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ALMOST-EINSTEIN HYPERSURFACES IN THE EUCLIDEAN SPACE

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ABSTRACT. We show that almost-Einstein hypersurfaces in the Euclidean space are homeomorphic to spheres. The proof relies on universal lower bounds in terms of the Betti numbers for the $L^{n/2}$-norms of the Ricci and traceless Ricci tensor of compact oriented n-dimensional hypersurfaces. Certain examples show that the assumption on the codimension is essential.

1. Introduction

One of the most fascinating problems in Riemannian geometry is to investigate relationships between topology and curvature of Riemannian manifolds. There are plenty of results in which certain restrictions on the curvatures of the metric g of a compact n-dimensional Riemannian manifold (M^n, g) yield information on the topology of M^n. The sphere theorem is an important result in this direction. The same problem can be raised from the point of view of submanifold geometry.

The aim of this paper is to study this problem for hypersurfaces in the Euclidean space. Let $(M^n, g), n \geq 3$, be a compact n-dimensional Riemannian manifold and let $f : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$. It is well known (cf. [4], [9], [11]) that if (M^n, g) is Einstein, then $f(M^n)$ is a round sphere. Recall that (M^n, g) is Einstein if the Ricci tensor satisfies $\mathrm{Ric}_g = kg$ for some constant k. The following question arises naturally: If (M^n, g) is almost-Einstein in the sense that $\mathrm{Ric}_g - kg$ is small in a suitable norm for a constant k, what can be said about the topology of M^n?

This question was studied by Roth [7] for the L_∞-norm of $\mathrm{Ric}_g - kg$, while in [8] the same author deals with the weaker L^q-norm of $\mathrm{Ric}_g - kg$ for specific k.

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In the present paper, we show that if the $L^{n/2}$-norm of the tensor $\mathrm{Ric}_g - kg$ is small, where $k \geq 0$ is a constant, then M^n is homeomorphic to the sphere S^n. Throughout the paper, all manifolds under consideration are assumed to be connected, without boundary and oriented. More precisely, we prove the following result.

THEOREM 1. *Let $(M^n, g), n \geq 3$, be a compact, n-dimensional Riemannian manifold with Ricci tensor Ric_g and volume element dM_g. For each constant $k \geq 0$, there exists a positive constant $\varepsilon(n,k)$, depending only on n and k, such that if (M^n, g) admits an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$ and*

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, dM_g < \varepsilon(n,k),$$

then M^n is homeomorphic to the sphere S^n.

Since the Einstein condition may be rewritten as $\mathrm{Ric}_g = \frac{s_g}{n} g$, where s_g denotes the scalar curvature of g, the $L^{n/2}$-norm of the traceless Ricci tensor $\mathrm{Ric}_g - \frac{s_g}{n} g$ may be viewed as a measure of how much a Riemannian manifold deviates from being Einstein. We may now state the following result.

THEOREM 2. *Let $(M^n, g), n \geq 3$, be a compact, n-dimensional Riemannian manifold with Ricci tensor Ric_g and scalar curvature $s_g \geq 0$ in the case where $n \geq 4$. There exists a positive constant $b(n)$, depending only on n, such that if (M^n, g) admits an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$ and*

$$\int_{M^n} \left\|\mathrm{Ric}_g - \frac{s_g}{n} g\right\|^{\frac{n}{2}} dM_g < b(n),$$

then M^n is homeomorphic to the sphere S^n.

The idea for the proofs is to relate the $L^{n/2}$-norm of the tensors $\mathrm{Ric}_g - kg$ and $\mathrm{Ric}_g - \frac{s_g}{n} g$ with the Betti numbers using well-known results of Chern and Lashof ([2], [3]). In fact, Theorems 1 and 2 follow immediately from the subsequent results.

THEOREM 3. *Let $(M^n, g), n \geq 3$, be a compact, n-dimensional Riemannian manifold with Ricci tensor Ric_g. There exists a positive constant $a(n)$, depending only on n, such that if (M^n, g) admits an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$, then*

$$\int_{M^n} \|\mathrm{Ric}_g\|^{\frac{n}{2}}\, dM_g \geq a(n) \sum_{i=0}^{n} \beta_i(M^n; \mathcal{F}),$$

where $\beta_i(M^n; \mathcal{F})$ is the ith Betti number of M^n with respect to an arbitrary coefficient field $\mathcal{F}$. In particular, if

$$\int_{M^n} \|\mathrm{Ric}_g\|^{\frac{n}{2}}\, dM_g < 3a(n),$$

then M^n is homeomorphic to the sphere S^n.

THEOREM 4. *Let $(M^n, g), n \geq 3$, be a compact, n-dimensional Riemannian manifold with Ricci tensor Ric_g. There exists a positive constant $c(n)$, depending only on n, such that if (M^n, g) admits an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$, then*

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}} \, dM_g \geq c(n) \sum_{i=1}^{n-1} \beta_i(M^n; \mathcal{F})$$

for any constant $k > 0$ and any coefficient field $\mathcal{F}$. Moreover, if

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}} \, dM_g < c(n),$$

then M^n is homeomorphic to the sphere S^n.

