MAKANIN–RAZBOROV DIAGRAMS OVER FREE PRODUCTS

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ABSTRACT. This paper is the first in a sequence on the first order theory of free products and further generalizations. In the first paper, we generalize the analysis of systems of equations over free and (torsion-free) hyperbolic groups, and analyze systems of equations over free products. To do that we introduce limit groups over the class of free products, and show that a finitely presented group has a canonical (finite) collection of maximal limit quotients. We further extend this finite collection and associate a Makanin–Razborov diagram over free products with every f.p. group. This MR diagram encodes all the quotients of a given f.p. group that are free products, all its homomorphisms into free products, and equivalently all the solutions to a given system of equations over a free product.

Sets of solutions to equations defined over a free group have been studied extensively. Considerable progress in the study of such sets of solutions was made by G. S. Makanin, who constructed an algorithm that decides if a system of equations defined over a free group has a solution [Ma], and showed that the universal and positive theories of a free group are decidable. A. A. Razborov was able to give a description of the entire set of solutions to a system of equations defined over a free group [Ra2], a description that was further developed by O. Kharlampovich and A. Myasnikov [Kh-My].

In [Se1], a geometric approach to the study of sets of solutions to systems of equations over a free group is presented. This was generalized in [Se3] for systems of equations over (torsion-free) hyperbolic groups, in [Al] to systems

of equations over limit groups, and in [Gro] to systems of equations over toral relatively hyperbolic groups.

In this paper, we generalize part of the techniques and the results that were obtained over free groups to study systems of equations over arbitrary free products. Let Σ be a system of equations which is defined over a free product, $A * B$:

$$w_1(x_1, \ldots, x_n) = 1,$$
$$\vdots$$
$$w_s(x_1, \ldots, x_n) = 1.$$

Following [Ra1], we set the associated f.p. group $G(\Sigma)$ to be:

$$G(\Sigma) = \langle x_1, \ldots, x_n | w_1, \ldots, w_s \rangle.$$

Clearly, every solution of the system Σ corresponds to a homomorphism $h : G(\Sigma) \to A * B$, and every such homomorphism corresponds to a solution of the system Σ. Therefore, the study of sets of solutions to systems of equations over the free product $A * B$ is equivalent to the study of all the homomorphisms from a fixed f.p. group G into $A * B$.

We further generalize our point of view, and instead of the set of homomorphisms from a given f.p. group $G(\Sigma)$ into a particular free product, we study the set of all the homomorphisms from the f.p. group $G(\Sigma)$ into all possible free products. By Kurosh subgroup theorem, this is equivalent to the study of all the quotients of a given f.p. group, $G(\Sigma)$, that are free products.

To analyze the set of free product quotients of a given f.p. group, we generalize the notion of limit groups (over free groups), and define *limit groups over free products*. The definition over free products (Definition 1) is a generalization of the definition of limit groups over free groups, but with each limit group over free products, L, there is an additional structure, a subset of conjugacy classes in the limit group L, that are called *elliptics*, that are forced to be mapped to conjugates of the factors in any homomorphism from the limit group into a free product.

After proving some basic properties of limit groups over free products, we associate with them a canonical virtually Abelian JSJ decomposition (Theorem 11). Limit groups over free products do not satisfy the d.c.c. that hold for limit groups over free and hyperbolic groups. Still, in Theorem 13, we prove a basic d.c.c. that holds for such limit groups, and applies to descending chain of limit groups over free products, in which the maps between successive limit groups are proper epimorphisms that do not map nontrivial elliptic elements to the identity element.

This d.c.c. allows us to associate a *resolution* with each limit group over free products (Theorem 18). We further define a natural partial order on the set of limit quotients over free products of a given f.p. group, and prove

that there are finitely many (equivalence classes of) maximal limit quotients (over free products) of a f.p. group. Finally, we extend each of the maximal limit quotients with finitely many resolutions and obtain a Makanin–Razborov diagram of a f.p. group over free products.

The diagram that we associate with a f.p. group encodes all the quotients of the given f.p. group that are free products. Unfortunately, our construction is not canonical, and we state a natural conjecture that if answered affirmatively will enable one to construct a canonical diagram. Also, the construction uses the finite presentability of the group in question in an essential way. Hence, encoding the set of free product quotients of a f.g. group is left open.

The Makanin–Razborov diagram over free products is the first step towards the analysis of the first order theory of free products that will appear in the sequel. This study was motivated by a question of the first author on the stability of a free product of stable groups, and we intend to address this question in the next paper in the sequence. The diagram that we construct provides the tools to deal with another well known and natural problem, the question of an elementary equivalence between two free products, $A_1 * A_2$ and $B_1 * B_2$, assuming that A_i is elementarily equivalent to B_i, for $i = 1, 2$ (where the A_i's and the B_j's are arbitrary groups). We intend to address this problem in the sequel as well.

We expect that some of the notions and constructions that appear in this paper (and in the sequel) can be generalized to other classes of groups, for example, acylindrical splittings of f.p. groups, and various classes of relatively hyperbolic groups. In particular, we hope that some of our constructions will serve as basic tools in constructing stable groups with various properties, for example, divisible or torsion groups.

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1. Limit groups over free products

We start the analysis of systems of equations over free products with the definition of a limit group over the set of free products. The definition generalizes the corresponding ones for free, hyperbolic, and relatively hyperbolic groups, but it associates with a limit group an additional structure—its collection of conjugacy classes of elliptic elements. Also, note that unlike the case of a free or a hyperbolic group, we consider limit groups over the entire class of free products, and not necessarily over a given one.

DEFINITION 1. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of groups (not necessarily finitely generated), and let G be a finitely generated group. We say that a sequence of homomorphisms, $\{h_n : G \to A_n * B_n\}$, is a convergent sequence, if the following conditions hold:

(i) For each $g \in G$ there exists some index $n_g > 0$, so that for every $n > n_g$, $h_n(g) = 1$, or for every $n > n_g$, $h_n(g) \neq 1$.
(ii) For each $g \in G$ there exists some index $n_g^e > 0$, so that for every $n > n_g^e$, $h_n(g)$ is elliptic in the free product $A_n * B_n$ (i.e., it is contained in a conjugate of A_n or B_n), or for every $n > n_g^e$, $h_n(g)$ is not elliptic in $A_n * B_n$.

With the convergent sequence, we associate its *stable kernel* that is defined to be:
$$K = \{g \in G \mid \exists n_g \; \forall n > n_g, h_n(g) = 1\}$$
and the associate limit group: $L = G/K$, which we call a limit group over (the collection of) free products, and set $\eta : G \to L$ to be the natural quotient map.

With the limit group L we associate an additional structure, its collection of conjugacy classes that are stably elliptic, that is:
$$E_L = \{\ell \in L \mid \exists g \in G, \eta(g) = \ell \; \exists n_g > 0 \; \forall n > n_g, h_n(g) \text{ is elliptic}\}$$

Note that by definition if $\eta(g_1) = \eta(g_2)$, then g_1 is stably elliptic iff g_2 is stably elliptic. Also, note that every f.g. group can be a limit group over free products, as given a finitely generated group G, we can look at the free product $G * B$, for some nontrivial group B, with the fixed sequence of homomorphisms that map G identically onto G in the free product $G * B$. Note that in this tautological case, the entire (limit) group G is set to be elliptic.

Given a convergent sequence of homomorphisms one can pass to a subsequence that converges into a (possibly trivial) action of the associated limit group on some real tree.

Let A and B be nontrivial groups (not necessarily finitely generated). With the free product, $A * B$, we can naturally associate its Bass–Serre tree. Let G be a f.g. group $G = \langle g_1, \ldots, g_m \rangle$, let $\{A_n, B_n\}$ be a sequence of pairs of nontrivial groups, and let $\{h_n : G \to A_n * B_n\}$, be a sequence of homomorphisms.

With the sequence of free products, $\{A_n * B_n\}$, we naturally associate their Bass–Serre trees that we denote, $\{T_n\}$, with a base point t_n (which is one of the vertices in T_n). Each homomorphism, $h_n : G \to A_n * B_n$, gives rise to an action λ_{h_n} of the group G on the Bass–Serre tree T_n. For each index n, we fix an element $\gamma_n \in A_n * B_n$, so that the homomorphism $\gamma_n h_n \gamma_n^{-1}$ has "minimal displacement," that is, the element γ_n satisfies the equality:
$$\max_{1 \leq u \leq m} d_{T_n}(t_n, \gamma_n h_n(g_u)\gamma_n^{-1}(t_n)) = \min_{\gamma \in A_n * B_n} \max_{1 \leq u \leq m} d_{T_n}(t_n, \gamma h_n(g_u)\gamma^{-1}(t_n)).$$
We further set μ_n to be:
$$\mu_n = \max_{1 \leq u \leq m} d_{T_n}(t_n, \gamma_n h_n(g_u)\gamma_n^{-1}).$$

First, suppose that the sequence of integers, $\{\mu_n\}$, is bounded. In that case, we can extract a subsequence of the homomorphisms $\{h_n\}$ (still denoted

$\{h_n\}$), that converges into a limit group (over free products) L, with an associated set of elliptics E_L. Furthermore, the sequence of homomorphisms $\gamma_n h_n \gamma_n^{-1}$ converges into a faithful action of L on some simplicial tree with trivial edge stabilizers, that we denote T. In that case either the entire group L is elliptic (i.e., $E_L = L$), or L is infinite cyclic, or it is freely decomposable and the stabilizer of each vertex group in T is elliptic. In this case, the limit group L is a free product of elliptic vertex groups (in T) with a (possibly trivial) free group.

Suppose that the sequence of integers, $\{\mu_n\}$, does not contain a bounded subsequence. We set $\{(X_n, x_n)\}_{n=1}^{\infty}$ to be the pointed metric spaces obtained by rescaling the metric on the Bass–Serre trees (T_n, t_n), by μ_n. (X_n, x_n) is endowed with a left isometric action of our f.g. group G via the homomorphisms $\gamma_n h_n \gamma_n^{-1}$. This sequence of actions of G on the metric spaces $\{(X_n, x_n)\}_{n=1}^{\infty}$ allows us to obtain an action of G on a real tree by passing to a Gromov–Hausdorff limit.

PROPOSITION 2 ([Pa], Proposition 2.3). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of δ_n-hyperbolic spaces with $\delta_{\infty} = \underline{\lim}\, \delta_n = 0$. Let H be a countable group isometrically acting on X_n. Suppose there exists a base point x_n in X_n such that for every finite subset P of H, the sets of geodesics between the images of x_n under P form a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a δ_{∞}-hyperbolic space X_{∞} endowed with a left isometric action of H.*

Our spaces $\{(X_n, x_n)\}_{n=1}^{\infty}$ endowed with the left isometric action of G, satisfy the assumptions of the proposition and they are all trees, so they are 0-hyperbolic. Hence, X_{∞} is a real tree endowed with an isometric action of G. By construction, the action of G on the real tree X_{∞} is nontrivial. Let $\{n_j\}_{j=1}^{\infty}$ be the subsequence for which $\{(X_{n_j}, x_{n_j})\}_{j=1}^{\infty}$ converges to the limit real tree X_{∞}, and let (Y, y_0) denote this (pointed) limit real tree.

For convenience, for the rest of this section we (still) denote the homomorphism $\gamma_{n_j} h_{n_j} \gamma_{n_j}^{-1} : G \to A_{n_j} * B_{n_j}$, by h_n. By passing to a further subsequence, we can assume that the sequence of homomorphisms $\{h_n\}$ converges into a limit group (over free products) that we denote L, with elliptic elements E_L, and an associated quotient map, $\eta : G \to L$, with kernel K. In the sequel, we call a limit group over free products that is obtained from a sequence of homomorphisms with unbounded stretching factors, a *strict* limit group over free products.

Before stating the following lemma, which describes some simple facts on the actions of strict limit groups over free products on real trees that are obtained from convergent sequences of homomorphisms, we review the basic notions of stable and super stable actions of groups on real trees. Recall that an action of a group G on a real tree T is called stable, if given any sequence of decreasing (closed) intervals, $\{I_j\}$, the increasing sequence of their

(pointwise) stabilizers stabilize, that is, there exists some index j_0 so that for every $j > j_0$, $\mathrm{stab}(I_j) = \mathrm{stab}(I_{j_0})$. The action is called super stable if for any interval I, for which $\mathrm{stab}(I)$ is nontrivial, and any nondegenerate subinterval, $J \subset I$, $\mathrm{stab}(I) = \mathrm{stab}(J)$ (for more details on the various stability properties of actions of groups on real trees, see Guirardel's paper [Gu]).

The following simple facts on the kernel of the action, K (see Definition 1), and the (strict) limit group L are important observations, and their proof is similar to the proof of Lemma 1.3 of [Se1].

LEMMA 3.
(i) *Elements in E_L fix points in Y.*
(ii) *L is f.g.*
(iii) *If Y is isometric to a real line then the limit group L has a subgroup of index at most 2, which is f.g. free Abelian.*
(iv) *If $g \in G$ stabilizes a tripod in Y then for all but finitely many n's, $g \in \ker(h_n)$ (recall that a tripod is a finite tree with 3 endpoints). In particular, if $g \in G$ stabilizes a tripod then $g \in K$.*
(v) *Let $g \in G$ be an element which does not belong to K. Then for all but finitely many n's, $g \notin \ker(h_n)$.*
(vi) *Every torsion element in L is elliptic, that is, it is in E_L.*
(vii) *Let $[y_1, y_2] \subset [y_3, y_4]$ be a pair of nondegenerate segments of Y and assume that the stabilizer of $[y_3, y_4]$ in L, $\mathrm{stab}([y_3, y_4])$, is nontrivial. Then $\mathrm{stab}([y_3, y_4])$ is an Abelian subgroup of L and:*

$$\mathrm{stab}([y_1, y_2]) = \mathrm{stab}([y_3, y_4]).$$

Hence, the action of L on the real tree Y is (super) stable.
(viii) *Let $H < G$ be a f.g. subgroup and suppose that $\eta(H) \subset E_L$. Then for all but finitely many n's, $h_n(H)$ is elliptic, that is, $h_n(H)$ is contained in a conjugate of A_n or B_n.*

Proof. Part (i) follows from the definition of the elliptic elements E_L. A limit group L is a quotient of a f.g. group, hence, it is finitely generated. If Y is a real line, then L contains a subgroup of index at most 2 that acts on the real line by isometries and preserves its orientation. Hence, this subgroup must be f.g. Abelian, and it contains no elliptic elements, so it contains no torsion. Therefore, L contains a f.g. Abelian subgroup of index at most 2. (iv), (v), and (vii), follow by the same argument that is used in the case of free and hyperbolic groups ([Se1], Lemma 1.3). Note that in analyzing limit groups over free groups, the action of the coefficient free group on its Cayley graph is trivial. Given a convergent sequence of homomorphisms into free products, the action of the target groups of the homomorphisms, $A_n * B_n$, on their associated Bass–Serre trees is not free, but stabilizers of edges in these trees are trivial. This is sufficient for the adapting the proof of Lemma 1.3 in [Se1] to prove parts (iv), (v), and (vii).

A torsion element in L is the image of an element $g \in G$, which is mapped to a torsion element by all the homomorphisms, $h_n : G \to A_n * B_n$, for large n. Hence, $h_n(g)$ is contained in a conjugate of A_n or B_n for large n, so g is mapped to an elliptic element in L, and (vi) follows. To prove (viii), let $H = \langle u_1, \ldots, u_m \rangle$. Since H is contained in E_L then there exists some n_0 so that for all $n > n_0$, all the elements $u_1, \ldots, u_m$ and the products $u_i u_j$, $i, j = 1, \ldots, m$, are mapped to elliptic elements by the homomorphism h_n. Therefore, by a standard argument of Serre [Ser], for all $n > n_0$, $h_n(H)$ is elliptic, that is, contained in a conjugate of A_n or B_n. $\qquad \square$

Recall that in limit groups over free and torsion-free hyperbolic groups, every nontrivial Abelian subgroup is contained in a unique maximal Abelian subgroup, and every maximal Abelian subgroup is f.g. and malnormal. This is clearly not the case in limit groups over free products, as every f.g. group is a limit group over free products. However, for the analysis of strict limit groups over free products, we are really interested only in nonelliptic Abelian subgroups, as only these occur as stabilizers of nondegenerate segments in real trees on which the strict limit groups act faithfully, and so that these real trees are obtained as a limit from a sequence of homomorphisms into free products. Nonelliptic Abelian subgroups have similar properties as Abelian subgroups in limit groups over free and torsion-free hyperbolic groups. The proof is similar to the proof of Lemma 1.4 in [Se1].

