerical semigroup $\langle 3, 8, 10\rangle$ is a PM-semigroup, but is not an M-semigroup.

Let us observe that it is easy to determine whether or not a numerical semigroup is a PM-semigroup via the previous characterization. On the other hand, this question is more complicated for M-semigroups. In [10], there is an algorithm to give the answer to this problem, but we have not got a good characterization for M-semigroups.

The purpose of this paper is to give explicit descriptions of all M-semigroups with embedding dimension equal to three. The content is summarized in the following way. After a section of preliminaries, in Section 3 we use the idea of numerical semigroup associated to an interval (see [9]) and give three families of M-semigroups with embedding dimension equal to three in an explicit way. In Section 4, we will prove that every M-semigroup with embedding dimension equal to three belongs to one of these families. Finally, in Section 5 we give another description by fixing the multiplicity and the ratio of the numerical semigroup.

2. Preliminaries

Let α, β be two positive rational numbers with $\alpha < \beta$ and let T be the submonoid of $(\mathbb{Q}_0^+, +)$ generated by the interval $[\alpha, \beta]$. Here we denote by $\mathbb{Q}$ the set of rational numbers and by $\mathbb{Q}_0^+$ the set of nonnegative rational numbers. In [9], it is shown that $\mathrm{T} \cap \mathbb{N}$ is a PM-semigroup and that every PM-semigroup is of this form. We will refer to $\mathrm{T} \cap \mathbb{N}$ as the PM-semigroup associated to the interval $[\alpha, \beta]$, and it will be denoted by $\mathrm{S}([\alpha, \beta])$. As a reformulation of [9, Corollary 9], we have the following result.

Lemma 2.1.

(1) *Let $c < a < b$ be positive integers. Then*

$$\{x \in \mathbb{N} \mid ax \bmod b \leq cx\} = \mathrm{T} \cap \mathbb{N},$$

where T is the submonoid of $(\mathbb{Q}_0^+, +)$ generated by $[\frac{b}{a}, \frac{b}{a-c}]$.

(2) *Conversely, let a_1, a_2, b_1, b_2 be positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ and let T be the submonoid of $(\mathbb{Q}_0^+, +)$ generated by $[\frac{b_1}{a_1}, \frac{b_2}{a_2}]$. Then*

$$\mathrm{T} \cap \mathbb{N} = \big\{x \in \mathbb{N} \mid a_1 b_2 x \bmod b_1 b_2 \leq (a_1 b_2 - a_2 b_1)x\big\}.$$

Since the inequality $ax \bmod b \leq cx$ has the same set of solutions as the inequality $(a \bmod b)x \bmod b \leq cx$, we will always assume that $a < b$. Besides, if $c \geq a$, then $\{x \in \mathbb{N} \mid ax \bmod b \leq cx\} = \mathbb{N}$. Therefore, the condition $c < a < b$ imposed in the previous lemma is not restrictive.

A sequence of rational numbers $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p}$ is a *Bézout sequence* if $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ are positive integers such that $a_i b_{i+1} - a_{i+1} b_i = 1$ for all $i \in \{1, 2, \ldots, p-1\}$. The fractions $\frac{b_1}{a_1}$ and $\frac{b_p}{a_p}$ are the *ends* of the sequence and p is the *length* of the sequence. We will say that a Bézout sequence is *proper* if $a_i b_{i+h} - a_{i+h} b_i \geq 2$ for all $h \geq 2$ such that $i, i+h \in \{1, 2, \ldots, p\}$.

The next result is [11, Theorem 12]. It shows the relation between Bézout sequences and PM-semigroups.

Lemma 2.2. *Let $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p}$ be a Bézout sequence. Then*

$$\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_p}{a_p}\right]\right) = \langle b_1, b_2, \ldots, b_p \rangle.$$

The following result is part of [3, Theorem 2.7].

Lemma 2.3. *Let a_1, a_2, b_1, b_2 be positive integers such that $\gcd\{a_1, b_1\} = \gcd\{a_2, b_2\} = 1$ and $\frac{b_1}{a_1} < \frac{b_2}{a_2}$. Then there exists a unique proper Bézout sequence with ends $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$.*

Let us observe that in [3] it is given an algorithm to compute the unique proper Bézout sequence with ends $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$.