THEOREM 5. *Let $(M^n, g), n \geq 3$, be a compact, n-dimensional Riemannian manifold with Ricci tensor Ric_g and scalar curvature $s_g \geq 0$ in the case where $n \geq 4$. There exists a positive constant $b(n)$, depending only on n, such that if (M^n, g) admits an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$, then*

$$\int_{M^n} \left\| \mathrm{Ric}_g - \frac{s_g}{n} g \right\|^{\frac{n}{2}} dM_g \geq b(n) \sum_{i=1}^{n-1} \beta_i(M^n; \mathcal{F})$$

for any coefficient field $\mathcal{F}$. Moreover, if

$$\int_{M^n} \left\| \mathrm{Ric}_g - \frac{s_g}{n} g \right\|^{\frac{n}{2}} dM_g < b(n),$$

then M^n is homeomorphic to the sphere S^n.

We note that our main results are not, in general, valid for compact Riemannian manifolds (M^n, g) that admit an isometric immersion into the $(n+m)$-dimensional Euclidean space $\mathbb{R}^{n+m}$ with codimension $m > 1$.

In fact, we consider the standard immersion of the torus $M^n(r) := S^1(r) \times S^{n-1}(\sqrt{1-r^2}), 0 < r < 1$, into the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, where $S^k(r)$ denotes the k-dimensional sphere of radius r. The principal curvatures of $M^n(r)$, with respect to some unit normal vector field in S^{n+1}, are $\sqrt{1-r^2}/r$ of multiplicity 1 and $-r/\sqrt{1-r^2}$ of multiplicity $n-1$. A straightforward computation shows that the Ricci tensor of $M^n(r)$ satisfies

$$\int_{M^n(r)} \|\mathrm{Ric}\|^{\frac{n}{2}} \, dM^n(r) = \frac{\alpha_n r}{\sqrt{1-r^2}},$$

where α_n is a positive constant depending only on n. Thus, the $L^{n/2}$-norm of the Ricci tensor of $M^n(r)$ is sufficiently close to zero, provided that r is small enough. On the other hand, the kth Betti number of $M^n(r)$ is equal to one when $k \in \{0, 1, n-1, n\}$ and zero, otherwise. Obviously, $M^n(r)$ may be viewed as a codimension 2 submanifold of $\mathbb{R}^{n+2}$. This example ensures that the assumption on the codimension in Theorem 1 is essential.

Moreover, Wallach [12] constructed an isometric immersion of an n-dimensional complex projective space $\mathbb{C}P^n$ of constant holomorphic curvature $2n/(n+1)$ into the sphere $S^{n(n+2)-1} \subset \mathbb{R}^{n(n+2)}$. Since $\mathbb{C}P^n$ is an Einstein manifold, this example justifies the necessity of the assumption on the codimension in Theorem 2.

We note that Shiohama and Xu [10] investigated the $L^{n/2}$-norm of the curvature tensor.

2. Preliminaries

Let $f:(M^n,g) \longrightarrow \mathbb{R}^{n+m}$ be an isometric immersion of a compact, connected, oriented n-dimensional Riemannian manifold (M^n,g) into the $(n+m)$-dimensional Euclidean space $\mathbb{R}^{n+m}$ equipped with the usual Riemannian metric $\langle\cdot,\cdot\rangle$. The normal bundle of f is given by

$$N_f = \{(p,\xi) \in f^*(T\mathbb{R}^{n+m}) : \xi \perp df_p(T_pM^n)\}$$

and the unit normal bundle of f is defined by

$$UN_f = \{(p,\xi) \in N_f : |\xi| = 1\}.$$

The generalized Gauss map $\nu : UN_f \longrightarrow S^{n+m-1}$ is given by $\nu(p,\xi) = \xi$, where S^{n+m-1} is the unit $(n+m-1)$-sphere in $\mathbb{R}^{n+m}$. For each $u \in S^{n+m-1}$, we consider the height function $h_u : M^n \longrightarrow \mathbb{R}$ defined by $h_u(p) = \langle f(p), u\rangle$, $p \in M^n$. Since h_u has a degenerate critical point if and only if u is a critical value of the generalized Gauss map, by Sard's theorem there exists a subset $E \subset S^{n+m-1}$ of zero measure such that h_u is a Morse function for all $u \in S^{n+m-1} \setminus E$. For every $u \in S^{n+m-1} \setminus E$, we denote by $\mu_i(u)$ the number of critical points of h_u of index i. We also set $\mu_i(u) = 0$ for any $u \in E$. According to Kuiper [5], the total curvature of index i of f is given by

$$\tau_i = \frac{1}{\mathrm{Vol}(S^{n+m-1})} \int_{S^{n+m-1}} \mu_i(u)\, dS_u^{n+m-1},$$

where dS^{n+m-1} denotes the volume element of the sphere S^{n+m-1}.