LEMMA 4. *With the notation of Definition* 1 *let* u_1, u_2, u_3 *be nontrivial elements of* L, *and suppose that at least one of the elements,* u_1, u_2, u_3, *is nonelliptic (i.e., not in* E_L), *and* $[u_1, u_2] = 1$ *and* $[u_1, u_3] = 1$. *Then:*

(i) u_1, u_2, u_3 *are nonelliptic and* $[u_2, u_3] = 1$.
(ii) *Let* $A < L$ *be a nonelliptic Abelian subgroup of* L. *Then* A *is contained in a unique maximal Abelian subgroup in* L, *which is its centralizer,* $C(A)$. *The centralizer of* A, $C(A)$, *intersects the set of elliptic elements,* E_L, *trivially.*
(iii) *Let* A *be a nonelliptic Abelian subgroup in* L. *Then the centralizer of* A, $C(A)$, *is almost malnormal in* L. $C(A)$ *is of index at most* 2 *in* $N(A)$, *the normalizer of* A, *and for each element* $\ell \in L$, $\ell \notin N(A)$, $\ell C(A) \ell^{-1}$ *intersects* A *trivially. Furthermore, if* $[N(A) : C(A)] = 2$ *then* $N(A)$ *is generated by* $C(A)$ *and an elliptic element of order* 2 *that conjugates each element in* $C(A)$ *to its inverse.*

Proof. Let g_1, g_2, g_3 be elements of G that are mapped to u_1, u_2, u_3. Since $[u_1, u_2] = 1$ and $[u_1, u_3] = 1$, and the elements u_1, u_2, u_3 are nontrivial, there exists some n_0, so that for all $n > n_0$, $[h_n(g_1), h_n(g_2)] = 1$ and $[h_n(g_1), h_n(g_3)] = 1$, and the elements, $h_n(g_1), h_n(g_2), h_n(g_3)$, are nontrivial. Since for some j, $1 \le j \le 3$, u_j is not elliptic, there exists some $n_j > n_0$, so that for all $n > n_j$, $h_n(g_j)$ is a hyperbolic element. Since for $n > n_j$, $h_n(g_1)$ is

nontrivial and commutes with $h_n(g_j)$, $h_n(g_1)$ is a hyperbolic element, and by the same argument so are $h_n(g_2)$ and $h_n(g_3)$. Hence, all the 3 elements, $h_n(g_1), h_n(g_2), h_n(g_3)$, are hyperbolic and have the same axis, so they all commute and $[u_2, u_3] = 1$.

By part (i) commutativity is transitive for nonelliptic elements of a limit group over free products. Hence, a nonelliptic Abelian subgroup is contained in a unique maximal Abelian subgroup, which is its centralizer, and the centralizer must be nonelliptic as well.

Let $A < L$ be a nonelliptic Abelian subgroup. Let $u \in N(A) \setminus C(A)$, and let $g \in G$ be an element that is mapped to u. Given a finite set of nontrivial elements $g_1, \dots, g_m$ that are mapped to $C(A)$, there exists some integer n_0, so that for every $n > n_0$, $h_n(g_j)$ are hyperbolic, $h_n(g_i)$ commutes with $h_n(g_j)$, and $h_n(g)$ does not commute with $h_n(g_i)$, for all $i, j = 1, \dots, m$, and $h_n(g)h_n(g_j)h_n(g)^{-1}$ commutes with all the elements $h_n(g_i)$, for $i, j = 1, \dots, m$. This imply that the elements $h_n(g_j)$ have the same axis in the Bass–Serre tree that is associated with the free product $A_n * B_n$, and the element $h_n(g)$ preserves this axis setwise, and must be an elliptic element. Hence, $h_n(g)$ is elliptic, and $h_n(g)h_n(g_j)h_n(g)^{-1} = h_n(g_j)^{-1}$, $j = 1, \dots, m$. Furthermore, $h_n(g)^2$ is an elliptic element that preserves the axis of the elements $h_n(g_j)$ pointwise, so $h_n(g)^2 = 1$. It follows that $ucu^{-1} = c^{-1}$ for every $c \in C(A)$, and $u^2 = 1$. By the same argument if $u_1, u_2 \in N(A) \setminus C(A)$, then $u_1 u_2 \in C(A)$, hence, $[N(A) : C(A)] = 2$.

Let $\ell \notin N(A)$, and let $t \in G$ be an element that is mapped to ℓ. Then there exists some index n_1, so that for all $n > n_1$, $h_n(t)$ maps the axis of $h_n(g_1), \dots, h_n(g_m)$ to a different axis that intersects the original axis in a bounded (or empty) set. Hence, $\ell C(A) \ell^{-1}$ intersects $C(A)$ trivially. $\square$

Lemma 3 shows that the action of L on the real tree Y is (super) stable. The original analysis of stable actions of groups on real trees applies to f.p. groups [Be-Fe], and the limit group L is only known to be f.g. by part (i) of Lemma 3. Still, given the basic properties of the action of L on the real tree Y that we already know, we are able to apply a generalization of Rips' work to f.g. groups obtained in [Se4] and [Gu]. In [Se4] and [Gu], the real tree Y is divided into distinct components, where on each component a subgroup of L acts according to one of several canonical types of actions. The theorem from [Se4] that we present, that was later corrected in [Gu], is going to be used extensively and its statement uses the notions and basic definitions that appear in the appendix of [Ri-Se1]. Hence, we refer a reader who is not yet familiar with these notions to that appendix and to [Be-Fe], [Be], and [Gu].

Note that the statement of Theorem 5 slightly generalizes Theorem 3.1 in [Se4], and also generalizes a special case of Theorem 5.1 in [Gu]. It does not require that the f.g. group G that acts on a real tree is freely indecomposable, but it rather associates with G a collection of conjugacy classes, that

we denote U, the elements in U are assumed to fix points when acting on the real tree, and we further assume that G admits no nontrivial free decomposition in which the elements in U can be conjugated into the factors. It is fairly straightforward to see that under the assumptions of trivial stabilizers of tripods, and the action of G being superstable, the proofs of Theorem 3.1 in [Se4] and that of Theorem 5.1 in [Gu], generalize to get the same conclusion under our assumptions.

THEOREM 5 (cf. [Se4], Theorem 3.1, [Gu], Theorem 5.1). *Let G be a f.g. group, let U be a collection of conjugacy classes in G, and suppose that G cannot be presented as a nontrivial free product in which the elements in U can be conjugated into the factors. Let G admits a super stable isometric action on a real tree Y, so that the elements in the collection U fix points in Y. Assume that the stabilizer of each tripod in Y is trivial.*

(1) *There exist canonical orbits of subtrees of Y: $Y_1, \ldots, Y_k$ with the following properties*:
   (i) *gY_i intersects Y_j at most in one point if $i \neq j$.*
   (ii) *gY_i is either identical with Y_i or it intersects it at most in one point.*
   (iii) *The action of $\operatorname{stab}(Y_i)$ on Y_i is either discrete or it is of axial type or IET type.*
(2) *G is the fundamental group of a graph of groups with*:
   (i) *Vertices corresponding to orbits of branching points with nontrivial stabilizer in Y.*
   (ii) *Vertices corresponding to the orbits of the canonical subtrees $Y_1, \ldots, Y_k$ which are of axial or IET type. The groups associated with these vertices are conjugates of the stabilizers of these components. To a stabilizer of an IET component is an associated 2-orbifold. All boundary components and branching points in this associated 2-orbifold stabilize points in Y. For each such stabilizer, we add edges that connect the vertex stabilized by it and the vertices stabilized by its boundary components and branching points.*
   (iii) *Edges corresponding to orbits of edges between branching points with nontrivial stabilizer in the discrete part of Y with edge groups which are conjugates of the stabilizers of these edges.*
   (iv) *Edges corresponding to orbits of points of intersection between the orbits of $Y_1, \ldots, Y_k$.*

Before concluding our preliminary study of limit groups over free products that appear as limits of sequences of homomorphisms with unbounded stretching factors $\{\mu_n\}$, and their actions on the limit real tree, we present the following basic facts that are necessary in the sequel.

PROPOSITION 6. *Suppose that L is a (strict) limit group over free products, that is obtained as a limit of homomorphisms into free products with*

unbounded stretching factors $\{\mu_n\}$. E_L *is its set of elliptic elements, and the limit real tree on which* L *acts that is obtained from this sequence of homomorphisms is* (Y, y_0). *Suppose further that* L *does not admit a nontrivial free decomposition in which all the elements in the set* E_L *can be conjugated into the various factors. Then:*

 (i) *Stabilizers of nondegenerate segments which lie in the complement of the discrete and axial parts of* Y *are trivial in* L.
 (ii) *The (set) stabilizer of an axial component in* Y *is either a maximal Abelian subgroup in* L, *or it contains a maximal Abelian subgroup in* L *as a subgroup of index 2.*
 (iii) *Let* A *be the maximal Abelian subgroup that is contained in the set stabilizer of an axial component in* Y. A *can be presented as a direct sum* $A = A_1 + A_2$, *where* A_1 *is the (possibly trivial) point stabilizer of the axial component, and* A_2 *is a f.g free Abelian group that acts freely on the axial component, and* A_2 *has rank at least 2.*

Proof. Since the elements in E_L fix points in the limit tree Y (part (i) in Lemma 3), the action of L on the real tree Y satisfies the conclusion of Theorem 5. Hence, the stabilizer of a segment in an IET component in Y fixes the entire IET component, and in particular it fixes a tripod. By part (iv) of Lemma 3 a stabilizer of a tripod is trivial, hence, so is the stabilizer of a nondegenerate segment in an IET component in Y.

Let Ax be an axis of an axial component in Y, and let $\mathrm{stab}(Ax)$ be its set stabilizer. Let $\mathrm{stab}^+(Ax)$ be the subgroup of $\mathrm{stab}(Ax)$ that preserve the orientation of Ax. By the same argument that was used in the proof of Lemma 4, $\mathrm{stab}^+(Ax)$, is Abelian. Since $\mathrm{stab}(Ax)$ normalizes $\mathrm{stab}^+(Ax)$, Lemma 4 implies that the index of $\mathrm{stab}^+(Ax)$ in $\mathrm{stab}(Ax)$ is bounded by 2.

Let $A = \mathrm{stab}^+(Ax)$, and let $A_1 < A$ be the point stabilizer of Ax. Then, by Theorem 5 (the structure of an axial component) there exists a short exact sequence: $1 \to A_1 \to A \to B \to 1$, where B is a f.g. free Abelian group. Since A is Abelian and B is free Abelian, the short exact sequence splits, and $A = A_1 + A_2$, where A_2 is isomorphic to B, hence, A_2 is f.g. free Abelian. $\square$

By Theorem 5 and Proposition 6, a nontrivial strict limit group over free products, which is not a cyclic group, admits a nontrivial virtually Abelian decomposition (i.e., a graph of groups with virtually Abelian edge groups). To further study the algebraic structure of a strict limit group, we need to construct its canonical virtually Abelian JSJ decomposition. However, unlike the case of limit groups over free and hyperbolic groups, in constructing the virtually Abelian JSJ decomposition of a strict limit group over free products, we will not be interested in all the virtually Abelian decompositions of L, but only in its *admissible* ones. Admissible virtually Abelian decompositions of a limit group over free products L, are those virtually Abelian decompositions in which all the elements in E_L are elliptic (i.e., can be conjugated into

nonvirtually-Abelian, non-QH vertex groups), and for which the nontrivial elements in the (nontrivial) maximal Abelian subgroups that are contained in the virtually Abelian edge groups are not in E_L. Note that since a nontrivial strict limit group over free products admits a virtually Abelian decomposition in which the elements E_L can be conjugated into non-QH, nonvirtually-Abelian vertex groups, and the (nontrivial) maximal Abelian subgroups of edge groups are not in E_L (except for the identity element), if a strict limit group over free products is not virtually Abelian nor a Fuchsian group, it admits an admissible (virtually Abelian) decomposition, hence, its (virtually) Abelian JSJ decomposition is nontrivial.

To construct the virtually Abelian JSJ decomposition of a strict limit group over free products, we need to study some basic properties of admissible splittings. We start with the following lemma, which is identical to Lemma 2.1 in [Se1] (the proofs are identical as well).

LEMMA 7. *Let L be a strict limit group over free products with set of elliptics E_L, and suppose that L admits no free product in which the elements in E_L can be conjugated into the various factors. Let A be a nonelliptic Abelian subgroup in L, and let M be the normalizer of A in L. Suppose that M is Abelian. Then:*

(i) *If $L = U *_A V$, and the elements in E_L can be conjugated into U and V, and M is not cyclic, then M can be conjugated into either U or V.*

(ii) *If $L = U*_A$, and the elements in E_L can be conjugated into U, and M is not cyclic, then either M can be conjugated into U, or M can be conjugated onto M', so that $L = U *_A M'$.*

Unlike limit groups over free and torsion-free hyperbolic groups in which normalizers of nontrivial Abelian subgroups are Abelian, by Proposition 6 the normalizers of nonelliptic Abelian subgroups in L are either Abelian or virtually Abelian, and if they are not Abelian, the Abelian centralizers of these (nonelliptic) Abelian subgroups are contained in their normalizers as subgroups of index 2. Lemma 7 deals with the case in which the normalizer of such an Abelian subgroup is Abelian. To construct the JSJ decomposition of limit groups over free products, we still need to analyze splittings over nonelliptic Abelian subgroups with virtually Abelian, non-Abelian normalizers.

LEMMA 8. *Let L be a limit group over free products, and let A be a nonelliptic Abelian subgroup in L. Let E_L be the set of elliptics in L, and suppose that L admits no free product decomposition in which the elements of E_L can be conjugated into the factors. Let $C(A)$ be the centralizer of A, let M be the normalizer of A, and suppose that $[M : C(A)] = 2$. Then:*

(i) *If $L = U *_A V$, and all the elements in E_L can be conjugated into U or V, then either M can be conjugated into either U or V, or M can be conjugated onto M', and M' inherits an Abelian amalgamation: $M' =$*

$U_1 *_A V_1$, *where* $U_1 < U$, $V_1 < V$, $[U_1 : A] = [V_1 : A] = 2$, *and both* U_1 *and* V_1 *are generated by* A *and an element of order* 2. *In this case,* M *is the semidirect product of* A *with an infinite dihedral group. In this case, we can modify the given Abelian decomposition, and obtain a virtually Abelian decomposition,* $L = U *_{U_1} M' *_{V_1} V$.

(ii) *If* $L = U*_A$, *and the elements in* E_L *can be conjugated into* U, *then either* M *can be conjugated into* U, *or* M *can be conjugated onto* M', *and* M' *inherits an Abelian amalgamation:* $M' = U_1 *_A V_1$, *where* $U_1 < U$, $V_1 < tUt^{-1}$, *where* t *is a Bass–Serre generator that is associated with the splitting,* $L = U*_A$. $[U_1 : A] = [V_1 : A] = 2$, *and both* U_1 *and* V_1 *are generated by* A *and an element of order* 2. *In this case,* M *is the semidirect product of* A *with an infinite dihedral group. In the HNN case,* $L = U*_A$, *we can modify the given Abelian decomposition, and obtain a virtually Abelian decomposition,* $L = (U *_{U_1} M')*_{V_1}$, *where with* V_1 *there are two associated embeddings, one into* M' *and one into* tUt^{-1}. *The graph of groups that is associated with this decomposition contains two vertices (with vertex groups,* U *and* M'*), and two edges with edge groups,* U_1 *and* V_1.

Proof. Let $L = U *_A V$ and suppose that M, the normalizer of A, that contains the centralizer of A as a subgroup of index 2, is not elliptic. Then M preserves (setwise) an axis in the Bass–Serre tree that is associated with the amalgamated product $U *_A V$. Since A preserves this axis pointwise, and M contains an (elliptic) element that acts on the axis as an inversion, M/A acts on the axis as an infinite dihedral group. Hence, it inherits from it a splitting: $M = U_1 *_A V_1$, where U_1 and V_1 contain A as a subgroup of index 2, and they are both obtained from A by adding to it an elliptic element of order 2. If we start with the Bass–Serre tree that is associated with $U *_A V$, add a vertex in the middle of the edge that is stabilized by A and connected to the vertices that are stabilized by U and V, and then fold the couple of edges that are stabilized by A and associated with the elements of order 2 in U_1 and V_1, we obtain the splitting: $L = U *_{U_1} M' *_{V_1} V$, that is, a splitting of L with 3 vertex groups U, V, and M', and two edge groups, U_1 and V_1.