We will say that two fractions $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ are *adjacent* if

$$\frac{b_2}{a_2+1} < \frac{b_1}{a_1} \quad \text{and} \quad \text{either } a_1 = 1 \text{ or } \frac{b_2}{a_2} < \frac{b_1}{a_1 - 1}.$$

The next result is [11, Theorem 20].

LEMMA 2.4. *Let* $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p}$ *be a proper Bézout sequence with adjacent ends. Then* $\{b_1, b_2, \ldots, b_p\}$ *is the minimal system of generators of the* PM*-semigroup* $\mathrm{S}([\frac{b_1}{a_1}, \frac{b_p}{a_p}])$.

We finish this section with a remark about the definition of PM-semigroup. In [12], it is shown that we can consider any type of interval for such definition. In fact, by [12, Proposition 5], we have that $\mathrm{S}(I) = T \cap \mathbb{N}$ is a PM-semigroup if T is the submonoid of $(\mathbb{Q}_0^+, +)$ generated by any (not necessarily closed) interval I with positive rational numbers as ends.

3. Three families of M-semigroups

The aim of this section is to prove Propositions 3.1, 3.2 and 3.6 in order to obtain M-semigroups with embedding dimension equal to three.

PROPOSITION 3.1. *Let* m_1, m_2 *be integers greater than or equal to three such that* $\gcd\{m_1, m_2\} = 1$. *Then* $S = \langle m_1, m_2, m_1m_2 - m_1 - m_2 \rangle$ *is an* M*-semigroup and* $\mathrm{e}(S) = 3$.

Proof. Because $\gcd\{m_1, m_2\} = 1$, there exist two positive integers u, v such that $\frac{m_1}{u} < \frac{m_2}{v}$ is a Bézout sequence. By a straightforward computation, it is easy to see that $\frac{m_1m_2-m_1-m_2}{m_2u-u-v} < \frac{m_1}{u} < \frac{m_2}{v} < \frac{m_1m_2-m_1-m_2}{m_1v-v-u}$ is also a Bézout sequence. Let us have $a = m_2u - u - v$ and $b = m_1m_2 - m_1 - m_2$. It is clear that $\frac{m_1m_2-m_1-m_2}{m_2u-u-v} = \frac{b}{a}$ and $\frac{m_1m_2-m_1-m_2}{m_1v-v-u} = \frac{b}{a-1}$. By Lemma 2.2, we have that $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ and, by Lemma 2.1, S is an M-semigroup. Since $\frac{b}{a} < \frac{m_2}{v}$ are adjacent fractions, we conclude from Lemma 2.4 that $\mathrm{e}(S) = 3$. □

The following result is deduced from [6, Corollary 6, Proposition 9].

PROPOSITION 3.2. *Let* λ, d, d' *be integers greater than or equal to* 2 *such that* $\gcd\{d, d'\} = \gcd\{\lambda, d + d'\} = 1$. *Then* $S = \langle \lambda d, d + d', \lambda d' \rangle$ *is an* M*-semigroup with* $\mathrm{e}(S) = 3$.

If S is a numerical semigroup, then the largest integer that does not belong to S is called the *Frobenius number* of S (see [5]) and denoted here by $\mathrm{F}(S)$. A numerical semigroup S is *symmetric* if $x \in \mathbb{Z} \setminus S$ implies $\mathrm{F}(S) - x \in S$ (as it is usual, $\mathbb{Z}$ is the set of integers). This type of semigroups has been widely studied and has relevance in Algebraic Geometry because they are those numerical semigroups whose semigroup ring is Gorenstein (see [4]). As a result of the study in [6], we have that the family of numerical semigroups given by Proposition 3.2 is precisely the family of all symmetric M-semigroups with embedding dimension equal to three.

Before showing the third family of M-semigroups, we need some lemmas. Firstly, we remember that, if a, b are two positive integers, then we denote by $a \mid b$ that a divides b.