Let $\mathcal{F}$ be a field and $\beta_i(M^n;\mathcal{F}) = \dim H_i(M^n;\mathcal{F})$ be the ith Betti number of M^n, where $H_i(M^n;\mathcal{F})$ is the ith homology group with coefficients in $\mathcal{F}$. From the weak Morse inequalities [6], we know that $\mu_i(u) \geq \beta_i(M^n;\mathcal{F})$ for every $u \in S^{n+m-1}$ such that h_u is a Morse function. Integrating over S^{n+m-1}, we obtain

$$\tau_i \geq \beta_i(M^n;\mathcal{F}). \tag{1}$$

For each $(p,\xi) \in UN_f$, we denote by A_ξ the shape operator of f associated with the direction ξ. There is a natural volume element $d\Sigma$ on the unit normal bundle UN_f. In fact, if dV is a $(m-1)$-form on UN_f such that its restriction to a fiber of the unit normal bundle at (p,ξ) is the volume element of the unit $(m-1)$-sphere of the normal space of f at p, then $d\Sigma = dM \wedge dV$. Furthermore, we have

$$\nu^*(dS^{n+m-1}) = G(p,\xi)\, d\Sigma = G(p,\xi)\, dM \wedge dV,$$

where $G(p,\xi) := (-1)^n \det A_\xi$ is the Lipschitz–Killing curvature at $(p,\xi) \in UN_f$.

A well-known formula due to Chern and Lashof [3] states that

$$\int_{UN_f} |\det A_\xi|\, d\Sigma = \sum_{i=0}^{n} \int_{S^{n+m-1}} \mu_i(u)\, dS_u^{n+m-1}. \tag{2}$$

The total absolute curvature $\tau(f)$ of f in the sense of Chern and Lashof is defined by

$$\tau(f) = \frac{1}{\mathrm{Vol}(S^{n+m-1})} \int_{UN_f} |\nu^*(dS^{n+m-1})| = \frac{1}{\mathrm{Vol}(S^{n+m-1})} \int_{UN_f} |\det A_\xi|\, d\Sigma.$$

The following result is due to Chern and Lashof ([2], [3]).

Theorem 6. *Let $f : (M^n, g) \longrightarrow \mathbb{R}^{n+m}$ be an isometric immersion of a compact, connected, oriented, n-dimensional Riemannian manifold (M^n, g) into the Euclidean space $\mathbb{R}^{n+m}$. Then the total absolute curvature of f satisfies the inequality*

$$\tau(f) \geq \sum_{i=0}^{n} \beta_i(M^n; \mathcal{F})$$

for any coefficient field $\mathcal{F}$. Moreover, if $\tau(f) < 3$, then M^n is homeomorphic to S^n.

For each $i \in \{0, \ldots, n\}$, we consider the subset $U^i N_f$ of the unit normal bundle of f defined by

$$U^i N_f = \{(p,\xi) \in UN_f : A_\xi \text{ has exactly } i \text{ negative eigenvalues}\}.$$

Shiohama and Xu [10, Lemma p. 381] refined formula (2) as follows

$$\int_{U^i N_f} |\det A_\xi|\, d\Sigma = \int_{S^{n+m-1}} \mu_i(u)\, dS_u^{n+m-1}. \tag{3}$$

We recall that the Ricci tensor of (M^n, g) is given by

$$\mathrm{Ric}(X,Y) = \sum_{\alpha=n+1}^{n+m} \big((\mathrm{tr}\, A_\alpha) g(A_\alpha X, Y) - g(A_\alpha^2 X, Y)\big), \tag{4}$$

where X, Y are tangent vector fields of M^n, $\{e_{n+1}, \ldots, e_{n+m}\}$ is a local orthonormal frame field in the normal bundle of f and $\mathrm{tr}\, A_\alpha$ stands for the trace of the shape operator A_α associated with e_α. Furthermore, from (4), we easily verify that the scalar curvature s_g is given by

$$s_g = \sum_{\alpha=n+1}^{n+m} \big((\operatorname{tr} A_\alpha)^2 - \operatorname{tr}(A_\alpha^2)\big). \tag{5}$$

3. Auxiliary results

This section is devoted to some algebraic results that are crucial for the proofs. For each real $n \times n$ matrix A we denote by $\|A\|$ the norm of A, that is, $\|A\| = \sqrt{\operatorname{tr}(AA^t)}$.

PROPOSITION 7. *For each integer $n \geq 3$, there exists a positive constant $c_1(n)$, depending only on n, such that every real $n \times n$ symmetric matrix A satisfies the inequality*

$$\|(\operatorname{tr} A)A - A^2\|^2 \geq c_1(n)|\det A|^{\frac{4}{n}}.$$

For the proof of Proposition 7, we need the following auxiliary result.