Let $L = U*_A$. The argument that we use in this case is similar. Suppose that M, the normalizer of A, is not elliptic. In this case, M preserves (setwise) an axis in the Bass–Serre tree that is associated with the HNN extension $U*_A$. A preserves this axis pointwise, and M contains an (elliptic) element that acts on the axis as an inversion, hence, M/A acts on the axis as an infinite dihedral group. Therefore, as in the amalgamated product case, M inherits from this action a splitting: $M = U_1 *_A V_1$, where U_1 and V_1 contain A as a subgroup of index 2, and they are both obtained from A by adding to it an elliptic element of order 2. $U_1 < U$, and $V_1 < tUt^{-1}$, for an appropriate Bass–Serre generator t. If we start with the Bass–Serre tree that is associated with $U*_A$,

add a vertex in the middle of the edge that is stabilized by A and connects the vertices that are stabilized by U and tUt^{-1}, and then fold the couple of edges that are stabilized by A and associated with the elements of order 2 in U_1 and V_1, we obtain the splitting: $L = (U *_{U_1} M') *_{V_1}$, where V_1 embeds into V_1 and into tUt^{-1}, that is, the limit group L admits a graph of groups decomposition with two vertex groups, stabilized by U and M', and two edges in between these two vertices, one edge is stabilized by U_1 and the second by V_1. $\qquad\square$

According to Lemma 7, we replace each Abelian splitting of L of the form $L = U *_A$, in which all the elements in E_L can be conjugated into U, A is a nonelliptic Abelian subgroup in L, and the centralizer of A which is also its normalizer is denoted by M, and M cannot be conjugated into U, by the amalgamated product $L = U *_A M'$ (where M' is a conjugate of M). According to part (i) of Lemma 8, we replace each Abelian splitting of L of the form $L = U *_A V$, in which all the elements in E_L can be conjugated into U and V, A is a nonelliptic subgroup in L, and M, the normalizer of A, contains the centralizer of A as a subgroup of index 2, and M cannot be conjugated into U nor V, by the amalgamated product $L = U *_{U_1} M' *_{V_1} V$, where M' is a conjugate of M. If $L = U*_A$, the elements in E_L can be conjugated into U, A is nonelliptic in L, the normalizer M of A in L contains the centralizer of A as a subgroup of index 2, and M cannot be conjugated into U, then we replace the given HNN extension by a graph of groups with two vertices and two edges between them, according to part (ii) of Lemma 8, $L = (U *_{U_1} M') *_{V_1}$.

By performing these replacements, we get that every nonelliptic Abelian subgroup of L with a noncyclic centralizer is contained in a vertex group in all the admissible virtually Abelian splittings of L, that is, splittings in which edge groups are nonelliptic Abelian, or edge groups that contain nonelliptic Abelian subgroups as subgroups of index 2, and all the elements in E_L can be conjugated into the vertex groups. This will allow us to use acylindrical accessibility in analyzing all the Abelian splittings of the limit group over free products L that are obtained from converging sequences of homomorphisms into free products.

DEFINITION 9 ([Se4], [De], [We]). A splitting of a group H is called k-*acylindrical* if for every element $h \in H$ which is not the identity, the fixed set of h when acting on the Bass–Serre tree corresponding to the splitting has diameter at most k. Following Delzant [De], and Weidmann [We], we say that a splitting of H is (k, C)-acylindrical if the stabilizer of a path of length bigger than k in the Bass–Serre tree corresponding to the splitting has stabilizer of cardinality at most C.

If a strict limit group over free products L can be written in the form $L = V_1 *_{A_1} V_2 *_{A_2} V_3 *_{A_3} V_4$, where A_1, A_2, A_3 are subgroups of a maximal Abelian subgroup M, which is its own normalizer, and M is a subgroup

of V_1, then one can modify the corresponding graph of groups to a tripod of groups with V_1 in the center, and V_2, V_3, V_4 at the 3 roots. Similarly if A_1, A_2, A_3 are subgroups of a maximal Abelian subgroup, which is of index 2 in its normalizer M, then if M is contained in one of the vertex groups V_i, $i = 1, \ldots, 4$, then one can modify the corresponding graph of groups to a tripod of groups in the same way. If M is not contained in one of the vertex groups, V_i, $i = 1, \ldots, 4$, then one can modify the given splitting to a virtually Abelian splitting which is a tree with one root and 4 vertices connected to it, where M is placed at the root, and the subgroups V_i, $i = 1, \ldots, 4$, are placed at the vertices that are adjacent to the root.

Since by Lemma 4 the centralizer of a nonelliptic Abelian subgroup is almost malnormal, the Bass–Serre trees corresponding to these tripods and quadruple of groups are $(2, 2)$-acylindrical. This folding and sliding operation generalizes to an arbitrary (finite) virtually Abelian splitting of a limit group over free products over normalizers of nonelliptic Abelian subgroups, where all the elements in E_L can be conjugated into non-Abelian, non-QH vertex groups.

LEMMA 10. *Let L be a limit group over free products that does not admit a free splitting in which all the elements in E_L can be conjugated into the various factors. A splitting of L, in which all the edge groups are nonelliptic Abelian subgroups in L, and in which all the elements in E_L can be conjugated into non-QH, non-Abelian vertex groups, can be modified using Lemmas 7 and 8 to a virtually Abelian splitting (of L) in which all the normalizers of nonelliptic Abelian subgroups with noncyclic centralizers can be conjugated into non-QH vertex groups, and so that the obtained virtually Abelian splitting is $(2, 2)$-acylindrical.*

Proof. Let L be a limit group over free products that admits no free factorization in which the elements of E_L can be conjugated into the factors. Let Λ be a graph of groups with fundamental group L with nonelliptic Abelian edge groups. If the normalizer of an Abelian edge group in Λ cannot be conjugated into a vertex group in Λ, we perform the modification that appears in part (ii) of Lemma 7 in case the normalizer of an edge group is Abelian, and the modification of parts (i) and (ii) in Lemma 8 in case the normalizer of an edge group is not Abelian, where these modifications are applicable. After performing these modifications, and sliding operations, so that the fixed set of a nonelliptic Abelian subgroup is star-like, we obtain a graph of groups Λ', with virtually Abelian edge groups, in which every nonelliptic Abelian subgroup with noncyclic centralizer can be conjugated into a non-QH vertex group. In the Bass–Serre tree that corresponds to Λ' the fixed set of every nonelliptic element has diameter bounded by 2. Since the centralizers of nonelliptic Abelian subgroups are almost malnormal, the stabilizers of paths

of length 3 in Λ' are either trivial, or a cyclic subgroup of order 2. Hence, Λ' is $(2, 2)$-acylindrical. $\square$

Lemma 10 shows that if in all the admissible virtually Abelian splittings of L, all the normalizers of noncyclic Abelian subgroups can be conjugated into vertex groups, these virtually Abelian splittings are $(2, 2)$-acylindrical. This acylindricity finally enables one to construct the canonical virtually Abelian JSJ decomposition of a strict limit group over free products (see Section 2 of [Se1]).

THEOREM 11 (cf. [Se1], Theorem 2.7). *Suppose that L is a strict limit group over free products with set of elliptics E_L, so that L admits no free decomposition in which the elements in E_L can be conjugated into the various factors. There exists a reduced unfolded splitting of L with virtually Abelian edge groups, which we call the virtually Abelian JSJ (Jaco–Shalen–Johannson) decomposition of L, with the following properties*:

(i) *Every canonical maximal QH subgroup (CMQ) of L is conjugate to a vertex group in the JSJ decomposition. Every QH subgroup of L, in which all the elements in E_L can be conjugated into vertex groups that are adjacent to the QH subgroup or into torsion elements in the QH subgroup, can be conjugated into one of the CMQ subgroups of L. Every vertex group in the JSJ decomposition which is not a CMQ subgroup of L is elliptic in any Abelian splitting of L under consideration.*

(ii) *Every CMQ subgroup is a Fuchsian group (in general, with torsion), where all its torsion elements are elliptic in L. The edge groups that are connected to a CMQ subgroup, that are all cyclic, may be elliptic.*

(iii) *Every edge group that is not connected to a CMQ vertex group in the JSJ decomposition, or an edge group that is connected to a virtually Abelian vertex group, contains an Abelian subgroup of index at most 2, and this Abelian subgroup is nonelliptic.*

(iv) *A one edge admissible Abelian splitting $L = D *_A E$ or $L = D*_A$ (i.e., a splitting in which A is a nonelliptic Abelian subgroup, and all the elements in E_L can be conjugated into D or E), which is hyperbolic in another such elementary admissible Abelian splitting, is obtained from the virtually Abelian JSJ decomposition of L by cutting a 2-orbifold corresponding to a CMQ subgroup of L along a weakly essential s.c.c.*

(v) *Let Θ be a one edge admissible splitting of L along a nonelliptic Abelian subgroup, $L = D *_A E$ or $L = D*_A$. Suppose that the given elementary admissible splitting is elliptic with respect to any other such elementary admissible Abelian splitting of L. Then Θ is obtained from the JSJ decomposition of L by a sequence of collapsings, foldings, conjugations, and finally possibly unfoldings that reverse the foldings that are performed according to part (i) of Lemma 7 and parts (i) and (ii) of Lemma 8.*

(vi) *If JSJ_1 is another JSJ decomposition of L, then JSJ_1 is obtained from the JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see Section 1 of [Ri-Se2] for these notions).*

Proof. By Lemma 10, the admissible splittings of the ambient limit group (over free products) L that we consider, have the property that all the elliptic elements E_L in L can be conjugated into non-QH, non-Abelian vertex groups, and every noncyclic, nonelliptic Abelian subgroup of L can also be conjugated into a non-QH vertex group. Since L admits no free decompositions in which the elements E_L can be conjugated into the factors, there is no pair of admissible elliptic-hyperbolic splittings, that is, all the admissible splittings along noncyclic, nonelliptic Abelian groups that we consider, are elliptic-elliptic with respect to all the admissible splittings along nonelliptic Abelian groups that we consider.

Since the modifications of admissible Abelian splittings along nonelliptic Abelian subgroups that are performed according to Lemmas 7 and 8, are performed only in case the centralizers of (nonelliptic) edge groups are noncyclic, in case centralizers of nonelliptic edge groups are (infinite) cyclic, we consider only cyclic edge groups (and not dihedral ones). Hence, the only admissible hyperbolic–hyperbolic splittings along nonelliptic Abelian edge groups are pairs of splittings along infinite cyclic groups. For these, we can apply [Ri-Se2] (this part of [Ri-Se2], the construction of the quadratic part (Section 5 in the paper), applies to f.g. groups, and not only to f.p. ones), that produces a finite collection of CMQ subgroups of L, and a quadratic decomposition of L with the CMQ subgroups as part of the vertex groups, so that every admissible splitting of L along a nonelliptic cyclic group, so that this admissible splitting is hyperbolic with respect to another admissible splitting, is obtained from the quadratic decomposition of L by cutting one of the CMQ subgroups along a s.c.c. and possibly collapsing the rest of the splitting.

Given the quadratic decomposition of L, to complete the construction of the (virtually Abelian) JSJ decomposition of L, we successively refine the quadratic decomposition using admissible splittings that are elliptic with respect to it. This refinement process terminates after finitely many steps, since all the obtained splittings are $(2, 2)$ acylindrical, and by [We] this implies a bound on the combinatorial complexity of the obtained splitting. All the properties of the obtained (virtually) Abelian JSJ decomposition of L follow in the same way as in Section 7 of [Ri-Se2]. $\qquad\square$

2. A descending chain condition

In Section 4 of [Se1], we were able to use the cyclic JSJ decomposition of a (F_k) limit group, in order to show that (F_k) limit groups are f.p. and that a f.g. group is a limit group if and only if it is ω-residually free. For limit groups

over a torsion-free hyperbolic group, we were able to prove similar d.c.c. and a.c.c. as in the case of a free group, even though a limit group over hyperbolic groups need not be finitely presented.

As every f.g. group can be a limit group over free products (where the ambient group is elliptic), limit groups over free products do not satisfy the d.c.c. and a.c.c. conditions that limit groups over free and hyperbolic groups do satisfy. However, weaker principles do hold for these limit groups, and these are enough to construct Makanin–Razborov diagrams, that encode sets of solutions to systems of equations over free products. As we will see one of the keys to formulate and prove the d.c.c. and a.c.c. principles that we present for limit groups over free products, is our consideration of limit groups over the entire class of free products, and not over a given one.

We start with a d.c.c. for limit groups over free products which is a key to our entire approach. It uses the techniques that were used to prove a general d.c.c. for limit groups over hyperbolic groups, but it is not as general as in the case of limit groups over free and hyperbolic groups.

DEFINITION 12. Let G be a f.g. group. On the set of limit groups over free products that are quotients of G, together with the quotient maps from G to these limit groups, we define a partial order. Let L_1, L_2 be two limit groups over free products that are quotients of G, with sets of elliptics, E_{L_1}, E_{L_2}, in correspondence, and with prescribed quotient maps $\eta_i : G \to L_i$, $i = 1, 2$. We write that $(L_1, \eta_1) > (L_2, \eta_2)$, if there exists an epimorphism: $\tau : L_1 \to L_2$, that maps the elliptics in L_1 into the elliptics of L_2, $\tau(E_{L_1}) \subset E_{L_2}$, and for which $\tau : L_1 \to L_2$ satisfies either:

(1) τ has a nontrivial kernel.
(2) τ is an isomorphism, and $\tau(E_{L_1})$ is a proper subset of E_{L_2}.

If there exists an isomorphism $\tau : L_1 \to L_2$ that maps the elliptics in L_1 onto the elliptics in L_2, and for which: $\eta_2 = \tau \circ \eta_1$, we say that (L_1, η_1) is in the same equivalence class as (L_2, η_2). Note that the relation that is defined on the limit quotients (over free products) of a f.g. group is a partial order.

THEOREM 13. *Let G a f.g. group. Every strictly decreasing sequence of limit groups over free products that are quotients of G:*

$$(L_1, \eta_1) > (L_2, \eta_2) > (L_3, \eta_3) > \cdots$$

for which:

(1) *the maps: $\tau_i : L_i \to L_{i+1}$, that satisfy: $\eta_{i+1} = \tau_i \circ \eta_i$, are proper quotient maps (i.e., have nontrivial kernels)*
(2) *the maps τ_i do not map nontrivial elements in E_{L_i} to the identity element in L_{i+1}*

terminates after finitely many steps.

Proof. The argument that we use is a modification of the argument that is used to prove Theorem 1.12 in [Se3]. Suppose that there exists a f.g. group G for which there exists an infinite decreasing sequence of limit groups over free products that are quotients of G: $L_1 > L_2 > L_3 > \cdots$ that satisfy the conditions of the theorem. W.l.o.g. we may assume that the f.g. group G is a free group F_d, for some integer d. We fix F_d, where d is the minimal positive integer for which there exists an infinite descending chain of limit groups over free products so that consecutive quotient maps, $\tau_i : L_i \to L_{i+1}$, have nontrivial kernels and do not map nontrivial elliptic elements to the identity element, and fix a free basis for F_d, $F_d = \langle f_1, \ldots, f_d \rangle$. We set C to be the Cayley graph of F_d with respect to the given generating set, and look at an infinite decreasing sequence constructed in the following way. We set R_1 to be a limit group over free products, which is a quotient of F_d, with the following properties:

(1) R_1 is a proper quotient of F_d.
(2) R_1 can be extended to an infinite decreasing sequence of limit groups over free groups: $R_1 > L_2 > L_3 > \cdots$, that satisfy the conditions of the theorem.
(3) The map $\eta_1 : F_d \to R_1$ maps to the identity the maximal number of elements in the ball of radius 1 in the Cayley graph C, among all possible maps from F_d to a limit group over free products L, that satisfy properties (1) and (2).

We continue iteratively. At step n, given the finite decreasing sequence $R_1 > R_2 > \cdots > R_{n-1}$, we choose the limit group over free products, R_n, to satisfy:

(1) R_n is a proper quotient of R_{n-1}.
(2) The finite decreasing sequence of limit groups over free products: $R_1 > R_2 > \cdots > R_n$ can be extended to an infinite decreasing sequence that satisfies the conditions of the theorem.
(3) The map $\eta_n : F_d \to R_n$ (that is obtained as a composition of the map $F_d \to R_1$ with the sequence of proper epimorphisms: $R_i \to R_{i+1}$, $i = 1, \ldots, n-1$), maps to the identity the maximal number of elements in the ball of radius n in the Cayley graph C, among all the possible maps from F_d to a limit group over free products, L_n, that satisfy the properties (1) and (2).

To prove Theorem 13, we will show that the last descending sequence we constructed terminates after finitely many steps. With the decreasing sequence $R_1 > R_2 > \cdots$ we associate a sequence of homomorphisms into free products: $\{h_n : F_d \to A_n * B_n\}$. For each index n, R_n is a quotient of F_d, hence, R_n is generated by d elements that are the image of the fixed generators of F_d under the quotient map η_n.

R_n is a limit group over free products. Hence, R_n, with its set of elliptics E_{R_n}, is obtained from a convergent sequence of homomorphisms $\{u_s : G_n \to$

$C_s * D_s$}, where G_n is a f.g. group. Since R_n is generated by the image of the elements $f_1, \ldots, f_d$ under the quotient map η_n, for large enough s, the images $u_s(G_n)$ are d-generated groups, and furthermore, they are generated by the images of d elements in the f.g. group G_n, that are mapped by the quotient map $\nu_n : G_n \to R_n$ onto the elements $\eta_n(f_1), \ldots, \eta_n(f_d)$. Hence, we may assume that the limit groups over free products, R_n, are obtained as limit groups from a sequence of homomorphisms $\{v_s : F_d \to C_s * D_s\}$, and the image of the fixed generating set of the free group F_d, is the set of elements $\eta_n(f_1), \ldots, \eta_n(f_d)$.