LEMMA 3.3. *Let a_1, a_2, b_1, b_2 be positive integers such that $\gcd\{a_1, b_1\} = \gcd\{a_2, b_2\} = 1$, $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, and $a_1 b_2 - a_2 b_1 = q$. Let t be a positive integer. If $\gcd\{b_1, b_2\} = 1$ and $q \mid (tb_1 + b_2)$, then $q \mid (ta_1 + a_2)$.*

Proof. If $q \mid (tb_1 + b_2)$, then there exists a positive integer k such that $tb_1 + b_2 = kq$. Therefore, $t = \frac{kq - b_2}{b_1}$, and consequently $ta_1 + a_2 = \frac{q(ka_1 - 1)}{b_1}$.

Because $\gcd\{b_1, q\} = \gcd\{b_1, a_1 b_2 - a_2 b_1\} = \gcd\{b_1, a_1 b_2\} = \gcd\{b_1, b_2\} = 1$ and $\frac{q(ka_1-1)}{b_1}$ is an integer, then $b_1 \mid (ka_1 - 1)$. Now we conclude that $q \mid (ta_1 + a_2)$. □

Let us observe that the previous lemma is not true in general if $\gcd\{b_1, b_2\} \neq 1$. In fact, if we consider $\frac{8}{5} < \frac{10}{3}$, then $q = 26$, $26 \mid (2*8+10)$, and $26 \nmid (2*5+3)$.

LEMMA 3.4. *Let m_1, m_2 be positive integers such that $\gcd\{m_1, m_2\} = 1$. Let q be a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$. Then $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$ is an M-semigroup.*

Proof. Let us have $b = \frac{m_2 - 1}{q} m_1$. Since $\gcd\{m_1, m_2\} = 1$, then there exist $s, t \in \mathbb{N} \setminus \{0\}$ such that $sm_2 - tm_1 = q$. So, let us have $a = \frac{m_2-1}{q} s$. In order to finish the proof, we will show that $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$.

In fact, by Lemma 3.3 and an easy computation, it is clear that $\frac{m_1}{s} < \frac{(m_1+m_2)/q}{(s+t)/q} < \frac{m_2}{t} < \frac{b}{a-1}$ is a Bézout sequence. Moreover, $\frac{b}{a} = \frac{m_1}{s}$. By Lemma 2.2, we deduce that $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$.

Finally, by Lemma 2.1, S is an M-semigroup. □

In the next result, we will see for what values of q the M-semigroups described in the previous lemma have embedding dimension equal to three.

LEMMA 3.5. *Let m_1, m_2 be positive integers such that $\gcd\{m_1, m_2\} = 1$. Let q be a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$. Then $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$ has embedding dimension equal to three if and only if $2 \leq q < \min\{m_1, m_2\}$.*

Proof. (*Necessity*) We have that $q \geq 2$. Let us suppose that $q \geq \min\{m_1, m_2\}$. Because $q \mid (m_2 - 1)$, then $q < m_2$. Moreover, $\gcd\{m_2 - 1, m_1 + m_2\} = \gcd\{m_2 - 1, m_1 + 1\}$. Therefore, $q = m_1 + 1$. Consequently, there exists $k \in \mathbb{N} \setminus \{0\}$ such that $m_2 = k(m_1 + 1) + 1$. Thus $S = \langle m_1, k+1, m_2 \rangle = \langle m_1, k+1 \rangle$, which is a contradiction to the fact that $\mathrm{e}(S) = 3$.

(*Sufficiency*) Let us have s, t as in the proof of the previous lemma. Then $\frac{m_1}{s} < \frac{(m_1+m_2)/q}{(s+t)/q} < \frac{m_2}{t}$ is a Bézout sequence with adjacent ends. By Lemma 2.4, we conclude that $\mathrm{e}(S) = 3$. □

As an immediate consequence of Lemmas 3.4 and 3.5, we have the last result of this section.

PROPOSITION 3.6. *Let m_1, m_2 be positive integers such that $\gcd\{m_1, m_2\} = 1$. Let q be a divisor of $\gcd\{m_2 - 1, m_1 + m_2\}$ such that $2 \le q < \min\{m_1, m_2\}$. Then $S = \langle m_1, \frac{m_1+m_2}{q}, m_2\rangle$ is an M-semigroup with embedding dimension equal to three.*

4. All M-semigroups with embedding dimension equal to three

In this section, we will see that every M-semigroup with embedding dimension equal to three belongs to one of the families described in Propositions 3.1, 3.2 and 3.6.