LEMMA 8. *Let $n \geq 3$ be an integer. There exists a positive constant $c_1(n)$, depending only on n, such that for all real numbers $x_1, \ldots, x_n$ the following inequality holds*

$$\sum_{i=1}^{n}\left(x_i \sum_{j=1}^{n} x_j - x_i^2\right)^2 \geq c_1(n)\left|\prod_{i=1}^{n} x_i\right|^{\frac{4}{n}}.$$

Proof. We consider the functions $\varphi, \psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = \sum_{i=1}^{n}\left(x_i \sum_{j=1}^{n} x_j - x_i^2\right)^2, \qquad \psi(x) = \prod_{i=1}^{n} x_i, \quad x = (x_1, \ldots, x_n).$$

We shall prove that φ attains a positive minimum on the level set $S = \{x \in \mathbb{R}^n : \psi(x) = \varepsilon\}$, where $\varepsilon = \pm 1$. Since $\varphi(S)$ is bounded from below by zero, there exists a sequence $\{x_m\}$ of points in S such that

$$\lim_{m\to\infty} \varphi(x_m) = \inf \varphi(S).$$

Then we may write $x_m = \rho_m a_m$, where $\rho_m := |x_m| > 0$ and a_m lies in the unit $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$.

We claim that the sequence $\{x_m\}$ is bounded. Assume to the contrary that there exists a subsequence of $\{x_m\}$, which is denoted again by $\{x_m\}$ for the sake of simplicity of notation, such that $\lim_{m\to\infty} \rho_m = +\infty$. Since the sequence $\{a_m\}$ is bounded, it converges to some $\overline{a} \in S^{n-1}$ by taking a subsequence if necessary. Using the fact that ψ is homogeneous of degree n and since $\{x_m\} \in S$, we get

$$\rho_m = \frac{1}{|\psi(a_m)|^{1/n}}. \tag{6}$$

Bearing in mind the fact that φ is homogeneous of degree 4, we obviously have $\varphi(a_m)=\varphi(x_m)/\rho_m^4$. Thus, we obtain $\lim_{m\to\infty}\varphi(a_m)=0$ and consequently $\overline{a}$ is a zero of φ. Therefore, at most one of the coordinates of $\overline{a}$ is nonzero. Since $\overline{a}\in S^{n-1}$, without loss of generality, we may suppose that $\overline{a}=(\varepsilon,0,\ldots,0)$, where $\varepsilon=\pm1$. We set $a_m=(a_{1,m},\ldots,a_{n,m})$. Using (6), we obtain

$$\varphi(x_m)=a_{1,m}^{2-\frac{4}{n}}\frac{(\sum_{i=2}^n a_{i,m})^2}{|\prod_{i=2}^n a_{i,m}|^{4/n}}+\frac{\sum_{i=2}^n a_{i,m}^2(\sum_{j\neq i}a_{j,m})^2}{|a_{1,m}|^{4/n}|\prod_{i=2}^n a_{i,m}|^{4/n}}.$$

From this we obviously get the inequality

$$\varphi(x_m)\geq\frac{\sum_{i=2}^n a_{i,m}^2(\sum_{j\neq i}a_{j,m})^2}{|a_{1,m}|^{4/n}|\prod_{i=2}^n a_{i,m}|^{4/n}}. \tag{7}$$

Now we set

$$\eta_m:=\left(\sum_{i=2}^n a_{i,m}^2\right)^{\frac{1}{2}}.$$

Since $\psi(a_m)\neq0$, we may write $(a_{2,m},\ldots,a_{n,m})=\eta_m\theta_m$, where

$$\theta_m=(\theta_{2,m},\ldots,\theta_{n,m})$$

lies in the unit $(n-2)$-sphere $S^{n-2}\subset\mathbb{R}^{n-1}$. By passing if necessary to a subsequence, we may assume that $\lim_{m\to\infty}\theta_m=(\overline{\theta}_2,\ldots,\overline{\theta}_n)\in S^{n-2}$. Observe that (7) becomes

$$\varphi(x_m)\geq\frac{\sum_{i=2}^n \theta_{i,m}^2(\sum_{j\neq i}a_{j,m})^2}{|a_{1,m}|^{4/n}\eta_m^{2(n-2)/n}|\prod_{i=2}^n \theta_{i,m}|^{4/n}}. \tag{8}$$

Using the fact that $\lim_{m\to\infty}a_{1,m}=\varepsilon$, $\lim_{m\to\infty}a_{i,m}=0$, $\lim_{m\to\infty}\theta_{i,m}=\overline{\theta}_i$ for any $i\in\{2,\ldots,n\}$ and $(\overline{\theta}_2,\ldots,\overline{\theta}_n)\in S^{n-2}$, we immediately get

$$\lim_{m\to\infty}\sum_{i=2}^n\theta_{i,m}^2\left(\sum_{j\neq i}a_{j,m}\right)^2=\sum_{i=2}^n\overline{\theta}_i^2=1.$$

Thus, by virtue of $\lim_{m\to\infty}\eta_m=0$, we obtain

$$\lim_{m\to\infty}\frac{\sum_{i=2}^n \theta_{i,m}^2(\sum_{j\neq i}a_{j,m})^2}{|a_{1,m}|^{4/n}\eta_m^{2(n-2)/n}|\prod_{i=2}^n \theta_{i,m}|^{4/n}}=+\infty.$$

This contradicts (8), since $\lim_{m\to\infty}\varphi(x_m)=\inf\varphi(S)\in\mathbb{R}$.