For each index n, we pick h_n to be a homomorphism $h_n : F_d \to A_n * B_n$, so that h_n is a homomorphism $v_s : F_d \to C_s * D_s$ for some large index s, so that h_n satisfies the following two conditions:

(i) Every element in the ball of radius n of C, the Cayley graph of F_d, that is mapped by the quotient map $\eta_n : F_d \to R_n$ to the trivial element, is mapped by h_n to the trivial element in $A_n * B_n$. Every such element that is mapped to a nontrivial element by η_n, is mapped by h_n to a nontrivial element in $A_n * B_n$.

(ii) Every element in the ball of radius n of C, the Cayley graph of F_d, that is mapped by the quotient map $\eta_n : F_d \to R_n$ to an elliptic element, is mapped by h_n to an elliptic element in $A_n * B_n$. Every such element that is mapped to a nonelliptic element by η_n, is mapped by h_n to a nonelliptic element in $A_n * B_n$.

From the sequence $\{h_n\}$, we can extract a subsequence that converges into a limit group over free products, that we denote R_∞. By construction, the limit group R_∞ is the direct limit of the sequence of (proper) epimorphisms:

$$F_d \to R_1 \to R_2 \to \cdots .$$

Let $\eta_\infty : F_d \to R_\infty$ be the canonical quotient map. Our approach towards proving the termination of given descending chains of limit groups over free products is based on studying the structure of the limit group R_∞, and its associated quotient map η_∞. We start this study by listing some basic properties of them.

LEMMA 14.

(i) R_∞ is not finitely presented.

(ii) R_∞ cannot be presented as the free product of a f.p. group and elliptic subgroups (factors).

(iii) Let $R_\infty = U_1 * \cdots * U_t * F$ be the most refined (Grushko) free decomposition of R_∞ in which the elliptic elements in R_∞, E_{R_∞}, can be conjugated into the various factors, and F is a f.g. free group. Then there exists an index j, $1 \leq j \leq t$, for which:

(1) U_j is not finitely presented nor elliptic.

(2) *If B is a f.g. subgroup of F_d for which $\eta_\infty(B) = U_j$, then U_j is a strict limit group over free products of a subsequence of the restricted homomorphisms, $h_n|_B$.*

Proof. To prove part (i), suppose that R_∞ is f.p., that is:

$$R_\infty = \langle g_1, \ldots, g_d | r_1, \ldots, r_s \rangle.$$

Then for some index n_0, and every index $n > n_0$, $h_n(r_j) = 1$ for $j = 1, \ldots, s$. This implies that for some index $n_1 > n_0$, and every index $n > n_1$, each of the groups R_n is a quotient of R_∞, by a quotient map that send the generating set $g_1, \ldots, g_d$ of R_∞ to the elements $\eta_n(f_1), \ldots, \eta_n(f_d)$, a contradiction.

Suppose that $R_\infty = V_1 * \cdots * V_t * M$ where M is f.p. and each of the factors V_j is elliptic. Let $B_1, \ldots, B_t$ and D be f.g. subgroups of F_d, for which $\eta_\infty(B_j) = V_j$ for $j = 1, \ldots, t$, and $\eta_\infty(D) = M$. W.l.o.g. we may assume that the free group F_d is generated by the collection of the subgroups $B_1, \ldots, B_t, D$.

Since the factors V_j, $j = 1, \ldots, t$, are elliptic, and since the subgroups, B_j, $j = 1, \ldots, t$, are f.g. for every index j, $j = 1, \ldots, t$, there exists an index n_j, so that for every index $n > n_j$, the image $\eta_n(B_j)$ is elliptic. Since the maps $\tau_i : R_i \to R_{i+1}$ do not map nontrivial elliptic elements (in E_{R_i}) to the identity element, $\eta_n(B_j)$ is isomorphic to $\eta_\infty(B_j) = V_j$ via the map $\eta_\infty \circ \eta_n^{-1}$.

The factor M is assumed f.p., hence, if $D = \langle d_1, \ldots, d_s \rangle$, then $M = \langle d_1, \ldots, d_s | r_1, \ldots, r_u \rangle$. There exists an index n_0, for which for every index $n > n_0$, $\eta_n(r_i) = 1$, for $i = 1, \ldots, u$.

Let $m_0 > n_j$ for $j = 0, \ldots, t$. By our arguments, from the universality of free products, all the relations that hold in R_∞ hold in $\eta_{m_0}(F_d) = R_{m_0}$. Hence, R_{m_0} is a quotient of R_∞, where the quotient map maps the prescribed generators of R_∞ to the prescribed generators of R_{m_0} (i.e., the corresponding images of the given set of generators $F_d = \langle f_1, \ldots, f_d \rangle$). Since R_{m_0+1} is a proper quotient of R_{m_0}, this implies that R_{m_0+1} is a proper quotient of R_∞, again by a map that maps the prescribed set of generators of R_∞ to the prescribed set of generators of R_{m_0+1}, which clearly contradicts our assumptions that the sequence of limit groups $\{R_j\}$ is strictly decreasing with R_{i+1} being a proper quotient of R_i for every index i, and the limit group R_∞ is the direct limit of this decreasing sequence. This concludes the proof of part (ii).

To prove part (iii), note that (1) in part (iii) follows from part (ii). Every factor U_j of the limit group L that is not elliptic is a strict limit group that is obtained from a sequence of homomorphisms of some f.g subgroup of F_d, and (2) follows. $\qquad\square$

R_∞ is a limit group over free products which is a (proper) quotient of all the limit groups over free products, $\{R_n\}$. For each index n, the limit group R_n was chosen to maximize the number of elements that are mapped to the identity in the ball of radius n of F_d by the quotient map $\eta_n : F_d \to R_n$, among all the proper limit (over free products) quotients of R_{n-1} that admit

an infinite descending chain of limit groups over free products that satisfy the conditions of Theorem 13. If R_∞ admits an infinite descending chain of limit groups over free products:

$$R_\infty \to L_1 \to L_2 \to \cdots$$

that satisfy the properties in Theorem 13, then the limit group (over free products) L_1 admits an infinite descending chain of limit groups that satisfy the conditions of Theorem 13, and since it is a proper quotient of R_∞, for large enough index n, the quotient map $\nu_n : F_d \to L_1$ maps to the identity strictly more elements of the ball of radius n in the Cayley graph of F_d, than the map $\eta_n : F_d \to R_n$, a contradiction. Hence, R_∞ does not admit an infinite descending chain of limit groups over free products that satisfy the conditions of Theorem 13.

To continue the proof of Theorem 13, that is, to contradict the existence of the infinite descending chain of limit groups over free products that satisfies the conditions of the theorem, we need a modification of the shortening procedure that was used in [Se1] for (F_k) limit groups, and in [Se3] for limit groups over hyperbolic groups. Since the description of the shortening procedure is rather long and involved, we prefer not to repeat it, and refer the interested reader to Section 3 of [Se1]. The same construction that appears in [Se1] applies to (strict) limit groups over free products.

Given a f.g. group G, and a sequence of homomorphisms into free products: $\{u_s : G \to A_s * B_s\}$, that converges into a (strict) limit group over free products, L, the shortening procedure constructs another (sub) sequence of homomorphisms from a free group F_d (where the f.g. group G is generated by d elements), $\{v_{s_n} : F_d \to A_{s_n} * B_{s_n}\}$, so that the sequence of homomorphisms v_{s_n} converges to a limit group over free products SQ, and there exists a natural epimorphism $L \to SQ$, that maps the elliptic elements in L, E_L, into the elliptic elements in SQ, E_{SQ}. Furthermore, although the map from L to SQ is not a monomorphism in general, no nontrivial elliptic element in E_L is mapped to the identity element in SQ.

DEFINITION 15. We call the limit group over free products, SQ, that is obtained by the shortening procedure, a shortening quotient of the limit group (over free products) L.

By construction, a shortening quotient of a limit group over free products is, in particular, a quotient of that limit group. In the case of freely indecomposable F_k-limit groups, a shortening quotient is always a proper quotient ([Se1], Claim 5.3). In the case of (strict) limit groups over free products the natural epimorphism from a limit group onto its shortening quotient may be an isomorphism, for example, if the limit group is a free product of elliptic factors.

If the limit group over free products that we start with, L, is strict, non-cyclic and admits no free decomposition in which the elements of E_L can be conjugated into the factors, a shortening quotient of L is a proper quotient of it. More generally, we have the following.

PROPOSITION 16. *Let G be a f.g. group, and let $\{u_s : G \to A_s * B_s\}$ be a sequence of homomorphisms that converges into an action of a noncyclic, strict limit group over free products, L, on some real tree Y, where L admits a (possibly trivial) free decomposition in which the elliptic elements, E_L, can be conjugated into the factors, and so that there exists at least one factor in this free decomposition, which is strict, noncyclic, and freely indecomposable relative to its intersection with E_L. Then every shortening quotient of L, obtained from the sequence $\{u_s\}$, is a proper quotient of L (in which nontrivial elliptic elements in L are not mapped to nontrivial elliptic elements of the shortening quotient).*

Proof. Suppose that the f.g. group G is generated by d elements. A shortening quotient SQ of L is obtained from a sequence of homomorphisms $\{v_{s_n} : F_d \to A_{s_n} * B_{s_n}\}$ that converges into SQ. Let L_1 be a factor in a (possibly trivial) free decomposition of L, in which all the elements E_L can be conjugated into the various factors, so that the factor L_1 is a noncyclic strict limit group (over free products), which is freely indecomposable relative to its intersection with E_L.

Let SQ_1 be the image of L_1 in the shortening quotient SQ. Note that SQ_1 is a shortening quotient of L_1. By construction, the shortening quotient SQ_1 is a quotient of the noncyclic, strict limit group over free products L_1, which is freely indecomposable relative to its elliptic elements, E_{L_1}. If the sequence of homomorphisms $\{v_{s_n}\}$, restricted to some f.g. preimage of L_1, has bounded stretching factors, that is, if the shortening quotient SQ_1 is not strict, SQ_1 cannot be (entirely) elliptic, hence, it must be freely decomposable or cyclic, so it is a proper quotient of L_1. If SQ_1 is a strict limit group over free products, then the shortening quotient SQ_1 is a proper quotient of L_1 by the shortening argument that is used in the proof of Claim 5.3 in [Se1]. $\square$

The shortening procedure, and Proposition 16, enable us to obtain a *resolution* of the limit group R_∞, with which we can associate a *completion*, into which R_∞ embeds. This completion enables us to present R_∞ as a f.g. group which is finitely presented over some of its elliptic subgroups. Since Theorem 13 assumes that the successive maps along the infinite descending chains under consideration, τ_i, do not map nontrivial elliptic elements to the identity element, it is implied that elliptic subgroups embed along the sequences under consideration. This implies that for large enough n, R_n is a proper quotient of R_∞, which contradicts the fact that R_∞ is a proper quotient of all the limit groups (over free products), $\{R_n\}$, that appear in the infinite descending chain we constructed. We now explain this reasoning in detail.

PROPOSITION 17. *Let R_∞ be the direct limit of the sequence, $\{R_n\}$, of limit groups over free products* (*that were constructed in order to prove Theorem* 13). *Then there exists a finite sequence of limit groups over free products*:

$$R_\infty \to L_1 \to L_2 \to \cdots \to L_s$$

for which:

(i) *L_1 is a shortening quotient of R_∞, and L_{i+1} is a shortening quotient of L_i, for $i = 1, \ldots, s-1$.*

(ii) *The epimorphisms along the sequence are proper epimorphisms, and nontrivial elliptic elements in L_i are mapped to nontrivial elliptic elements in L_{i+1}.*

(iii) *$L_s = H_1 * \cdots * H_r * F_t$ where the factors, $H_1, \ldots, H_r$, are elliptic, and the entire elliptic set, E_{L_s}, is the union of the conjugates of $H_1, \ldots, H_r$. F_t is a* (*possibly trivial*) *free group.*

(iv) *The resolution: $R_\infty \to L_1 \to L_2 \to \cdots \to L_s$ is a strict resolution* ([Se1], *Section* 5), *that is, in each level non-QH, nonvirtually-Abelian vertex groups in the virtually Abelian JSJ decomposition are mapped monomorphically into the limit group in the next level, and QH vertex groups are mapped into nonvirtually-Abelian, nonelliptic subgroups.*

(v) *The Abelian edge groups in the virtually Abelian JSJ decompositions* (*over free products*) *of the limit groups, $L_1, \ldots, L_{s-1}$, are f.g. and torsion-free.*

(vi) *The constructed resolution is well-structured* (*see Definition* 1.11 *in* [Se2] *for a well-structured resolution*).

Proof. By Lemma 14 and Proposition 16, a shortening quotient of R_∞ is a proper quotient of it. Furthermore, nontrivial elliptic elements in R_∞ are mapped to nontrivial elliptic elements in the shortening quotient. Hence, we set L_1 to be a shortening quotient of R_∞. If from the sequence of (shortened) homomorphisms that was used to construct L_1, it's possible to extract a subsequence that satisfy the properties of Lemma 14, we continue with this subsequence, and use it to get a shortening quotient L_2 of L_1, which by Proposition 16 is a proper quotient of L_1. Continuing this process iteratively, and recalling that every descending chain of limit groups over free products that starts with R_∞ and satisfies the assumptions of the statement of Theorem 13, terminates after finitely many steps, we finally get the sequence of proper epimorphisms:

$$R_\infty \to L_1 \to L_2 \to \cdots \to L_s.$$

Parts (i) and (ii) follow immediately from the construction of the descending finite sequence of shortening quotients, and part (iii) follows, since by Lemma 14 and Proposition 16, the descending sequence of shortening quotients terminates, precisely when the obtained limit group is the free product

of elliptic factors and a free group. Part (iv) follows from the way homomorphisms are shortened along the shortening procedure. Each shortening quotient in the sequence is constructed from a sequence of (shortened) homomorphisms, which are obtained from the sequence of homomorphisms that converge into the previous limit group in the resolution. The homomorphisms that converge into the previous limit group in the resolution, are modified (shortened), keeping the images of non-QH, nonvirtually-Abelian vertex groups, in the virtually Abelian JSJ decomposition of the previous limit group (in the resolution), conjugate to their images under the original homomorphisms. Hence, these vertex groups are embedded into the shortening quotient (that appears) in the next level of the resolution. The same argument explains why the images of QH vertex groups are nonvirtually-Abelian in that shortening quotient.

The virtually Abelian JSJ decomposition (over free products) of the various factors in L_s is trivial, since they are elliptic. Since the edge groups in the virtually Abelian JSJ decomposition (over free products) of L_{s-1} are embedded into L_s, by part (iv), and they are nonelliptic, they must be infinite cyclic, as these are the only nonelliptic Abelian subgroups in L_s. Continuing this argument by climbing along the levels of the resolution proves that every nonelliptic Abelian subgroup in one of the limit groups, $L_1, \ldots, L_s$, is f.g. and is necessarily torsion-free, which implies part (v). Part (vi) follows since like in the case of free and hyperbolic groups, every strict Makanin–Razborov resolution, that is, a resolution that is obtained from a sequence of shortening quotients, is well-structured (see Definition 1.11 in [Se2]). $\square$

Proposition 17 constructs from a subsequence of the homomorphisms, $\{h_n : F_d \to A_n * B_n\}$, a well-structured resolution of the limit group over free products, R_∞, that terminates in a limit group L_s, which is a free product of elliptic subgroups and a (possibly trivial) free group. In Section 1 of [Se2], a *completion* is constructed from a given well-structured resolution (see Definition 1.11 in [Se2] for a well-structured resolution). This construction that generalizes in a straightforward way to well-structured resolutions of limit groups over torsion-free hyperbolic groups in [Se3], generalizes in a straightforward way to well-structured resolutions of limit groups over free products. For the detailed construction of the completion, see Definition 1.12 in [Se2].

We denote the completion of the well-structured resolution that is constructed in Proposition 17, $Comp(Res)$. By Definition 1.12 and Lemma 1.13 in [Se2], each of the limit groups, $R_\infty, L_1, \ldots, L_s$ is embedded into the completion of the constructed resolution, $Comp(Res)$. All the (virtually Abelian) edge groups that are not connected to boundary elements of QH vertex groups (edge groups that are connected to QH vertex groups are always cyclic), and all the virtually Abelian vertex groups, contain Abelian subgroups as subgroups of index at most 2. Furthermore, these Abelian groups are nonelliptic

subgroups of the associated limit groups, $R_\infty, L_1, \ldots, L_s$. Since the only nonelliptic Abelian subgroups of the terminal limit group, L_s, are infinite cyclic, all the edge groups, and all the vertex groups, that appear in all the levels of the completion, $Comp(Res)$, are finitely generated. In particular, all the edge groups and all the vertex groups that appear in the virtually Abelian JSJ decompositions of the limit groups over free products, $R_\infty, L_1, \ldots, L_s$, are finitely generated.