Let us remember (see Lemma 2.1) that a numerical semigroup S is an M-semigroup if and only if there exist two integers a, b ($2 \le a < b$) such that $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$. We begin with Proposition 4.6, where we prove that, if $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$ and $\mathrm{e}(S) = 3$, then S is one of the M-semigroups described in Proposition 3.1.

Let us denote by $]\alpha, \beta[= \{x \in \mathbb{Q} \mid \alpha < x < \beta\}$, that is, the opened interval with ends α and β. From [12, Proposition 8, Theorems 11 and 20], we deduce the following result.

LEMMA 4.1. *Let a, b be integers such that $2 \le a < b$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$. Then*

(1) $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[) = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}]) \setminus \{b\}$.

(2) *b is the Frobenius number of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.*

(3) *$\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$ is a symmetric numerical semigroup.*

The next result is a consequence of [10, Proposition 29].

LEMMA 4.2. *Let a, b be integers such that $2 \le a < b$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$. Then b is the largest minimal generator of $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$. Moreover, $b = \mathrm{F}(S) + \mathrm{m}(S)$.*

If S is a numerical semigroup and $n \in S \setminus \{0\}$, then the *Apéry set of n in S* (see [1]) is the set $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. It is well known (see [7], [8]) that $\mathrm{Ap}(S, n) = \{\omega(0) = 0, \omega(1), \ldots, \omega(n)\}$, where $\omega(i)$ is the least element in S congruent to i modulo n. It is evident that $\max\{\mathrm{Ap}(S, n)\} = \mathrm{F}(S) + n$ and that, if $m_1 < m_2 < \cdots < m_p$ is the minimal system of generators of S, then $\{m_2, \ldots, m_p\} \subseteq \mathrm{Ap}(S, m_1)$.

LEMMA 4.3. *Let S be a symmetric numerical semigroup with* $\mathrm{m}(S) \geq 3$ *and let* $\{m_1 < m_2 < \cdots < m_p\}$ *be its minimal system of generators. Then* $m_p < \mathrm{F}(S)$.

Proof. Let us have $\omega \in \mathrm{Ap}(S, m_1)$. Then $\omega - \mathrm{m}(S) = \omega - m_1 \notin S$. Because S is symmetric, then $\mathrm{F}(S) - \omega + m_1 \in S$. Moreover, since $(\mathrm{F}(S) - \omega + m_1) - m_1 = \mathrm{F}(S) - \omega \notin S$, then $\omega' = \mathrm{F}(S) - \omega + m_1 \in \mathrm{Ap}(S, m_1)$.

Now, let us have $\omega = m_p$. Then $\mathrm{F}(S) - m_p + m_1 = \omega' \in \mathrm{Ap}(S, m_1)$. If $\omega' \neq 0$, we have the conclusion. In other case, $m_p = \mathrm{F}(S) + m_1$ and $\mathrm{F}(S) - m_2 + m_1 = m_p - m_2 \in \mathrm{Ap}(S, m_1)$, which is a contradiction to the fact that $\{m_1, m_2, \ldots, m_p\}$ is a minimal system of generators. □

From this result it follows the next lemma.

LEMMA 4.4. *Let S be a numerical semigroup with minimal system of generators given by* $m_1 < m_2 < \cdots < m_p$. *If* $p \geq 3$, $m_p = \mathrm{F}(S) + m_1$, *and* $S \setminus \{m_p\}$ *is symmetric, then* $S \setminus \{m_p\} = \langle m_1, \ldots, m_{p-1} \rangle$.

Proof. Let us observe that $\mathrm{F}(S \setminus \{m_p\}) = m_p$. Moreover, it is obvious that $\langle m_1, \ldots, m_{p-1} \rangle \subseteq S \setminus \{m_p\}$. Let us suppose that there exists $x \in S \setminus \{m_p\}$ such that $x \notin \langle m_1, \ldots, m_{p-1} \rangle$. Then, by Lemma 4.3, $x < m_p$. Therefore, such x must be a minimal generator of S, in contradiction with the hypothesis. □

The next one is a classic result by Sylvester [13].

LEMMA 4.5. *Let m_1, m_2 be positive integers such that* $\gcd\{m_1, m_2\} = 1$. *Then* $\mathrm{F}(\langle m_1, m_2 \rangle) = m_1 m_2 - m_1 - m_2$.