Consequently our claim is proved, that is, the sequence $\{x_m\}$ is bounded. Thus, there exits a convergent subsequence $\{x_{k_m}\}$ of $\{x_m\}$. We set $x_0=\lim_{m\to\infty}x_{k_m}$. Then we have $\inf\varphi(S)=\lim_{m\to\infty}\varphi(x_{k_m})=\varphi(x_0)$. Obviously, $x_0\in S$. This means that φ attains a minimum $c_1(n)$ on S which obviously depends only on n. We recall the fact that the zeros of φ are precisely the points where at most one of its coordinates is nonzero. Since $x_0\in S$, we obviously have $c_1(n)=\varphi(x_0)>0$.

Now let $x \in \mathbb{R}^n$. Assume that $\psi(x) \neq 0$ and set $\tilde{x} = x/|\psi(x)|^{\frac{1}{n}}$. Clearly, $\tilde{x} \in S$ and consequently $\varphi(\tilde{x}) \geq c_1(n)$. Then the desired inequality follows from the fact that φ is homogeneous of degree 4. In the case where $\psi(x) = 0$, the inequality holds trivially. $\square$

Proof of Proposition 7. Let A be a real $n \times n$ symmetric matrix. There exists an orthogonal matrix P such that $A = P^t DP$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Since

$$\|(\operatorname{tr} A)A - A^2\|^2 = \sum_{i=1}^{n} \left(\lambda_i \sum_{j=1}^{n} \lambda_j - \lambda_i^2 \right)^2,$$

the desired inequality follows immediately from Lemma 8. $\square$

PROPOSITION 9. *For each integer $n \geq 3$, there exists a positive constant $c_2(n)$, depending only on n, such that every real $n \times n$ symmetric matrix A satisfies the inequality*

$$\left\| (\operatorname{tr} A)A - A^2 - \frac{2}{n}\sigma_2(A) I_n \right\|^2 \geq c_2(n) |\det A|^{\frac{4}{n}},$$

where I_n is the $n \times n$ identity matrix, provided that the eigenvalues of A are not all of the same sign and the second elementary symmetric function $\sigma_2(A)$ of the eigenvalues of A satisfies $\sigma_2(A) \geq 0$. Moreover, for $n = 3$ the above inequality holds under the single assumption that the eigenvalues of A are not all of the same sign.

The proof of this proposition relies on the following result.

LEMMA 10. *Let $n \geq 3$ be an integer. There exists a positive constant $c_2(n)$, depending only on n, such that the following inequality holds*

$$\sum_{i=1}^{n} x_i^2 \left(\sum_{j \neq i} x_j \right)^2 - \frac{1}{n} \left(\sum_{i \neq j} x_i x_j \right)^2 \geq c_2(n) \left| \prod_{i=1}^{n} x_i \right|^{\frac{4}{n}},$$

provided that $\sum_{i<j} x_i x_j \geq 0$ in the case where $n \geq 4$, and $x_1, \ldots, x_n$ are not all of the same sign.

Proof. We define the functions $\varphi, \psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = \sum_{i=1}^{n} x_i^2 \left(\sum_{j \neq i} x_j \right)^2 - \frac{1}{n} \left(\sum_{i \neq j} x_i x_j \right)^2, \qquad \psi(x) = \prod_{i=1}^{n} x_i, \quad x = (x_1, \ldots, x_n)$$

and set $U_n = \mathbb{R}^n \setminus (-\infty, 0)^n \cup (0, +\infty)^n$, $D_n = \{(x_1, \ldots, x_n) \in U_n : \sum_{i<j} x_i x_j \geq 0\}$ for $n \geq 4$ and $D_n = U_n$ for $n = 3$. We shall prove that φ

attains a positive minimum on the level set $S=\{x\in D_n : \psi(x)=\varepsilon\}$, where $\varepsilon=\pm 1$. An easy computation shows that

$$\varphi(x)=\sum_{i=1}^{n}\left(x_i\sum_{j=1}^{n}x_j-x_i^2-\frac{1}{n}\sum_{r\neq s}x_rx_s\right)^2, \tag{9}$$

from which we see that $\varphi(S)$ is bounded from below by zero. So there exists a sequence $\{x_m\}$ of points in S such that

$$\lim_{m\to\infty}\varphi(x_m)=\inf\varphi(S).$$

Then we may write $x_m=\rho_m a_m$, where

$$\rho_m:=|x_m|>0 \quad\text{and}\quad a_m=(a_{1,m},\dots,a_{n,m})$$

lies in the unit $(n-1)$-sphere $S^{n-1}\subset\mathbb{R}^n$.