Let $\rho \colon R_\infty \to Comp(Res)$ be the embedding of the limit group over free products, R_∞, into the completion of the constructed resolution: $R_\infty \to L_1 \to \cdots \to L_s$. $\rho(R_\infty)$ being a f.g. subgroup of $Comp(Res)$ inherits a (finite) virtually Abelian decomposition from the virtually Abelian decomposition that is associated with the top level of $Comp(Res)$. The edge groups in that inherited (finite) virtually Abelian decomposition are subgroups of f.g. virtually Abelian groups, hence, f.g. virtually Abelian groups. The vertex groups in that virtually Abelian decomposition can either be conjugated into subgroups of a lower level of the completion, or they can be conjugated into QH groups or into f.g. virtually Abelian groups. f.g. subgroups of virtually Abelian groups are again f.g. virtually Abelian. f.g. subgroups of Fuchsian groups are free products of f.g. Fuchsian groups and f.g. virtually free groups. Hence, R_∞ can be reconstructed from finitely many f.g. groups that can be conjugated into lower level of the completion, $Comp(Res)$, and finitely many f.g. Fuchsian groups, f.g. virtually free groups, and f.g. virtually Abelian groups, by performing free products and free products with amalgamation and HNN extensions along f.g. virtually Abelian groups.

Continuing with this decomposition procedure along the lower levels of the completion, $Comp(Res)$, we get that the subgroup $\rho(R_\infty)$ (that is isomorphic to R_∞) can be reconstructed from finitely many f.g. elliptic subgroups in $Comp(R_\infty)$, and finitely many f.g. Fuchsian groups, f.g. virtually free groups, and f.g. virtually Abelian groups, by performing finitely many operations of free products and free products with amalgamation and HNN extensions along f.g. virtually Abelian groups. In particular, R_∞ is obtained from finitely many elliptic subgroups of R_∞ by adding finitely many generators and relations.

By construction, the limit group (over free products), R_∞, is the direct limit of the decreasing sequence of limit groups, $\{R_n\}$, which are all quotients of some free group, F_d. Every f.g. subgroup of F_d that is mapped to an elliptic subgroup in R_∞, is mapped to elliptic subgroups in all the limit groups, R_n, for all $n > n_0$ for some index n_0. R_∞ is generated by finitely many f.g. elliptic subgroups and finitely many virtually Abelian, virtually free, and f.g. Fuchsian groups together with finitely many Bass–Serre generators that are added in each of the performed HNN extensions (along f.g. virtually Abelian subgroups). Since these last groups are all f.p. and elliptic subgroups in each of the limit groups R_n are mapped monomorphically into R_∞ by our assumptions on the decreasing sequence, $\{R_n\}$, there exists some index n_1, so that

for all $n > n_1$, the limit group R_n is generated by finitely many elements that are mapped to the Bass–Serre elements that are used in constructing R_∞, finitely many subgroups that are isomorphic to the f.g. virtually Abelian, f.g. virtually free, and f.g. Fuchsian groups, and finitely many elliptic subgroups that are isomorphic to the f.g. elliptic subgroups that altogether generate R_∞. Since R_∞ is generated by these subgroups and the Bass–Serre elements by imposing finitely many relations, there exists some index n_2, so that for every $n > n_2$ these relations hold in R_n, which implies that R_n is a quotient of R_∞ using a quotient map that maps the fixed generating set of R_∞ (the images of a fixed basis of F_d) to the fixed generating set of R_n. This implies that R_{n+1} is a proper quotient of R_∞ by a quotient map that maps the fixed generating set of R_∞ to the fixed generating set of R_{n+1}, which contradicts the construction of R_∞ as the direct limit of the decreasing sequence of limit groups over free products, $\{R_n\}$. This finally implies the d.c.c. that is stated in Theorem 13.   □

Theorem 13 proves a basic d.c.c. that holds for limit groups over free products. This d.c.c. is weaker than the ones proved for limit groups over free and hyperbolic groups ([Se1], [Se3]). Indeed, it is stated only for decreasing sequences of limit groups over free products for which the successive maps do not map nontrivial elliptic elements to the identity. Still, this d.c.c. is the basis for our analysis of limit groups over free products, and for the analysis of solutions to systems of equations over free products.

We start with the following theorem, which is a rather immediate corollary of the d.c.c. that is stated in Theorem 13, that associates a resolution with a given limit group over free products, a resolution that has similar properties to the resolution described in Proposition 17.

THEOREM 18. *Let L be a limit group over free products. Then there exists a finite sequence of limit groups over free products*:

$$L \to L_1 \to L_2 \to \cdots \to L_s$$

for which:

(i) *L_1 is a shortening quotient of L, and L_{i+1} is a shortening quotient of L_i, for $i = 1, \ldots, s-1$. In particular, elliptic elements in L_i are mapped monomorphically to elliptic elements in L_{i+1}.*

(ii) *The epimorphisms along the sequence are proper epimorphisms.*

(iii) *$L_s = H_1 * \cdots * H_r * F_t$ where the factors, $H_1, \ldots, H_r$, are elliptic, and the entire elliptic set, E_{L_s}, is the union of the conjugates of $H_1, \ldots, H_r$. F_t is a (possibly trivial) free group.*

(iv) *The resolution: $L \to L_1 \to L_2 \to \cdots \to L_s$ is a strict resolution ([Se1], Section 5), i.e., in each level non-QH, nonvirtually-Abelian vertex groups and edge groups in the virtually Abelian JSJ decomposition are mapped*

monomorphically into the limit group in the next level, and QH vertex groups are mapped into nonvirtually-Abelian, nonelliptic subgroups.

(v) *The Abelian edge groups in the virtually Abelian JSJ decompositions (over free products) of the limit groups, $L_1, \ldots, L_{s-1}$, are f.g. and torsion-free.*

(vi) *The constructed resolution is well-structured (see Definition 1.11 in [Se2] for a well-structured resolution). As a corollary, the limit group (over free products) L is embedded into the completion of the well-structured resolution:*

$$L \to L_1 \to L_2 \to \cdots \to L_s$$

so that all the elliptic elements in L are mapped into conjugates of the elliptic subgroups, $H_1, \ldots, H_r$, of L_s.

Proof. Theorem 18 generalizes the resolution that was constructed for the limit group (over free products) R_∞, to general limit groups over free products. To prove Proposition 17 we used the d.c.c. for resolutions of R_∞ for which the epimorphisms that are associated with them do not map nontrivial elliptic elements to the identity element, that follows from the construction of R_∞. Theorem 13 proves that such a d.c.c. holds for resolutions of an arbitrary limit group over free products, for which the associated epimorphisms do not map nontrivial elliptic elements to the identity element. With this general d.c.c., the proof of Proposition 17 generalizes to general limit groups over free products. □

3. Finitely presented groups

Theorem 13 proves the basic d.c.c. for limit groups over free products, and Theorem 18 associates a resolution with each such limit group, hence, it embeds each limit group over free products into a completion, where this completion is a tower over a limit group which is a free product of a (possibly trivial) free group with a (possibly empty) finite collection of f.g. elliptic subgroups.

When considering limit groups over free products we analyzed sequences of homomorphisms from a f.g. group into free products. Since our goal is to obtain a structure theory for sets of solutions to systems of equations, and the group that is associated formally with a finite system of equations is f.p. and not only f.g., we may assume that the limit groups over free products that we are considering are obtained from sequences of homomorphisms from a f.p. group into free products (and not only from a f.g. one).

As we will see in the sequel, if we attempt to construct a Makanin–Razborov diagram that is associated with a f.p. group, we will need to consider only f.g. limit groups over free products that are recursively presented, that is, limit groups that can be embedded into f.p. groups. A modification or a strengthening of the existence of such an embedding is a key for obtaining further d.c.c.

that will eventually allow the construction of a Makanin–Razborov diagram over free products for a given f.p. group. We start with the following simple observation.

PROPOSITION 19. *Let G be a f.p. group, and let L be a limit group over free products which is a quotient of G. Then there exists a limit group over free products $\hat{L}$ with the following properties*:

(1) *There is a f.p. completion, Comp, which is a tower over a free product of finitely many f.p. elliptic subgroups and a free group, so that $\hat{L}$ embeds into Comp, and the elliptic elements in $\hat{L}$ are mapped into conjugates of the finitely many elliptic factors in the free decomposition that is associated with the limit group that appears in the terminal level of the completion Comp.*

(2) *Either $\hat{L} = L$ or $\hat{L} > L$ (see Definition 12 for the relation $>$ on limit groups over free products).*

Proof. By Theorem 18, the limit group (over free products) L admits a well-structured resolution:

$$L \to L_1 \to L_2 \to \cdots \to L_s$$

and L_s admits a free product decomposition: $L_s = H_1 * \cdots * H_r * F_t$ where the factors, $H_1, \ldots, H_r$, are elliptic, and the entire elliptic set, E_{L_s}, is the union of the conjugates of $H_1, \ldots, H_r$. F_t is a (possibly trivial) free group.

Furthermore, with this resolution it is possible to associate a completion, $Comp_1$, and the limit group L embeds into this completion, so that all the elliptic elements in L are mapped into conjugates of the elliptic subgroups, $H_1, \ldots, H_r$, of L_s (the groups $L_1, \ldots, L_s$ admit natural embeddings into the various levels of the completion, $Comp_1$, and the elliptics in each of these limit groups are mapped into conjugates of $H_1, \ldots, H_r$ in the completion $Comp_1$).

Since L is embedded into the completion $Comp_1$, G is naturally mapped into $Comp_1$. By construction, the completion $Comp_1$ is built as a tower over the terminal limit group L_s. If $Comp_1$ is f.p., we obtained the conclusion of the proposition, as we can take $\hat{L} = L$, and $\hat{L}$ is embedded into the f.p. completion $Comp_1$. Hence, we may assume that $Comp_1$ is not finitely presented, i.e., at least one of the factors, $H_1, \ldots, H_r$, is not finitely presented. In that case, we gradually replace $Comp_1$ by a f.p. completion into which G is mapped.

Each of the factors of L_s, $H_1, \ldots, H_r$, is f.g. so it is a quotient of some f.g. free group. Let $F^1, \ldots, F^r$ be f.g. free groups that $H_1, \ldots, H_r$ are quotients of. We start the construction of a f.p. completion that replaces the completion $Comp_1$, with a tower T_2 that has in its base level the free group $F^1 * \cdots * F^r * F_t$, and the next (upper) levels are connected to the lower levels of the constructed tower, precisely as they are connected in the completion, $Comp_1$, that is, using the same graphs of groups, just that the group that is associated

with the lowest level in $Comp_1$, which is $L_s = H_1 * \cdots * H_r * F_t$, is replaced by the free group, $F^1 * \cdots * F^r * F_t$.

Note that T_2 is a tower, but it is not necessarily a completion (see Definition 1.12 in [Se2]), as in general there are no retractions from a group that is associated with a certain level onto the group that is associated with the level below it. Each of the levels above the base level in T_2 is constructed using a (finite) graph of groups, in which some vertex groups are the groups that are associated with the lower level in T_2. Hence, the group that is associated with a level above the base level, is obtained from a free product of the group that is associated with the lower level with a f.p. group by imposing finitely many relations. Furthermore, the graphs of groups that are associated with the different levels in T_2 are similar to the graphs of groups that are associated with the corresponding levels in the completion $Comp_1$, and differ from $Comp_1$ only in the groups that are associated with the base level.

Since each of the groups that are associated with the upper levels in T_2 is obtained from a free product with a f.p. group by imposing finitely many relations, and since the graphs of groups that are associated with the upper levels have similar structure as the corresponding graphs of groups that are associated with the levels of the completion $Comp_1$, and these graphs of groups differ only in the structure of the group that is associated with the base level, it is enough to impose only finitely many relations from the defining relations of the various factors of the limit groups that is associated with the base level of $Comp_1$, L_s, $H_1, \ldots, H_r$, on the associated free groups, $F^1, \ldots, F^r$, so that if we replace the group that is associated with the base level of T_2, $F^1 * \cdots * F^r * F_t$, with the obtained f.p. quotient, $V_1 * \cdots * V_r * F_t$, and construct from it a tower, T_3, by imitating the construction of $Comp_1$ and T_2 (i.e., with similar graphs of groups in all the upper levels), T_3 will be a completion.

T_3 is a completion, but it may be that the f.p. group G is not mapped into it. G is mapped into the completion $Comp_1$. Hence, once again, since G is f.p. it is enough to impose only finitely many relations from the defining relations of the various factors, $H_1, \ldots, H_r$, on the factors, $V_1, \ldots, V_r$, so that if we replace the group that is associated with the base level in T_3 with the obtained f.p. quotient, $M_1 * \cdots * M_r * F_t$, and construct from it a tower T_4 by imitating the construction of the towers $Comp_1$, T_2, and T_3, T_4 is a f.p. completion, and G maps into it.

Furthermore, the map from G into the completion $Comp_1$, is a composition of the maps from G to T_4, composed with the natural quotient map from T_4 to $Comp_1$. Hence, if we denote the image of G in T_4 by $\hat{L}$, then $\hat{L}$ is a limit group over free products, its set of elliptics is precisely the intersection of $\hat{L}$ with the set of conjugates of $M_1, \ldots, M_r$, and either $\hat{L}$ is isomorphic to L and the natural isomorphism from $\hat{L}$ onto L maps the elliptics in $\hat{L}$ monomorphically onto the elliptics in L, or the natural epimorphism from $\hat{L}$

onto L has a nontrivial kernel, and this epimorphism maps the elliptics in $\hat{L}$ onto the elliptics in L, in which case $\hat{L} > L$. $\square$

Proposition 19, the d.c.c. proved in Theorem 13, and the resolution that is associated with a limit group over free products in Theorem 18, enable us to prove that there are maximal elements in the set of all limit groups over free products that are all quotients of a (fixed) f.p. group G, and that there are only finitely many equivalence classes of such maximal elements. The existence of maximal elements in the set of limit quotients is valid even for f.g. groups.

PROPOSITION 20. *Let G be a f.g. group. Let $R_1, R_2, \ldots$ be a sequence of limit groups over free products that are all quotients of the f.g. group G, and for which*:
$$R_1 < R_2 < \cdots .$$
Then there exists a limit group over free products R that is a quotient of G, so that for every index m, $R > R_m$.

Proof. Identical to the proof in the free and hyperbolic groups cases (see Proposition 1.20 in [Se3]). $\square$

Proposition 20 proves that given an ascending chain of limit quotients (over free products) of a f.g. group G, there exists a limit quotient of G that bounds all the limit groups in the sequence. Hence, we can apply Zorn's lemma (it is enough to consider countable ascending chains in case of quotients of a f.g. group), and obtain maximal limit quotients (over free products) of any given f.g. group, and every limit quotient of a f.g. group is dominated by a maximal limit quotient of that group.

Proposition 19 proves that if G is in addition f.p. then if R is a limit quotient of G (over free products), then there exists a limit group over free products L, that is either isomorphic to R or $L > R$, so that L embeds in a f.p. completion. Hence, if we are interested in maximal limit quotients (over free products) of a f.p. group G, it is enough to consider limit quotients of G that embed in f.p. completions, and there are clearly at most countably many such limit quotients.

In case a group G is f.p. the existence of maximal limit quotients, and the existence of an embedding of maximal limit quotients of a f.p. group G into f.p. completions, imply the finiteness of the (equivalence classes of) maximal limit quotients (over free products) of a f.p. group.

THEOREM 21. *Let G be a f.p. group. Then there are only finitely many equivalence classes of maximal elements in the set of limit quotients (over free products) of G, and each of these maximal elements embeds in a f.p. completion.*

Proof. Let G be a f.p. group. Since all its maximal limit quotients over free products can be embedded into f.p. completions, there are at most countably many maximal limit quotients of G (over free products). Suppose that there are infinitely many nonequivalent maximal limit quotients of G, and let $R_1, R_2, \ldots$ be the infinite sequence of (non-equivalent) maximal limit quotients (over free products) of G. Each R_i is equipped with a given quotient map $\eta_i : G \to R_i$, hence, fixing a generating set for G, we fix a generating set in each of the R_i's, that is, we have maps $\nu_i : F_d \to R_i$ (assuming G has rank d), that factor through the epimorphism $F_d \to G$.

For each index i, we look at the collection of words of length 1 in F_d that are mapped to the identity, and those that are mapped to elliptic elements by ν_i. There is a subsequence of the R_i's for which this (finite) collection of words is identical. Starting with this subsequence, for each R_i (from the subsequence) we look at the collection of words of length 2 in F_d that are mapped to the identity and those that are mapped to elliptic elements by ν_i, and again there is a subsequence for which this (finite) collection is identical. We continue with this process for all lengths ℓ of words in F_d, and look at the diagonal sequence (that we denote $R_{i_1}, R_{i_2}, \ldots$).