We are now ready to prove the announced result.

PROPOSITION 4.6. *Let* $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ *be a numerical semigroup such that* $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$ *and* $\mathrm{e}(S) = 3$. *Then there exist two integers m_1, m_2 greater than or equal to three such that* $\gcd\{m_1, m_2\} = 1$ *and* $S = \langle m_1, m_2, m_1 m_2 - m_1 - m_2 \rangle$.

Proof. By Lemma 4.2, we deduce that there exist two integers m_1, m_2 greater than or equal to three such that $m_1 < m_2 < b$ is the minimal system of generators of S. By Lemma 4.1, we know that $S \setminus \{b\}$ is symmetric. Therefore, by Lemma 4.4, we have that $S \setminus \{b\} = \langle m_1, m_2 \rangle$ and, by Lemma 4.1 again, that $b = \mathrm{F}(\langle m_1, m_2 \rangle)$. We finish the proof using Lemma 4.5. □

Our next aim will be Proposition 4.9, where we prove that, if $\gcd\{a, b\} \neq 1$, $\gcd\{a-1, b\} \neq 1$, and $\mathrm{e}(S) = 3$, then $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ is one of the M-semigroups described in Proposition 3.2.

The following result is [11, Lemma 4].

LEMMA 4.7. *Let* a_1, a_2, b_1, b_2, x, y *be positive integers such that* $\frac{b_1}{a_1} < \frac{b_2}{a_2}$. *Then* $\frac{b_1}{a_1} < \frac{x}{y} < \frac{b_2}{a_2}$ *if and only if* $\frac{x}{y} = \frac{\lambda b_1 + \mu b_2}{\lambda a_1 + \mu a_2}$ *for some* $\lambda, \mu \in \mathbb{N} \setminus \{0\}$.

LEMMA 4.8. *Let* $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \frac{b_3}{a_3}$ *be a proper Bézout sequence. If* $q = a_1 b_3 - a_3 b_1$, *then* $a_2 = \frac{a_1+a_3}{q}$ *and* $b_2 = \frac{b_1+b_3}{q}$.

Proof. By Lemma 4.7, there exist three positive integers λ, μ, t such that $a_2 = \frac{\lambda a_1 + \mu a_3}{t}$ and $b_2 = \frac{\lambda b_1 + \mu b_3}{t}$. Because $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ and $\frac{b_2}{a_2} < \frac{b_3}{a_3}$ are Bézout sequences, then $\mu q = t$ and $\lambda q = t$. Therefore, $\lambda = \mu$ and the conclusion is obvious. □

Now, we can prove the above mentioned result.

PROPOSITION 4.9. *Let* $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ *be a numerical semigroup such that* $\gcd\{a,b\} = d \neq 1$, $\gcd\{a-1,b\} = d' \neq 1$, *and* $\mathrm{e}(S) = 3$. *Then there exists an integer* λ *greater than or equal to two such that* $S = \langle \lambda d, d+d', \lambda d' \rangle$ *and* $\gcd\{d,d'\} = \gcd\{\lambda, d+d'\} = 1$.

Proof. By straightforward computations, $\frac{b/d}{a/d} < \frac{b/d'}{(a-1)/d'}$ are adjacent fractions and $\frac{a}{d}\frac{b}{d'} - \frac{a-1}{d'}\frac{b}{d} = \frac{b}{dd'}$. Since $S = \mathrm{S}([\frac{b/d}{a/d}, \frac{b/d'}{(a-1)/d'}])$ and $\mathrm{e}(S) = 3$, we apply Lemma 2.2 to deduce that $\frac{b}{dd'} \neq 1$. Moreover, by Lemmas 2.3 and 2.4, there exist two positive integers x, y such that $\frac{b/d}{a/d} < \frac{x}{y} < \frac{b/d'}{(a-1)/d'}$ is a proper Bézout sequence. By Lemma 4.8, it follows that $x = d + d'$. Finally, by Lemma 2.2, if $\lambda = \frac{b}{dd'}$, then $S = \langle \lambda d, d+d', \lambda d' \rangle$. □

In the following proposition we show that, if $\gcd\{a,b\} \neq 1$, $\gcd\{a-1,b\} = 1$, and $\mathrm{e}(S) = 3$, then $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ is one of the M-semigroups described in Proposition 3.6. Before this, we remember [11, Lemma 17].