We claim that the sequence $\{x_m\}$ is bounded. Assume to the contrary that there exists a subsequence of $\{x_m\}$, which by abuse of notation is denoted again by $\{x_m\}$, such that $\lim_{m\to\infty}\rho_m=+\infty$. Since the sequence $\{a_m\}$ is bounded, we may assume by taking a subsequence if necessary that $\lim_{m\to\infty}a_m=\overline{a}\in S^{n-1}$. Using the fact that ψ is homogeneous of degree n and since $\{x_m\}\in S$, we get

$$\rho_m=\frac{1}{|\psi(a_m)|^{1/n}}. \tag{10}$$

Due to the fact that φ is homogeneous of degree 4, we obviously have $\varphi(a_m)=\varphi(x_m)/\rho_m^4$. Thus, we find $\lim_{m\to\infty}\varphi(a_m)=0$ and consequently $\overline{a}=(\overline{a}_1,\dots,\overline{a}_n)$ is a zero of φ.

Using (9), we deduce that the zeros of φ are precisely the points $x=(x_1,\dots,x_n)$ such that each of $x_1,\dots,x_n$ is a root of the quadratic equation $t^2-S_1t+\frac{2}{n}S_2=0$, where $S_1=\sum_{i=1}^n x_i$ and $S_2=\sum_{i<j}x_ix_j$. This shows that at most two of $x_1,\dots,x_n$ are distinct. Hence, either $x_1=\dots=x_n$ or, after an eventual re-enumeration, $x_1=\dots=x_p, x_{p+1}=\dots=x_n, x_1\neq x_n, 1\leq p\leq n-1$. Moreover, in the latter case, from the quadratic equation we get $x_1+x_n=S_1$, that is, $(p-1)x_1+(n-p-1)x_n=0$.

Therefore, either all coordinates of $\overline{a}$ are equal, that is, $\overline{a}_1=\dots=\overline{a}_n$ or, after an eventual re-enumeration, $\overline{a}_1=\dots=\overline{a}_p, \overline{a}_{p+1}=\dots=\overline{a}_n, \overline{a}_1\neq\overline{a}_n$, where

$$(p-1)\overline{a}_1+(n-p-1)\overline{a}_n=0, \quad 1\leq p\leq n-1. \tag{11}$$

We distinguish two cases.

Case 1. Assume that $\overline{a}_1=\dots=\overline{a}_n$. Using the fact that $\overline{a}\in S^{n-1}$, we have $\overline{a}_1=\dots=\overline{a}_n=\pm 1/\sqrt{n}$. Letting m go to infinity in (10), we reach a contradiction since by our assumption $\lim_{m\to\infty}\rho_m=+\infty$.

Case 2. Assume that $\overline{a}_1=\dots=\overline{a}_p, \overline{a}_{p+1}=\dots=\overline{a}_n, \overline{a}_1\neq\overline{a}_n$, where $1\leq p\leq n-1$. In this case, we have either $\overline{a}_1=0$ and $p=n-1$ or $\overline{a}_n=0$ and $p=1$. In fact, for $n=3$, (11) immediately yields $\overline{a}_1=0$ and $p=2$ or $\overline{a}_3=0$ and $p=1$.

For $n \geq 4$, since $\{x_m\} \in D_n$, we obtain $\sum_{i<j} a_{i,m}a_{j,m} \geq 0$ and consequently, passing to the limit, we get $\sum_{i<j} \overline{a}_i\overline{a}_j \geq 0$. This together with (11) shows that either $\overline{a}_1 = 0$ and $p = n-1$ or $\overline{a}_n = 0$ and $p = 1$. Without loss of generality, we suppose that $\overline{a}_n = 0$ and $p = 1$. Then $\overline{a}_1 = \varepsilon = \pm 1$, since $\overline{a} \in S^{n-1}$. Taking (10) into account, from (9) we obtain

$$\varphi(x_m) = \frac{1}{|\prod_{i=1}^n a_{i,m}|^{4/n}} \left(a_{1,m} \sum_{j=2}^n a_{j,m} - \frac{1}{n} \sum_{r \neq s} a_{r,m}a_{s,m} \right)^2 + \frac{1}{|\prod_{i=1}^n a_{i,m}|^{4/n}} \sum_{i=2}^n \left(a_{i,m} \sum_{j \neq i} a_{j,m} - \frac{1}{n} \sum_{r \neq s} a_{r,m}a_{s,m} \right)^2.$$

From this, we obviously get the inequality

$$\varphi(x_m) \geq \frac{1}{|\prod_{i=1}^n a_{i,m}|^{4/n}} \sum_{i=2}^n \left(a_{i,m} \sum_{j \neq i} a_{j,m} - \frac{1}{n} \sum_{r \neq s} a_{r,m}a_{s,m} \right)^2. \tag{12}$$