We choose homomorphisms $h_j : F_d \to A_j * B_j$, that factor through the map $F_d \to G$, so that for words w of length at most j, $h_j(w) = 1$ iff $\nu_{i_j}(w) = 1$, and $h_j(w)$ is elliptic iff $\nu_{i_j}(w)$ is elliptic (we can choose such homomorphisms since R_{i_j} is a limit quotient of G). After passing to a subsequence, the homomorphisms h_j converge into a limit group over free products M, which is a limit quotient of G. Note that in the (canonical) map $F_d \to M$, the elements of length at most j that are mapped to the identity, and those that are mapped to be elliptic, are precisely those that are mapped to the identity and those that are mapped to be elliptic by the map $\nu_{i_j} : F_d \to R_{i_j}$.

$R_1, R_2, \ldots$ form the entire list of maximal limit quotients of G over free products. We construct a new sequence of homomorphisms: $f_j : F_d \to C_j * D_j$ that factor through the quotient map $F_d \to G$. First, f_j has the same property as h_j, i.e., the elements of length at most j that are mapped to the identity by f_j are precisely those that are mapped to the identity by $\nu_{i_j} : F_d \to R_{i_j}$, and the elements of length at most j that are mapped to be elliptic by f_j are precisely those that are mapped to be elliptic by $\nu_{i_j} : F_d \to R_{i_j}$. Second, since R_{i_j} is maximal and is not equivalent to $R_1, \ldots, R_{i_j - 1}$, there must exist some elements $u_1, \ldots, u_{i_j - 1} \in F_d$ so that for each index s, $1 \le s \le i_j - 1$, either u_s is mapped to the identity in R_s, but u_s is mapped to a nontrivial element in R_{i_j} by ν_{i_j}, or u_s is mapped to an elliptic element in R_s, but u_s is mapped to a nonelliptic element in R_{i_j} by ν_{i_j}. If the first holds, we require that $f_j(u_s) \neq 1$, and if the second holds we require that $f_j(u_s)$ is not elliptic.

The sequence of homomorphisms, $\{f_j\}$, converges into the limit group (over free products) M. We look at a subsequence of the homomorphisms $\{f_j\}$, so that the subsequence and its shortenings converge into a resolution of M that

satisfy the properties that are listed in Theorem 18, $M \to L_1 \to L_2 \to \cdots \to L_s$ (we still denote the subsequence $\{f_j\}$).

With the resolution $M \to L_1 \to L_2 \to \cdots \to L_s$, which is a well-structured resolution by construction, we can naturally associate a completion. Let $Comp$ be this completion. Since G is naturally mapped onto the limit group M, there exists a natural map, $\rho : G \to Comp$, that factors through the map $G \to M$.

Note that by construction, the completion $Comp$ is obtained from the terminal limit group, L_s, of the given resolution of M, by adding finitely many generators and relations. Since the group G is f.p. we can repeat the argument that was used to prove Proposition 19, and replace the terminal limit group L_s with a (possibly the same) f.p. group L_s^1 that maps onto L_s, and starting with L_s^1 construct a completion, $Comp^1$, that has the same structure as the completion $Comp$, except that the terminal limit group (over free products) of $Comp^1$ is L_s^1, whereas the terminal limit group of the completion $Comp$ is L_s. Furthermore, the group G maps into $Comp^1$, and since $Comp^1$ is finitely presented, there exists a subsequence of the sequence of homomorphisms $\{f_j\}$, that factor through the completion $Comp^1$.

Let M^1 be the image of G in $Comp^1$. M^1 is a limit quotient of G (over free products), so there must exist some maximal limit quotient of G, that we denote R_b, so that R_b is either equivalent to M^1 or $R_b > M^1$. Now, there exists a subsequence of the homomorphisms $\{f_j\}$ that factor through the limit group M^1, hence, this subsequence of homomorphisms factor through the maximal limit quotient R_b. By construction, each of the homomorphisms f_j does not factor through any of the maximal limit groups, $R_1, \ldots, R_{i_j-1}$. Hence, for large enough j, none of the homomorphisms f_j factor through the maximal limit quotient R_b, a contradiction. Therefore, G admits only finitely many maximal limit quotients (over free products), and by Proposition 19, each of the maximal limit quotients of G embeds into a f.p. completion. $\square$

Theorem 21 proves the existence of finitely many limit quotients of a given f.p. group. Hence, it gives the first level of a Makanin–Razborov diagram of a f.p. group over free products, and it proves that the groups that appear in the first level of the Makanin–Razborov diagram of a f.p. group over free products are canonical (i.e., they are an invariant of the f.p. group). Still, the construction of maximal limit groups over free products, and the proof that there are only finitely many (equivalence classes of) maximal quotients of a f.p. group (over free products), does not generalize in a straightforward way to allow the construction of the next levels in the Makanin–Razborov diagram.

Furthermore, Theorem 13 proves the basic d.c.c. that is required for analyzing limit groups over free products. However, it is not sufficient for the construction of a Makanin–Razborov diagram as it guarantees the termination of strict resolutions, but not of general resolutions in the diagram (if we try to imitate the construction over free and hyperbolic groups). Hence, to

construct a finite Makanin–Razborov diagram we will need to construct the
next levels in the diagram, and in addition to prove an additional d.c.c. that
will guarantee the termination of the construction after finitely many steps.

Let G be a f.p. group. We start the construction of the Makanin–Razborov
diagram over free products of G with the finite collection of (equivalence
classes of) maximal limit quotients of G, according to Theorem 21. We con-
tinue by studying the homomorphisms of each of the maximal limit quotients
of G into free products. As in the construction of Makanin–Razborov dia-
grams over free and hyperbolic groups, we continue by modifying (shortening)
these homomorphisms using the modular groups that are associated with the
maximal limit quotients (over free products) of the given f.p. group G.

Recall that the group of modular automorphisms that is associated with a
limit group (over free products), is the group of automorphisms that includes
inner automorphisms in addition to those automorphisms that are encoded
by the virtual Abelian JSJ decomposition (over free products). That is, Dehn
twists along edges in the JSJ, elements from the Teichmuller modular groups
of the QH vertex groups, and subgroups of the general linear group that act
on (virtual) Abelian vertex groups and for each virtually Abelian vertex group
preserve the subgroup that is generated by the edge groups that are connected
to the vertex group (see Definition 5.2 in [Se1]).

Let L be one of the maximal limit quotients (over free products) of G,
and let E_L be its set of elliptics. First, we factor L into its most refined
free decomposition in which the elements in E_L are elliptic (i.e., contained
in conjugates of the factors), $L = U_1 * \cdots * U_m * F_t$, where F_t is a (possibly
trivial free group, and the elements in E_L can be conjugated into the various
factors, $U_1, \ldots, U_m$.

(L, E_L) is a (maximal) limit quotient of G (over free products), hence,
L is obtained as a limit of a sequence of homomorphisms $\{h_n : G \to A_n *
B_n\}$. G is f.p. and is mapped onto L, and L admits the free decomposition,
$L = U_1 * \cdots * U_m * F_t$, where the elliptic elements in E_L can be embedded
into the various factors $U_1, \ldots, U_m$. By the argument that is used to prove
Proposition 19, there exist finitely presented groups $M_1, \ldots, M_m$ so that the
map $G \to L$ factors as:

$$G \to M_1 * \cdots * M_m * F_t \to U_1 * \cdots * U_m * F_t,$$

where the two maps are onto, and for each index i, $1 \leq i \leq m$, M_i is mapped
onto U_i. Since the sequence of homomorphisms $\{h_n\}$ of G converges into
(L, E_L), and the group $M_1 * \cdots * M_m * F_t$ is f.p. and the map from G to L fac-
tors through it, for large enough n the homomorphisms $\{h_n\}$ factor through
the map $G \to M_1 * \cdots * M_m * F_t$. Now, if we apply the proof of Proposi-
tion 19, it follows that there are m f.p. completions (over free products),
$Comp_1, \ldots, Comp_m$, so that each of the factors U_i is embedded into the com-
pletion $Comp_i$ so that the elliptics in U_i are mapped into elliptics in $Comp_i$

(and only elliptics in U_i are mapped into elliptics in $Comp_i$), and there exist maps:

$$M_1 * \cdots * M_m * F_t \to U_1 * \cdots * U_m * F_t \to Comp_1 * \cdots * Comp_m * F_t$$

that extend the embeddings from U_i to $Comp_i$, for $1 \le i \le m$.

Hence, we may continue with each of the factors U_i of L in parallel. U_i is a maximal limit quotient (over free products) of the f.p. group M_i, and by Proposition 21 it is embedded into a f.p. completion $Comp_i$.

Therefore, we may assume that in the sequel, we are given a f.p. group G, and a maximal limit quotient of it, that we still denote, (L, E_L), and the limit quotient L is freely indecomposable relative to the elliptic subset E_L (i.e., L admits no nontrivial free decomposition in which the elements in E_L can be conjugated into the factors).

With (the factor) L and E_L, we naturally associate its virtually Abelian JSJ decomposition over free products (Theorem 11). We also associate with (L, E_L) the collection of homomorphisms of G into free products that factor through (L, E_L).

Fixing a (finite) generating set of a limit group (over free products) L, and given a homomorphism $h: L \to A * B$, we look at a shortest homomorphism among those that are obtained by precomposing h with a modular automorphism of L that is contained in the modular group of automorphisms of L that is associated with the virtually Abelian JSJ decomposition over free products of L (relative to E_L). A limit group over free products that is the limit of a sequence of such shortest homomorphisms is called a shortening quotient, and denoted SQ. Note that this definition of a shortening quotient is different than the more restricted one given in Definition 15, as in particular, the natural map from a limit group over free products, L, onto a shortening quotient SQ of L, is not always monomorphic on the set of elliptic elements in L, E_L. Still, like in Proposition 16, if a shortening quotient is not elliptic it is a proper quotient of the limit group L.

LEMMA 22 (cf. Proposition 16). *Let L be a limit group over free products, and let E_L be its set of elliptics. Suppose that L admits no free decompositions in which the elements in E_L can be conjugated into the factors. Then every shortening quotient of L which is not (entirely) elliptic is a proper quotient of it.*

Proof. Identical to the proof of Proposition 16.   □

Like limit quotients (over free products) of a f.g. group, every ascending sequence of shortening quotient of a limit group over free products is bounded by a shortening quotient of that limit group. The proof is identical to the proof of Proposition 20.

LEMMA 23. *Let L be a f.g. limit group over free products. Let $SQ_1, SQ_2, \dots$ be a sequence of shortening quotients of L, for which:*

$$SQ_1 < SQ_2 < \cdots .$$

Then there exists a shortening quotient SQ of L, so that for every index m, $SQ > SQ_m$.

By Zorn's lemma and Lemma 23, it follows that there are maximal elements in the set of shortening quotients of a f.g. limit group over free products. We call such a maximal element, a *maximal shortening quotient*. By Lemma 22, if the limit group (over free products) L does not admit a free product in which the elliptic elements in L, E_L, can be conjugated into the factors, every maximal shortening quotient of L that is not entirely elliptic is a proper quotient of L.

4. Covers of limit quotients and their resolutions

The first level in the Makanin–Razborov diagram over free products of a f.p. group G consists of the finitely many maximal limit quotients of G (Theorem 21). Over free and hyperbolic groups, we continued to the next level in the diagram by proving that there are only finitely many (equivalence classes of) maximal shortening quotients. Over free product, we need to prove a finiteness result for shortening quotients and their (strict) resolutions, that will enable us to continue to the next level, and so that the next levels will be constructed in a way for which a termination can be proved.

In order to prove that there are only finitely many maximal limit quotients over free products of a f.p. group over free products (Theorem 21), we first showed that any maximal limit quotient can be embedded into a f.p. completion (Proposition 19). For maximal shortening quotients of a f.g. limit group over free products, we were not able to prove such a statement. For the continuation of the diagram, we first prove an observation that holds for all the (proper) limit quotients of a given limit group over free products, that still allows us to construct the Makanin–Razborov diagram over free products for a f.p. group, although we lose some of the canonical properties of the diagrams over free and hyperbolic groups.

Given a limit group over free products, L, and a limit quotient M of L, Theorem 24 associates a cover, CM, with M. CM is a limit quotient of L, if $L > M$, then $L > CM$ and M is a limit quotient of CM. The main property of the cover CM that is used in the sequel (and is not always true for M) is that CM can be embedded into a completion, $Comp_{CM}$, and $Comp_{CM}$ is f.p. relative to the elliptic subgroups of the given limit group L, that is, $Comp_{CM}$ is generated from the elliptic subgroups in L by adding finitely many generators and relations. In particular, this implies that if L is recursively presented so is the cover CM.

THEOREM 24. *Let L be a f.g. limit group over free products, and let E_L be its set of elliptics. Let M be a limit quotient of L (over free products), with set of elliptics, E_M, and with a quotient map, $\eta : L \to M$ that maps E_L into E_M.*

Suppose that $L > M$, that is, that the map η has a nontrivial kernel, or that there exists a nonelliptic element in L that is mapped to an elliptic element in M by η.

Let $M \to M_1 \to \cdots \to M_s$ be a (well-structured) resolution of M, that is, a resolution of M that satisfies the properties of the resolution that is associated with a limit group over free products in Theorem 18. Then there exists a f.g. limit quotient of L, CM, with a set of elliptics, E_{CM}, and a well-structured resolution of CM, $CM \to CM_1 \to \cdots \to CM_s$, that satisfies the properties of the resolutions in Theorem 18, and a quotient map: $\tau : L \to CM$, that maps E_L into E_{CM}, so that:

(1) *There exists a quotient map: $\nu : CM \to M$, that maps E_{CM} onto E_M, so that $\eta = \nu \circ \tau$.*

(2) *If $\eta : L \to M$ has a nontrivial kernel, then $\tau : L \to CM$ has a nontrivial kernel. If there exists a nonelliptic element in L that is mapped to an elliptic element in M by η, then there exists a nonelliptic element in L that is mapped to an elliptic element in CM by τ. If M_{i+1} is a proper quotient of M_i, then CM_{i+1} is a proper quotient of CM_i.*

(3) *If η maps an elliptic element in L to the identity, then τ maps an elliptic element in L to the identity.*

(4) *If M is a free product of finitely many elliptic subgroups and a free group, so is CM. More generally, CM_j is mapped onto M_j, $1 \leq j \leq s$, where elliptics in CM_j are mapped onto elliptics in M_j.*

(5) *All the homomorphisms of the given limit group L that factor through the given well-structured resolution of M, factor through the resolution $CM \to CM_1 \to \cdots \to CM_s$.*

(6) *With the given well-structured resolution, $M \to M_1 \to \cdots \to M_s$, we can naturally associate a completion, $Comp_M$ (see Definition 1.12 in [Se2]), and with the resolution $CM \to CM_1 \to \cdots \to CM_s$ we can naturally associate a completion, $Comp_{CM}$. CM is embedded into $Comp_{CM}$, and the elliptic elements in CM are mapped into the terminal limit group CM_s.*

(7) *By Theorem 18, the elliptic elements, E_L, in the limit group L are conjugates of finitely many (possibly none) f.g. subgroups, $E_1, \ldots, E_r$ in L. Then the completion, $Comp_{CM}$, is obtained from (copies of the subgroups) $E_1, \ldots, E_r$ by adding finitely many generators and relations, i.e., $Comp_{CM}$ is f.p. relative to the subgroups $E_1, \ldots, E_r$.*

(8) *If M admits a free decomposition, $M = V_1 * \cdots * V_u * F_t$, where F_t is a free group, and this free decomposition is respected by the given resolution of M, then CM has a similar free decomposition, $CM = CV_1 * \cdots * CV_u *$*

F_t, which is respected by the constructed resolution of CM, where the map ν respects this free decomposition, i.e., $\nu(CV_i) = V_i$, $i = 1, \ldots, u$, and $\nu(F_t) = F_t$. In particular, the completion, $Comp_{CM}$, admits a similar free decomposition, $Comp_{CM} = Comp_1 * \cdots * Comp_u * F_t$, where CV_i embeds into $Comp_i$.

Proof. Let L be a limit group over free products, with set of elliptics E_L. By Theorem 18, there are finitely many subgroups, $E_1, \ldots, E_r$, in L, so that the set of elliptic elements in L, E_L, is the union of the conjugacy classes of $E_1, \ldots, E_r$. Let M be a limit quotient of L, and let $M \to M_1 \to \cdots \to M_s$ be a well-structured resolution of M, where M_s is a free product of finitely many elliptic factors and a possibly trivial free group.

With the given well-structured resolution of M we associate a completion, $Comp_M$. M is a limit quotient of L, and M is a subgroup of the completion, $Comp_M$, so L is mapped into $Comp_M$. Hence, the elliptic subgroups in L, $E_1, \ldots, E_r$, are mapped into conjugates of the elliptic subgroups, that are factors in the terminal limit group M_s, in $Comp_M$. If the terminal limit group M_s is f.p. relative to the subgroups, $E_1, \ldots, E_r$, the theorem follows (by taking the cover CM to be M and $Comp_{CM}$ to be $Comp_M$). Otherwise, we modify the argument that was used to prove Proposition 19.