LEMMA 4.10. *Let* $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_p}{a_p}$ *be a proper Bézout sequence. Then*

$$\max\{b_1, b_2, \ldots, b_p\} = \max\{b_1, b_p\}.$$

PROPOSITION 4.11. *Let* $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ *be a numerical semigroup such that* $\gcd\{a,b\} = d \neq 1$, $\gcd\{a-1,b\} = 1$, *and* $\mathrm{e}(S) = 3$. *Then there exist three positive integers* m_1, m_2, q *such that* $\gcd\{m_1, m_2\} = 1$, q *is a divisor of* $\gcd\{m_2 - 1, m_1 + m_2\}$, $2 \leq q < \min\{m_1, m_2\}$, *and* $S = \langle m_1, \frac{m_1+m_2}{q}, m_2 \rangle$.

Proof. Firstly, by Lemma 4.10, if $\frac{b/d}{a/d} < \frac{b_1}{a_1} < \cdots < \frac{b_e}{a_e} < \frac{b}{a-1}$ is a proper Bézout sequence, then $b_e \leq b$. Consequently, by an easy computation, we have that $\frac{b/d}{a/d} < \frac{b_1}{a_1} < \cdots < \frac{b_e}{a_e}$ is a proper Bézout sequence with adjacent ends.

From this observation, Lemmas 2.2, 2.3, 2.4, and that $\mathrm{e}(S) = 3$, we deduce that there exists a proper Bézout sequence of the form $\frac{b/d}{a/d} < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \frac{b}{a-1}$. Because $b_2 \frac{a}{d} - a_2 \frac{b}{d} = \frac{1}{d}(b_2 + (a-1)b_2 - a_2 b) = \frac{b_2 - 1}{d}$, by Lemma 4.8, we have that $b_1 = \frac{b/d + b_2}{(b_2-1)/d}$. Therefore, by Lemma 2.2, $S = \langle \frac{b}{d}, \frac{b/d+b_2}{(b_2-1)/d}, b_2 \rangle$. Finally, since $\gcd\{b_2, b\} = 1$, then $\gcd\{b_2, \frac{b}{d}\} = 1$ and, taking $q = \frac{b_2-1}{d}$, we finish the proof using Lemma 3.5. □

By [10, Lemma 3], we know that $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}]) = \mathrm{S}([\frac{b}{b+1-a}, \frac{b}{b-a}])$. It is obvious that $\gcd\{a,b\} = \gcd\{b-a,b\}$ and $\gcd\{a-1,b\} = \gcd\{b+1-a,b\}$. Therefore, if $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ is a numerical semigroup such that $\gcd\{a,b\} = 1$, $\gcd\{a-1,b\} = d \neq 1$, and $\mathrm{e}(S) = 3$, and if $a' = b+1-a$, then $S = \mathrm{S}([\frac{b}{a'}, \frac{b}{a'-1}])$ with $\gcd\{a',b\} = d \neq 1$, $\gcd\{a'-1,b\} = 1$, and $\mathrm{e}(S) = 3$. In consequence, S is one of the M-semigroups described in Proposition 3.6 too.

We summarize the results of Section 3 and Section 4 in the next theorem.

THEOREM 4.12. *S is an* M-*semigroup with* $\mathrm{e}(S) = 3$ *if and only if it is one of the following types.*

(T1) $S = \langle m_1, m_2, m_1m_2 - m_1 - m_2\rangle$ *where m_1, m_2 are integers greater than or equal to three such that* $\gcd\{m_1, m_2\} = 1$.

(T2) $S = \langle \lambda d, d+d', \lambda d'\rangle$ *where λ, d, d' are integers greater than or equal to two such that* $\gcd\{d, d'\} = \gcd\{\lambda, d+d'\} = 1$.

(T3) $S = \langle m_1, \frac{m_1+m_2}{q}, m_2\rangle$ *where m_1, m_2, q are positive integers such that* $\gcd\{m_1, m_2\} = 1$, *q is a divisor of* $\gcd\{m_2-1, m_1+m_2\}$, *and* $2 \leq q < \min\{m_1, m_2\}$.

A natural question that arises after Theorem 4.12 is whether the three types are disjoint. Let us see the answer.