Now, we set

$$\eta_m := \left(\sum_{i=2}^n a_{i,m}^2 \right)^{\frac{1}{2}}.$$

Since $\psi(a_m) \neq 0$, we may write $(a_{2,m}, \ldots, a_{n,m}) = \eta_m \theta_m$, where

$$\theta_m = (\theta_{2,m}, \ldots, \theta_{n,m})$$

lies in the unit $(n-2)$-sphere $S^{n-2} \subset \mathbb{R}^{n-1}$. By taking a subsequence if necessary, we may assume that $\lim_{m \to \infty} \theta_m = (\overline{\theta}_2, \ldots, \overline{\theta}_n) \in S^{n-2}$. Observe that (12) becomes

$$\varphi(x_m) \geq \delta_m, \tag{13}$$

where the sequence $\{\delta_m\}$ is given by

$$\delta_m = \frac{\sum_{i=2}^n \left[\theta_{i,m} \sum_{j \neq i} a_{j,m} - \frac{1}{n}(2a_{1,m} \sum_{r=2}^n \theta_{r,m} + \eta_m \sum_{r \neq s \geq 2} \theta_{r,m}\theta_{s,m}) \right]^2}{|a_{1,m}|^{4/n} \eta_m^{2(n-2)/n} |\prod_{i=2}^n \theta_{i,m}|^{4/n}}.$$

Using the fact that $\lim_{m \to \infty} a_{1,m} = \varepsilon$, $\lim_{m \to \infty} a_{i,m} = 0$, $\lim_{m \to \infty} \theta_{i,m} = \overline{\theta}_i$ for any $i \in \{2, \ldots, n\}$ and $\lim_{m \to \infty} \eta_m = 0$, we immediately get

$$\lim_{m \to \infty} \sum_{i=2}^n \left[\theta_{i,m} \sum_{j \neq i} a_{j,m} - \frac{1}{n} \left(2a_{1,m} \sum_{r=2}^n \theta_{r,m} + \eta_m \sum_{r,s \geq 2, r \neq s} \theta_{r,m}\theta_{s,m} \right) \right]^2 = \sum_{i=2}^n \left(\varepsilon \overline{\theta}_i - \frac{2\varepsilon}{n} \sum_{r=2}^n \overline{\theta}_r \right)^2 > 0,$$

since $(\overline{\theta}_2, \ldots, \overline{\theta}_n) \in S^{n-2}$. Therefore, we find $\lim_{m \to \infty} \delta_m = +\infty$. This contradicts (13), since $\lim_{m \to \infty} \varphi(x_m) = \inf \varphi(S) \in \mathbb{R}$.

Since D_n is closed, we easily deduce that $x_0 \in S$. Hence, φ attains a minimum $c_2(n)$ on S which depends only on n. We recall that the zeros of φ are precisely the points $x = (x_1, \dots, x_n)$ where either $x_1 = \dots = x_n$ or, after an eventual re-enumeration, $x_1 = \dots = x_p, x_{p+1} = \dots = x_n, x_1 \neq x_n, 1 \leq p \leq n-1$ and $(p-1)x_1 + (n-p-1)x_n = 0$. It is clear that the zeros of φ do not lie in D_n. Since $x_0 \in S$, we obviously have $c_2(n) = \varphi(x_0) > 0$.

Now let $x \in D_n$. Assume that $\psi(x) \neq 0$ and set $\tilde{x} = x/|\psi(x)|^{\frac{1}{n}}$. Clearly $\tilde{x} \in S$, and consequently $\varphi(\tilde{x}) \geq c_2(n)$. Since φ is homogeneous of degree 4, the desired inequality is obviously fulfilled. In the case where $\psi(x) = 0$, the inequality is trivial. □

Proof of Proposition 9. Let A be a real $n \times n$ symmetric matrix. There exists an orthogonal matrix P such that $A = P^t D P$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Furthermore, we have $\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j$. Then a straightforward computation shows that

$$\left\|(\operatorname{tr} A)A - A^2 - \frac{2}{n}\sigma_2(A) I_n\right\|^2 = \sum_{i=1}^{n} \lambda_i^2 \Big(\sum_{j \neq i} \lambda_j\Big)^2 - \frac{1}{n}\Big(\sum_{i \neq j} \lambda_i \lambda_j\Big)^2.$$

In view of our assumption, and appealing to Lemma 10, we immediately get the desired inequality. □

PROPOSITION 11. *For each integer $n \geq 3$, there exists a positive constant $c_3(n)$, depending only on n, such that every real $n \times n$ symmetric matrix A satisfies the inequality*

$$\|(\operatorname{tr} A)A - A^2 - kI_n\|^2 \geq c_3(n) |\det A|^{\frac{4}{n}}$$

for any constant $k > 0$, provided that the eigenvalues of A are not all of the same sign.