Since M is embedded into the completion $Comp_M$, L is naturally mapped into $Comp_M$. Each of the factors of the terminal limit group of $Comp_M$, M_s, is f.g. so it is a quotient of some conjugates of (copies of) the elliptic subgroups of L, $E_1, \ldots, E_r$ and a f.g. free group. We start the construction of a the completion $Comp$ that covers the completion $Comp_M$, with a tower T_1 that has in its base level the free product of a free group (isomorphic to the free factor in the free decomposition of M_s), the free products of corresponding conjugates of $E_1, \ldots, E_r$ with free groups (so that each of the factors of M_s is a quotient of each of these free products). The next (upper) levels are connected to the lower levels of the constructed tower T_1, precisely as they are connected in the completion, $Comp_M$, that is, using the same graphs of groups, just that the group that is associated with the lowest level in $Comp_M$, which is M_s, is replaced by the prescribed free products.

T_1 is a tower, but it is not necessarily a completion (see Definition 1.12 in [Se2]), as in general there are no retractions from a group that is associated with a certain level onto the group that is associated with the level below it. Each of the levels above the base level in T_1 is constructed using a (finite) graph of groups, in which some vertex groups are the groups that are associated with the lower level in T_1. Hence, the group that is associated with a level above the base level, is obtained from a free product of the group that is associated with the lower level with a f.p. group by imposing finitely many relations. Furthermore, the graphs of groups that are associated with the different levels in T_1 are similar to the graphs of groups that are associated with

the corresponding levels in the completion $Comp_M$, and differ from $Comp_M$ only in the groups that are associated with the base level.

Each of the groups that are associated with the upper levels in T_1 is obtained from the groups that appear in the lower level of T_1 by a free product with a f.g. free group and further imposing finitely many relations. The graphs of groups that are associated with the upper levels in T_1 have similar structure as the corresponding graphs of groups that are associated with the levels of the completion $Comp_M$, that is, the graphs of groups differ only in the vertex groups that are associated with lower levels. Furthermore, these vertex groups differ only in the groups that are associated with the base levels of T_1 and $Comp_M$. Hence, it is enough to impose only finitely many (additional) relations from the defining relations of the various factors of the limit groups that is associated with the base level of $Comp_M$, M_s, on the subgroup that is associated with the base level of T_1. This means imposing finitely many (additional) relations on the associated free products of free groups and conjugates of (copies of) the subgroups, $E_1, \ldots, E_r$, that form the group which is associated with the base level of T_1, so that if we replace the group that is associated with the base level of T_1, with the obtained quotient, and construct from the obtained base subgroup a tower, T_2, by imitating the construction of $Comp_M$ and T_1 (that is, with similar graphs of groups in all the upper levels), T_2 will be a completion (that is, it is a tower with retractions between consecutive levels).

T_2 is a completion, but it may be that the limit group L is not mapped into it. L is mapped into the completion $Comp_M$, and as a limit group it is finitely presented relative to its elliptic subgroups. Hence, once again, it is enough to impose only finitely many relations from the defining relations of the various factors of M_s, so that if we replace the group that is associated with the base level in T_2 with the obtained quotient, and construct from it a tower T_3 by imitating the construction of the towers $Comp_M$, T_1, and T_2, T_3 is a completion, it is f.p. relative to the elliptic subgroups, $E_1, \ldots, E_r$, and L maps into it.

We denote the images of the limit group L into the various levels of the completion T_3, by $CM, CM_1, \ldots, CM_s$. By imposing finitely many additional relations on the base subgroup of T_3 from the relations of the base subgroup of $Comp_M$, M_s, one can further guarantee that if M is a proper quotient of L, then CM is a proper quotient of L, if $L > M$ then $L > CM$, and similarly, if M_{j+1} is a proper quotient of M_j then CM_{j+1} is a proper quotient of CM_j, and if $M_{j+1} > M_j$ then $CM_{j+1} > CM_j$. We denote the obtained completion by $Comp_{CM}$ and its associated resolution as the resolution that is associated with CM (the obtained resolution is a well-structured resolution by construction). All the other properties of the limit groups, and the associated resolution and completion, $Comp_{CM}$, that are listed in the statement of the theorem follow easily from the construction. $\qquad\square$

Given a f.g. limit group over free products L, and its limit quotient M with an associated well-structured resolution, $M \to M_1 \to \cdots \to M_s$, that satisfy the assumptions of Theorem 24, and for which $L > M$, we call a limit quotient CM of L, that satisfies the conclusion of the theorem, a *cover* of the limit quotient M, its associated well-structured resolution, $CM \to CM_1 \to \cdots \to CM_s$, a *cover* of the given resolution of M, and the associated completion, *Comp*, into which CM is embedded, that was constructed from the given well-structured resolution of M, a *cover completion*.

In constructing the Makanin–Razborov diagrams of a f.p. or a f.g. group over a free or a hyperbolic group, we were able to show that the set of shortening quotients of a limit group over these groups contain finitely many equivalence classes of maximal shortening quotients. In studying limit groups over free products, we are not able to prove a similar theorem. Over free products, we prove that given a limit group L, and fixing a cover for each pair of a shortening quotient and its associated well-structured resolution, there exists a finite subcollection of covers which is good for all the shortening quotients of L. As we will see in the sequel, a similar statement on the existence of a finite subcollection of cover completions (with a similar proof) is sufficient for the construction of the Makanin–Razborov diagram over free products.

THEOREM 25. *Let L be a f.g. limit group over free products, suppose that L is not (entirely) elliptic and that L admits no free product decomposition in which the elliptic elements in L, E_L, can be conjugated into the factors.*

With each pair of a shortening quotient M of L, and a well-structured resolution of M, there is an associated quotient map, $\eta_M : L \to M$, that satisfies the assumptions of Theorem 24. Hence, by the conclusion of Theorem 24, for each pair of a shortening quotient M of L, and its associated well-structured resolution we can choose a cover $CM(M)$ together with a completion, $Comp_{CM}$, into which CM embeds.

From the entire collection of covers of shortening quotients of L and their associated well-structured resolutions, it is possible to choose a finite subcollection of covers, $CM_1, \ldots, CM_e$, so that for every maximal shortening quotient M, there exists an index i, $1 \le i \le e$, for which the quotient map, $\eta : L \to M$, is a composition of the two quotient maps: $L \to CM_i \to M$ (where elliptics are mapped to elliptics in these two maps).

Proof. The argument that we use is similar to the proof of the finiteness of the number of equivalence classes of maximal shortening quotients (over free products) of a f.p. group. Let L be a f.g. limit group. By Theorem 24, given a shortening quotient of it, M, and a well-structured resolution of that shortening quotient, there exists a cover CM of M, and CM can be embedded into a completion, *Comp*, that is obtained from the finitely many (conjugates of) elliptic subgroups of L, by adding finitely many generators and relations.

Therefore, there are at most countably many such completions, *Comp*, and hence, at most countably many such covers, CM.

Note that by Lemma 22 each shortening quotient M of L is either entirely elliptic, or it is a proper quotient of L. In case the shortening quotient M is not entirely elliptic, it follows by Theorem 24, that the associated cover, CM is a proper quotient of L (like the shortening quotient M).

Suppose that the countable collection of covers does not contain a finite subcover, that is, there is no finite subcollection of the constructed covers, $CM_1, \ldots, CM_e$, so that for every shortening quotient M, the quotient map $L \to M$ factors as a composition of quotient maps of limit groups over free products: $L \to CM_i \to M$, for some index i, $1 \leq i \leq e$.

To contradict the lack of a finite subcover, we start by ordering the collection of covers, $CM_1, CM_2, \ldots$. Since there is no finite subcover, there must be a sequence of indices, $i_1, i_2, \ldots$, so that a shortening quotient, M_{i_j}, which is covered by CM_{i_j}, is not covered by any of the previous covers, $CM_1, \ldots, CM_{i_j-1}$.

For each index j, the shortening quotient M_{i_j} is a limit of shortest homomorphisms, and it is not covered by any of the covers, $CM_1, \ldots, CM_{i_j-1}$. Hence, for each index j, there exists a shortest homomorphism $h_j : L \to A_j * B_j$, that does not factor through any of the covers, $CM_1, \ldots, CM_{i_j-1}$.

We look at the sequence of homomorphisms $\{h_j\}$. A subsequence of this sequence converges into a limit group (over free products) R, which is a quotient of the limit group L. Unless the limit group R is the (possibly trivial) free product of finitely many elliptic factors and a (possibly trivial) free group, a subsequence of the shortenings of these homomorphisms converges into a shortening quotient R_1 of R, where the elliptics in R are mapped monomorphically into the elliptics in R_1, and R_1 is a proper quotient of R. By continuing iteratively and applying the d.c.c. for decreasing sequences of limit groups over free products (Theorem 13), we obtain a finite (strict) resolution $R \to R_1 \to \cdots \to R_s$, where R_s is a free product of finitely many f.g. elliptic subgroups and a (possibly trivial) free group. For brevity, we still denote the obtained subsequence of shortest homomorphisms, $\{h_j\}$.

The pair of the shortening quotient R, and its (strict) resolution, $R \to R_1 \to \cdots \to R_s$, is one of the pairs of a shortening quotient of the limit group L, and its associated strict resolutions, with which we have associated the covers, $CM_1, CM_2, \ldots$. Hence, one of these covers, CM_r, is a cover that is associated with this pair. Since a cover completion is finitely presented relative to the elliptic subgroups of L, for large enough indices j, the homomorphisms $\{h_j\}$ factor through cover completion and hence factor through the cover CM_r. That contradicts the choice of the homomorphisms $\{h_j\}$, as for large j, h_j is supposed not to factor through the covers, $CM_1, \ldots, CM_{j-1}$. $\square$

Theorem 25 proves that given a f.g. limit group L that admits no free decomposition in which the elliptic elements, E_L, can be conjugated into the factors, it is possible to find finitely many limit quotients of L, one which is isomorphic to L and is entirely elliptic, and the rest which are proper quotients of L, that cover all its shortening quotients. This finite collection of covers is not canonical, but in principle it can be taken as the next step in the Makanin–Razborov diagram. Except for the entirely elliptic cover that is isomorphic to L, the other covers that are associated with L are all proper quotients of it, hence, in principle, we can continue with the construction iteratively. However, the d.c.c. that we proved is valid only for sequences of strictly decreasing limit quotients, for which the quotients are proper and are monomorphic when restricted to elliptic elements (Theorem 13).

Therefore, to complete the construction of the Makanin–Razborov diagram of a f.p. group over free products, we use a different approach. Instead of constructing a finite cover of all the shortening quotients of a given limit group (over free products), we construct a finite cover for all the (strict) resolutions of the given limit group. With each strict resolution of the given limit group, we associate a cover of that resolution (which is a resolution by itself), and there are only countably many such covers, as the completion that is associated with the cover resolution is f.p. relative to the elliptic subgroups of the original limit group. Then we use a similar argument to the one that was used in proving Theorem 25 to prove that there exists a finite subcollection of the collection of cover resolutions, that is, that there exists a finite subcollection so that every homomorphisms of the given limit group into free products factors through at least one of the resolutions from the finite subcollection of cover resolutions.

THEOREM 26. *Let L be a f.g. limit group over free products. Then there exist finitely many well-structured resolutions of quotients of L, so that every homomorphism from L into a free product factors through at least one of these well-structured resolutions. Furthermore, with each of these (finitely many) well-structured resolutions we can naturally associate a completion, and these completions are f.p. relative to the (finitely many) elliptic subgroups in the given limit group L.*

Proof. The proof is similar to the proof of Theorem 25. First, we factor the limit group over free products L into a maximal free decomposition in which the elliptic elements of L, E_L, can be conjugated into the factors. We continue with each of the factors separately. Hence, we may assume that the limit group L is freely indecomposable with respect to its set of elliptics, E_L. By Theorem 24, given a limit quotient of L, that we denote T, and a well-structured resolution of T, $T \to T_1 \to \cdots \to T_s$, that is obtained by taking successive shortening quotients (see Theorem 18), there exists a cover of T, that we denote CT, which is a limit quotient of L, and a cover of the resolution that is associated with T, which is a well-structured resolution,

with an associated cover completion, $Comp_{CT}$, that is obtained from the finitely many (conjugates of) elliptic subgroups of L, by adding finitely many generators and relations. Therefore, there are at most countably many such triples of a cover of a limit quotient, an associated (well-structured) cover resolution, and the corresponding cover completion.

Suppose that the countable collection of cover resolutions does not contain a finite subcover, that is, there is no finite subcollection of the constructed covers, $CT_1, \ldots, CT_e$, with associated cover completions, $Comp_1, \ldots, Comp_e$, so that for each homomorphism h of L into a free product (that maps the elliptics in L, E_L, into elliptic elements), the homomorphism h factors through at least one of the cover resolutions that is associated with the cover completions, $Comp_1, \ldots, Comp_e$.

To obtain a contradiction to the lack of finiteness of covering resolutions, we start by ordering the collection of covering completions and their associated resolutions, $Comp_1, Comp_2, \ldots$. Since there is no finite subcover for the entire collection of homomorphisms of the given limit group L into free products, for each index i, there exists a homomorphism, $h_i : L \to A_i * B_i$, that does not factor through the resolutions that are associated with the completions, $Comp_1, \ldots, Comp_{i-1}$.

Like in the proof of Theorem 25, a subsequence of the sequence of homomorphisms, $\{h_i\}$, converges into a limit group (over free products) R, which is a quotient of the limit group L. Unless R is a (possibly trivial) free product of elliptic subgroups and a (possibly trivial) free group, a subsequence of the shortenings of these homomorphisms converges into a shortening quotient R_1 of R, where the elliptics in R are mapped monomorphically into the elliptics in R_1, and R_1 is a proper quotient of R. By continuing iteratively and applying the d.c.c. for decreasing sequences of limit groups over free products (Theorem 13), we obtain a finite well-structured resolution $R \to R_1 \to \cdots \to R_s$, where R_s is a free product of finitely many f.g. elliptic subgroups and a (possibly trivial) free group. For brevity, we still denote the obtained subsequence of shortest homomorphisms, $\{h_i\}$.

The pair of the limit quotient R of the given limit group (over free products) L, and its (well-structured) resolution, $R \to R_1 \to \cdots \to R_s$, is one of the pairs of a limit quotient of L, and its associated well-structured resolutions, with which we have associated the covers, $Comp_1, Comp_2, \ldots$. Hence, one of these completions, $Comp_r$, is a cover that is associated with this pair. Therefore, for large enough index i, the homomorphism $\{h_i\}$ factors through the cover resolution that is associated with the completion, $Comp_r$. That contradicts the choice of the homomorphisms $\{h_i\}$, as for each i, h_i is supposed not to factor through the cover resolutions that are associated with the completions, $Comp_1, \ldots, Comp_{i-1}$. $\qquad\square$

5. Makanin–Razborov diagrams of finitely presented groups

Theorem 21 on the finiteness of the number of equivalence classes of maximal limit quotients (over free products) of a f.p. group, together with Theorem 26 on the existence of finitely many (cover) resolutions of some quotients of a given f.g. limit group over free products, so that every homomorphism of the given f.g. limit group into free products factors through at least one of the resolutions, allow us to construct a Makanin–Razborov diagram of a f.p. group over free products.

Given a f.p. group G, we start with its (canonical) finite collection of maximal limit quotients over free products (Theorem 21). With each maximal limit quotient, we associate a finite collection of well-structured resolutions of it (according to Theorem 26), so that each homomorphism of the original maximal limit quotient into free products, factors through at least one of its associated resolutions. We construct the diagram by mapping the given f.p. group G into the f.g. limit group that appears in the top level of each of the (finitely many) well-structured resolutions that are associated with its collection of maximal limit quotients (in parallel). Since every homomorphism of G into free products, factors through at least one of its maximal limit quotients, every homomorphism of G into free products factors through at least one of the resolutions in its Makanin–Razborov diagram over free products. That is for every homomorphism of the f.p. group G, there exists at least one resolution in the Makanin–Razborov diagram, so that the homomorphism can be written as a successive composition of the epimorphisms between the groups that appear in the various levels of the resolutions, modular automorphisms of the limit groups that appear in the various levels (that are encoded by the virtually Abelian decompositions that are associated with these groups), and finally a homomorphism from the terminal group of the resolution (which is a free product of elliptic factors and a free group), that sends every elliptic factor into a conjugate of a factor in the image free product.

At this stage, we slightly improve the diagram. The virtually Abelian decompositions that are associated with each of the limit groups that appear in the various levels of the well-structured resolutions in the Makanin–Razborov diagrams, are decompositions that are inherited from the free and virtually Abelian JSJ decompositions of the limit groups that appear along the well-structured resolutions that the resolutions in the Makanin–Razborov diagram cover, according to the construction that appears in Theorem 24. However, these may not be the Grushko and virtually Abelian decompositions of the limit groups in the Makanin–Razborov diagram themselves. To fix that, and make sure that all the decompositions in the limit groups that appear in the Makanin–Razborov diagram are indeed Grushko and virtually Abelian JSJ decompositions (over free products), we slightly modify the construction of a cover.