(1) If we take $m_1 = m_1'm_2' - m_1' - m_2'$, $m_2 = m_2'$, and $q = m_2' - 1$ in the third type, then we obtain the first one.

(2) If $\gcd\{m_1, m_2\} = 1$, it is clear that $\gcd\{m_1, m_1m_2 - m_1 - m_2\} = \gcd\{m_2, m_1m_2 - m_1 - m_2\} = 1$. Therefore, there is not any relation between the first two types.

(3) Let us suppose, without loss of generality, that $d < d'$ and $m_1 < m_2$. Then $\lambda d < d + d' < \lambda d'$ or $d + d' < \lambda d < \lambda d'$ in the second type, and $m_1 < \frac{m_1+m_2}{q} < m_2$ or $\frac{m_1+m_2}{q} < m_1 < m_2$ in the third one.

    If we consider $m_1 = \lambda d$ and $m_2 = \lambda d'$, then $\gcd\{m_1, m_2\} = \lambda$. On the other hand, if we consider $\frac{m_1+m_2}{q} = \lambda d$ and $m_2 = \lambda d'$, then $m_1 = \lambda(qd - d')$, and consequently $\lambda \mid \gcd\{m_1, m_2\}$. Because $\gcd\{m_1, m_2\} = 1$, we conclude that there is no relation between the last two types.

Therefore, S is an M-semigroup with embedding dimension equal to three if and only if it is $(T2)$ or $(T3)$. Moreover, this is a disjoint classification.

5. Multiplicity and ratio fixed

Let $\{n_1 < n_2 < n_3\}$ be the minimal system of generators of a numerical semigroup S. The aim of this section is to describe all M-semigroups with embedding dimension equal to three when we fix the multiplicity and the ratio of S, that is, when n_1 and n_2 are fixed. In what follows, we will suppose that n_1, n_2 are integers such that $3 \leq n_1 < n_2$ and $\gcd\{n_1, n_2\} = 1$. Moreover, to simplify the notation we will use the following sets:

* $A(n_1) = \{2, \ldots, n_1 - 1\}$;
* $A(n_1, n_2) = \{\lceil \frac{2n_2}{n_1} \rceil, \ldots, n_2 - 1\}$;
* $D(n) = \{k \in \mathbb{N} \text{ such that } k \mid n\}$.

Here, if $q \in \mathbb{Q}$, then $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$.

Since an M-semigroup is a PM-semigroup, by Lemma 1.1, we have two cases.

(1) $S = \langle n_1, n_2, n_3 \rangle$ such that $n_1 < n_2 < n_3$ and $(n_1 + n_3) \equiv 0 \bmod n_2$. Since $\mathrm{e}(S) = 3$, then $n_3 = kn_2 - n_1$ with $k \in A(n_1)$.

(2) $S = \langle n_1, n_2, n_3 \rangle$ such that $n_1 < n_2 < n_3$ and $(n_2 + n_3) \equiv 0 \bmod n_1$. Since $\mathrm{e}(S) = 3$, then $n_3 = tn_1 - n_2$ with $t \in A(n_1, n_2)$.

REMARK 5.1. It is easy to show that S is a PM-semigroup such that $\mathrm{e}(S) = 3$ if and only if it has a minimal system of generators given by one of the previous cases. Moreover, both of these cases coincide if and only if $k = n_1 - 1$ and $t = n_2 - 1$, that is, when we consider the M-semigroup of type $(T1)$ given by $S = \langle n_1, n_2, n_1n_2 - n_1 - n_2 \rangle$.

The main result of this section is Theorem 5.6. We need some preliminary lemmas in order to prove it.

LEMMA 5.2. *Let $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ be a numerical semigroup such that $k \in A(n_1)$. Then S is $(T2)$ if and only if $k \mid n_1$.*

Proof. If $k \mid n_1$, then S is $(T2)$ for $d = \frac{n_1}{k}$, $d' = n_2 - \frac{n_1}{k}$, and $\lambda = k$.

For the opposite implication, let us suppose, without loss of generality, that $d < d'$. Then we have two possibilities to relate $(n_1, n_2, kn_2 - n_1)$ and (λ, d, d'). The first one is given by $n_1 = \lambda d$, $n_2 = d + d'$, and $kn_2 - n_1 = \lambda d'$. This election is valid if $k(d + d') - \lambda d = \lambda d'$, and consequently if $\lambda = k$. We conclude that $k \mid n_1$.

The second choice is $n_1 = d + d'$, $n_2 = \lambda d$, and $kn_2 - n_1 = \lambda d'$. In this case, $k\lambda d - (d + d') = \lambda d$, and then $d + d' = \lambda(kd - d')$. But this is not possible because we need that $\gcd\{\lambda, d + d'\} = 1$. □

LEMMA 5.3. *Let $S = \langle n_1, n_2, kn_2 - n_1 \rangle$ be a numerical semigroup such that $k \in A(n_1) \setminus \{n_1 - 1\}$. Then S is $(T3)$ if and only if $k \mid (n_1 - 1)$ or $k \mid (n_1 + 1)$.*

Proof. On the one hand, if $k \mid (n_1 - 1)$, then S is $(T3)$ for $m_1 = kn_2 - n_1$, $m_2 = n_1$, and $q = k$. On the other hand, if $k \mid (n_1 + 1)$, then S is $(T3)$ for $m_1 = n_1$, $m_2 = kn_2 - n_1$, and $q = k$.

For the opposite implication, let us suppose, without loss of generality, that $m_1 < m_2$, but we accept that $q \mid (m_1 - 1)$ or $q \mid (m_2 - 1)$. Then we have two possibilities to relate $(n_1, n_2, kn_2 - n_1)$ and (m_1, m_2, q). The first one is given by $n_1 = m_1$, $n_2 = \frac{m_1 + m_2}{q}$, and $kn_2 - n_1 = m_2$. This option is valid if $k\frac{m_1 + m_2}{q} - m_1 = m_2$, and consequently if $q = k$. If $q \mid \gcd\{m_2 - 1, m_1 + m_2\}$, since $\gcd\{m_2 - 1, m_1 + m_2\} = \gcd\{n_1 + 1, kn_2\}$, we conclude that $k \mid (n_1 + 1)$.

If $q \mid \gcd\{m_1 - 1, m_1 + m_2\}$, since $\gcd\{m_1 - 1, m_1 + m_2\} = \gcd\{n_1 - 1, kn_2\}$, we conclude that $k \mid (n_1 - 1)$.

The second choice is $n_1 = \frac{m_1+m_2}{q}$, $n_2 = m_1$, and $kn_2 - n_1 = m_2$. But then we have $q = \frac{(k+1)n_2}{n_1} - 1$, which is not possible because $\gcd\{n_1, n_2\} = 1$ and $k \leq n_1 - 2$. □

Using similar arguments to those of Lemmas 5.2 and 5.3, we have the next two lemmas.

LEMMA 5.4. *Let $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ be a numerical semigroup such that $t \in A(n_1, n_2)$. Then S is $(T2)$ if and only if $t \mid n_2$.*

LEMMA 5.5. *Let $S = \langle n_1, n_2, tn_1 - n_2 \rangle$ be a numerical semigroup such that $t \in A(n_1, n_2) \setminus \{n_2 - 1\}$. Then S is $(T3)$ if and only if $t \mid (n_2 - 1)$ or $t \mid (n_2 + 1)$.*

From Remark 5.1 and Lemmas 5.2, 5.3, 5.4, and 5.5, we deduce the announced theorem.

THEOREM 5.6. *Let n_1, n_2, n_3 be integers such that $3 \leq n_1 < n_2 < n_3$, $\gcd\{n_1, n_2\} = 1$, and $n_3 \notin \langle n_1, n_2 \rangle$. Then $\langle n_1, n_2, n_3 \rangle$ is an* M*-semigroup if and only if n_3 belongs to one of the following sets.*

(1) $B_1 = \{kn_2 - n_1 \mid k \in A(n_1) \cap [D(n_1 - 1) \cup D(n_1) \cup D(n_1 + 1)]\}$.

(2) $B_2 = \{tn_1 - n_2 \mid t \in A(n_1, n_2) \cap [D(n_2 - 1) \cup D(n_2) \cup D(n_2 + 1)]\}$.

Moreover, $B_1 \cap B_2 = \{n_1 n_2 - n_1 - n_2\}$.

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