THEOREM 27. *Let L be a f.g. limit group over free products, let M be a limit quotient of L, and let $M \to M_1 \to \cdots \to M_s$, be a well-structured resolution of M, so that M_s is a free product of finitely many elliptic factors and a possibly trivial free group. Suppose that the free products that are associated with the various limit groups along the resolution, $M, M_1, \ldots, M_s$, are their Grushko free decompositions with respect to their elliptic subgroups (i.e., the resolution respects the Grushko free decompositions of the groups along it), and that the virtually Abelian decompositions that are associated with the limit groups $M, M_1, \ldots, M_s$ are their virtually Abelian JSJ decompositions over free products.*

Then there exists a cover CM of M, with a cover resolution, $CM \to CM_1 \to \cdots \to CM_s$, that satisfies the properties of a cover that are listed in Theorem 24, and for which the free decompositions along the cover resolution are the Grushko free decompositions of the limit groups, $CM, CM_1, \ldots, CM_s$, and the virtually Abelian JSJ decompositions of these groups over free products have the same structure as the virtually Abelian decompositions that are associated with them along the resolution, that is, the same structure as the virtually Abelian JSJ decompositions of the limit groups, $M, M_1, \ldots, M_s$.

Proof. The proof that we use is a modification of the argument that we used to prove Theorem 24. Let L be a limit group over free products, with a set of elliptics E_L. Recall that by Theorem 18, the set of elliptics E_L is the union of conjugates of some (elliptic) subgroups, $E_1, \ldots, E_r$, in L. Let M be a limit quotient of L, and let $M \to M_1 \to \cdots \to M_s$ be a well-structured resolution of M, where M_s is a free product of finitely many elliptic factors and a possibly trivial free group.

With the given well-structured resolution of M, we associate a completion, $Comp_M$. Given the well-structured resolution of M, and its associated completion, $Comp_M$, we use the construction that was used in proving Theorem 24, and construct a completion, $Comp$, which is f.p. relative to the elliptic subgroups, $E_1, \ldots, E_r$, and for which the images of the limit group L into the various levels of $Comp$, that were denoted, $CM, CM_1, \ldots, CM_s$, satisfy the list of properties that is presented in Theorem 24.

By adding finitely many relations to the base subgroup of $Comp$ from the set of relations that are defined on the base subgroup, M_s, of the completion $Comp_M$, we may assume that the Abelian decompositions that are inherited by the subgroups, $CM, CM_1, \ldots, CM_s$, from the Abelian decompositions that are associated with the various levels of the completion $Comp$, are similar to the Abelian decompositions that are inherited by the various Abelian decompositions of the subgroups, $M_1, \ldots, M_s$ from the Abelian decompositions that are associated with the various levels of $Comp_M$.

Suppose that the Grushko free decomposition of the limit group M with respect to its elliptic subgroups is $M = M^1 * \cdots * M^b * F_v$, and this free decomposition together with the virtually Abelian JSJ decompositions of the factors, M^j, over free products with respect to the elliptic subgroups of M, give rise to an Abelian decomposition, Δ_M. Note that by our assumptions, the completion, $Comp_M$, respects the Grushko free decomposition of M, and the Abelian decompositions that are associated with the various levels of $Comp_M$ are the virtually Abelian JSJ decompositions over free products of the subgroups, $M, M_1, \ldots, M_{s-1}$.

We order the relations that the terminal limit group M_s of $Comp_M$ satisfy, and sequentially impose them on the terminal limit group of the completion, $Comp$. We claim that after adding finitely many of these relations, the free product decomposition, and the virtually Abelian JSJ decomposition of the corresponding subgroup CM (after adding the relations) will be similar to those of the subgroup M.

The cover CM, which is the image of the limit group L in the completion, $Comp$, admits a free decomposition $CM = CM^1 * \cdots * CM^b * F_v$, in which the elliptic subgroups in CM can be conjugated into the factors. This free decomposition is inherited from the structure of the completion, $Comp$, as the completions $Comp$ and $Comp_M$ have the same structure, and $Comp_M$ respects the Grushko decomposition (relative to elliptic subgroups) of the limit quotient M, $M = M^1 * \cdots * M^b * F_v$.

Let $CM(n)$ be the image of L in the completion, $Comp(n)$, that is obtained from $Comp$ by imposing on the terminal level in $Comp$ the first n relations in M_s, the terminal limit group in $Comp_M$. $CM(n)$ inherits a free decomposition from $Comp(n)$, $CM(n) = CM(n)^1 * \cdots * CM(n)^b * F_v$, a free decomposition in which the elliptic subgroups in $CM(n)$ can be conjugated into the factors (note that the elliptic subgroups in $CM(n)$ can be conjugated into the factors of the terminal limit group of $Comp(n)$). If this free decomposition is not the Grushko free decomposition of $CM(n)$ with respect to its elliptic subgroups, then at least one of the factors admits a further nontrivial free decomposition with respect to the elliptic subgroups.

Suppose that there exists a sequence of indices (still denoted n) for which the free decomposition of $CM(n)$ that is inherited from $Comp(n)$ is not the Grushko free decomposition of $CM(n)$ with respect to the elliptic subgroups in $CM(n)$. By passing to a subsequence (still denoted n) we may assume that one of the factors, w.l.o.g. $CM(n)^1$ admits a nontrivial free decomposition $CM(n)^1 = A_n * B_n$, where each of the elliptic subgroups in $CM(n)$ can be conjugated into one of the other factors in the given free decomposition of $CM(n)$, to A_n or to B_n.

In that case, we look at the actions of the groups $CM(n)^1$ on the (pointed) Bass–Serre trees, (T_n, t_n), that correspond to the (nontrivial) free products, $A_n * B_n$. Note that these actions are faithful actions of the groups, $CM(n)^1$,

that the elliptic subgroups in $CM(n)$ that can be conjugated into $CM(n)^1$ can be conjugated into A_n or B_n, and that by construction, the direct limit of the groups, $CM(n)^1$, is the factor M^1 of the limit group M which is assumed to be freely indecomposable relative to its elliptic subgroups.

CM^1 is f.g. so we fix a generating set for it, $\langle g_1, \ldots, g_d \rangle$, and since the groups $CM(n)^1$ are (limit) quotients of CM^1, it gives us a generating set for each of the groups, $CM(n)^1$. Given the action of $CM(n)^1$ on the Bass–Serre tree, (T_n, t_n), we precompose this action with a (modular) automorphism ϕ_n of $CM(n)^1$, that is, an automorphism that can be expressed as a composition of an automorphism that comes from the virtually Abelian decomposition that $CM(n)^1$ inherits from the virtually Abelian decomposition that is associated with the top level in $Comp(n)$ and an inner automorphism, so that the maximal displacement of the base point t_n by the action of the tuple of elements, $\phi_n(g_1), \ldots, \phi_n(g_d)$, is minimal among all such (modular) automorphisms ϕ.

Since we modify the actions of the groups, $CM(n)^1$, by precomposing them with (modular) automorphisms, and since the actions are all faithful, there is a subsequence of twisted actions that converge into an action of the direct limit of the groups, $CM(n)^1$, that is, the factor M^1 of M, on a real tree. Since the automorphisms ϕ_n were chosen to minimize the displacement of the base points under the corresponding twisted actions, and since the virtual Abelian JSJ decomposition of the limit group M^1 has the same structure as the virtually Abelian decomposition that is inherited by $CM(n)^1$ from the virtually Abelian decomposition that is associated with the top level of the completions, $Comp(n)$, the set of displacements of the base points under the twisted actions has to be bounded. Hence, the factor M^1 of M inherits a nontrivial free decomposition from the limit action, a free decomposition in which all the elliptic subgroups in M^1 can be conjugated into the factors. This contradicts the assumption that M^1 admits no such nontrivial free decomposition. Therefore, there must exist some index n_0, so that for all $n > n_0$, the limit groups $CM(n)^1$ admit no free decomposition in which the elliptic subgroups of $CM(n)^1$ can be conjugated into the factors.

By passing to a subsequence, we may assume that all the factors in the free decomposition of the limit groups, $CM(n)$, are freely indecomposable relative to their elliptic subgroups. Suppose that there exists a sequence of indices (still denoted n) for which the virtually Abelian decomposition that at least one of the factors of the the groups, $CM(n)$, $CM(n)^1, \ldots, CM(n)^b$, inherits from the virtually Abelian decomposition that is associated with the top level of the completion, $Comp(n)$, is not the virtually Abelian JSJ decomposition over free products of that factor. Wlog we may assume that this factor is $CM(n)^1$.

Let $\Delta(n)$ be the virtually Abelian decomposition that $CM(n)^1$ inherits from the virtually Abelian decomposition that is associated with the top level

of the completion $Comp(n)$. Let $\text{JSJ}(n)$ be the virtually Abelian JSJ decomposition over free products of $CM(n)^1$, and let Δ_M be the virtually Abelian decomposition that M inherits from the virtually Abelian decomposition that is associated with the top level of the completion, $Comp_M$, which by our assumptions is the virtually Abelian JSJ decomposition of M over free products. Since we assumed that the virtually Abelian decompositions, $\Delta(n)$, are not identical to the virtually Abelian decompositions, $\text{JSJ}(n)$, the virtually Abelian JSJ decompositions, $\text{JSJ}(n)$, must be proper refinements of the virtually Abelian decompositions, $\Delta(n)$. Note that the structure of the virtually Abelian decompositions, $\Delta(n)$, is similar to that of the Abelian decomposition, Δ_M.

For every index n, the virtually Abelian JSJ decomposition $\text{JSJ}(n)$ is a proper refinement of the virtually Abelian decomposition $\Delta(n)$. Hence, if needed we can cut some of the QH subgroups in $\text{JSJ}(n)$ along s.c.c. and obtain a new decomposition, $\Theta(n)$, of $CM(n)^1$ that refines $\Delta(n)$, in which all the edge groups and all the QH vertex groups in $\Delta(n)$ are elliptic, and at least one of the non-QH nonvirtually-Abelian vertex groups in $\Delta(n)$ is not elliptic. Hence, at least one of these vertex groups inherits a nontrivial virtually Abelian decomposition from $\Theta(n)$, a decomposition in which all the edge groups that are connected to that vertex group are elliptic.

By passing to a further subsequence (still denoted n), we may assume that the vertex group that inherits a nontrivial virtually Abelian decomposition from $\Theta(n)$ is a vertex group $V(n)$ in $\Delta(n)$ that is mapped to the same vertex group V in Δ_M, the virtually Abelian JSJ decomposition of the limit group M.

We fix a free group F_r, where r is the rank of the limit group CM, and an epimorphism, $\tau : F_r \to CM$. We fix a finite generating set for F_r. We may assume that this generating set contains elements that are mapped to elements that generate the edge groups and the vertex groups in the virtually Abelian decomposition of CM that is inherited from the top level of the completion, $Comp$.

For each index n, we look at a homomorphism $h_n : F_r \to A_n * B_n$ that approximates the limit group $CM(n)$. This means that h_n maps each element in the ball of radius n in the Cayley graph of F_r (with respect to the given set of generators), to an elliptic element or to a trivial element if and only if the element is trivial or elliptic in $CM(n)$. It maps the elements from the generating sets that are mapped to the edge groups in $\Delta(n)$ to nonelliptic elements. Furthermore, let $S < F_r$ be the subgroup that is generated by those elements in the fixed generating set of F_r whose image generate the vertex group in the virtually Abelian decomposition of CM that is mapped to the vertex group V in Δ_M, and the edge groups that are connected to that vertex group. The vertex group $V(n)$ is not elliptic in the virtually Abelian decomposition of the factor $CM(n)^1$, $\Theta(n)$, and the edge groups that

are connected to $V(n)$ in $\Delta(n)$ are elliptic in $\Theta(n)$. Hence, we may further modify each of the homomorphisms h_n, by precomposing each of them with Dehn twists along edge groups that lie in the graph of groups that is inherited by $V(n)$ from the graph of groups $\Theta(n)$. We apply this modification, so that for the obtained homomorphism, $\hat{h}_n$, when restricted to the subgroup $S < F_r$ (which is mapped onto $V(n)$), $\hat{h}_n : S \to A_n * B_n$, the minimal displacement of a point in the Bass–Serre tree, that is associated with the free product $A_n * B_n$, under the action of the fixed set of generators of S, will be at least n times larger than the minimal displacement of a point in that Bass–Serre tree, under the action of the fixed set of generators that are mapped to any given edge group that is connected to $V(n)$ in $\Delta(n)$.

By construction, the homomorphisms, $\{\hat{h}_n : S \to A_n * B_n\}$, converge into a nontrivial action of the vertex group V in Δ_M on some real tree (where the convergence is into V as a limit group over free products). All the edge groups that are connected to V in Δ_M fix points in that real tree and they are all nonelliptic subgroups (i.e., each element in these groups is mapped to nonelliptic element in $A_n * B_n$ for large n). With this action, it is possible to associate a nontrivial graph of groups decomposition of V, with Abelian edge groups, in which all the edge groups that are connected to V are contained in vertex groups in that graph of groups decomposition. Hence, using this graph of groups decomposition it is possible to further refine the graph of groups, Δ_M, and this clearly contradict the assumption that Δ_M is the virtually Abelian JSJ decomposition of the limit group M.

Therefore, for large n, the Abelian decompositions, $\Delta(n)$, are the virtually Abelian JSJ decompositions of the limit groups over free products, $CM(n)$. The same argument implies the same results for the next limit groups in the constructed resolution, $CM_1(n), \ldots, CM_{s-1}(n)$, and the theorem follows. $\square$

The Makanin–Razborov diagram of a f.p. group G over free products is uniform, that is, it encodes all the homomorphisms from G into arbitrary free products. Equivalently, it encodes all the quotients of a f.p. group that are free products. As we will see in the sequel, the Makanin–Razborov diagram that we constructed suffices in order to modify the results and the techniques that were used to study the first order theory of a free or a hyperbolic group, in order to study the first order theory of a free product. We also believe that modifications of it can be applied for studying homomorphisms of a f.p. group into groups with more general splittings (notably k-acylindrical splittings), and probably homomorphisms into (some classes of) relative hyperbolic groups.

Unfortunately, the diagram that we constructed is not canonical, as it uses finite covers (Theorems 25 and 26), and these are not unique. To construct a canonical diagram, we believe that it's better to study only maximal homomorphisms into free products.

DEFINITION 28. Let G be a f.g. group. On the set of homomorphisms of G into free products, we define a partial order. Let $h_i : G \to A_i * B_i$, $i = 1, 2$, be two homomorphisms. Note that the images of the homomorphisms h_i inherit (possibly trivial) free products from the free product decompositions $A_i * B_i$, $i = 1, 2$. We write that $h_1 > h_2$, if there exists an epimorphism with nontrivial kernel: $\tau : h_1(G) \to h_2(G)$, that maps the elliptics in $h_1(G)$ into the elliptics in $h_2(G)$, so that for every $g \in G$, $h_2(g) = \tau(h_1(g))$.

If τ is an isomorphism and it maps the elliptics in $h_1(G)$ onto the elliptics in $h_2(G)$, and for every $g \in G$, $h_2(g) = \tau(h_1(g))$, we say that h_1 is in the same equivalence class as h_2.

Note that this relation on homomorphisms into free products, which is a partial order on homomorphisms, is a special case of the partial order that was defined in Definition 12 for limit groups over free products.

To construct a canonical Makanin–Razborov diagram, it seems that one needs to prove the existence of maximal homomorphisms with respect to the above partial order. The existence of maximal homomorphisms allows one to construct a canonical (finite) collection of maximal shortening quotients of a f.g. limit group over free products, and then prove a d.c.c. that allows the termination of the construction of a diagram, using somewhat similar construction to the one used over free and hyperbolic groups. To prove the existence of maximal homomorphisms (with respect to the prescribed partial order), one needs to prove the following natural conjecture:

CONJECTURE. *Let G be a f.g. group. Let $h_1, h_2, \ldots$ be a sequence of homomorphisms of G into free products, for which:*

$$h_1 < h_2 < \cdots.$$

Then there exists a homomorphism h from G into a free product, so that for every index m, $h > h_m$ (one may even assume that the homomorphisms, $\{h_m\}$, do not factor through an epimorphism onto a group of the form $M * F$ for some nontrivial free group F).

Finally, we note that the Makanin–Razborov diagram over free products that we constructed is associated with a f.p. group. Some of our arguments are not valid for f.g. groups. In particular, although there exist maximal elements in the set of limit quotients over free products of a f.g. group, it is not clear if there are only finitely many maximal limit quotients. Therefore, the problem of the existence of a Makanin–Razborov diagram that describes the collection of homomorphisms from a given f.g. group (and not only a f.p. one) into free products remains open.

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