Proof. We easily verify that

$$\begin{aligned} &\|(\operatorname{tr} A)A - A^2 - kI_n\|^2 \\ &= \left\|(\operatorname{tr} A)A - A^2 - \frac{2}{n}\sigma_2(A) I_n\right\|^2 + \frac{1}{n}\big(2\sigma_2(A) - nk\big)^2. \end{aligned} \tag{14}$$

If $\sigma_2(A) \geq 0$, then by virtue of Proposition 9 and (14) we get

$$\|(\operatorname{tr} A)A - A^2 - kI_n\|^2 \geq c_2(n) |\det A|^{\frac{4}{n}}.$$

If $nk^2 \leq \frac{1}{4}\|(\operatorname{tr} A)A - A^2\|^2$, then

$$\begin{aligned} \|(\operatorname{tr} A)A - A^2 - kI_n\|^2 &\geq \frac{1}{2}\|(\operatorname{tr} A)A - A^2\|^2 - nk^2 \\ &\geq \frac{1}{4}\|(\operatorname{tr} A)A - A^2\|^2 \end{aligned}$$

and appealing to Proposition 7, we immediately get

$$\|(\operatorname{tr} A)A - A^2 - kI_n\|^2 \geq \frac{1}{4}c_1(n)|\det A|^{\frac{4}{n}}.$$

If $\sigma_2(A) \leq 0$ and $\frac{1}{4}\|(\operatorname{tr}A)A - A^2\|^2 \leq nk^2$, then in view of (14), we have

$$\|(\operatorname{tr} A)A - A^2 - kI_n\|^2 \geq \frac{1}{n}\big(2\sigma_2(A) - nk\big)^2 \geq nk^2 \geq \frac{1}{4}\|(\operatorname{tr} A)A - A^2\|^2$$

and the desired inequality follows from Proposition 7. Obviously, the constant $c_3(n)$ is given by $c_3(n) = \min\{c_2(n), \frac{1}{4}c_1(n)\}$. □

Remark 1. It is worth noticing that the assumptions in Propositions 9 and 11 are essential. In fact, it is easy to see that the inequality in Proposition 9 is not fulfilled for $A = \lambda I_n, \lambda \in \mathbb{R} \setminus \{0\}$, or $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_n)$, where $\lambda_1 = \cdots = \lambda_p, \lambda_{p+1} = \cdots = \lambda_n, 1 \leq p \leq n-1, \lambda_1, \lambda_n \in \mathbb{R} \setminus \{0\}$ and $(p-1)\lambda_1 + (n-p-1)\lambda_n = 0$. Moreover, the inequality in Proposition 11 does not hold for the matrix $A = \sqrt{\frac{k}{n-1}}I_n$.

Remark 2. The constant $c_1(n)$ that appears in Proposition 7 is not computed explicitly here, although one can apply the Lagrange multiplier method to compute $c_1(n)$. In fact, from the proof of Lemma 8, we know that $c_1(n)$ is the minimum of the function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = \sum_{i=1}^{n}\left(x_i \sum_{j=1}^{n} x_j - x_i^2\right)^2, \quad x = (x_1, \ldots, x_n),$$

subject to the constraint $\prod_{i=1}^{n} x_i = \varepsilon$, where $\varepsilon = \pm 1$. For example, the Lagrange multiplier method, after some tedious computations, yields $c_1(3) = 3\sqrt[3]{3/2}$ and $c_1(4) = 4$. In a similar way, one can compute the constant $c_2(n)$. For example, the Lagrange multiplier method for $n = 3$ yields $c_2(3) = 3/\sqrt[3]{2}$.

4. Proofs

We are now ready to give the proofs of the main results.

Proof of Theorem 3. Let $f : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion with shape operator A with respect to a global unit normal vector field and unit normal bundle $UN_f = \{(p, \xi) \in N_f : |\xi| = 1\}$. According to (4), the Ricci tensor of (M^n, g) is given by

$$\operatorname{Ric}(X, Y) = (\operatorname{tr} A)g(AX, Y) - g(A^2X, Y),$$

where X, Y are arbitrary tangent vector fields of M^n. Appealing to Proposition 7, we have

$$\|\operatorname{Ric}_g\|^{\frac{n}{2}} \geq (c_1(n))^{\frac{n}{4}}|\det A|.$$

Integrating over M^n, we obtain

$$\int_{M^n} \|\mathrm{Ric}_g\|^{\frac{n}{2}}\, dM_g \geq (c_1(n))^{\frac{n}{4}} \int_{M^n} |\det A|\, dM_g.$$

Bearing in mind the definition of the total absolute curvature and the fact that

$$\int_{M^n} |\det A|\, dM_g = \frac{1}{2} \int_{UN_f} |\det A_\xi|\, d\Sigma,$$

we finally get

$$\int_{M^n} \|\mathrm{Ric}_g\|^{\frac{n}{2}}\, dM_g \geq a(n)\tau(f), \tag{15}$$

where $\tau(f)$ is the total absolute curvature of f and $a(n) := \frac{1}{2}(c_1(n))^{\frac{n}{4}} \operatorname{Vol}(S^n)$. Thus, part (i) of the theorem follows immediately from inequality (15) and Theorem 6.

Now assume that $\int_{M^n} \|\mathrm{Ric}_g\|^{\frac{n}{2}}\, dM_g < 3a(n)$. Then, (15) yields $\tau(f) < 3$ and consequently, by virtue of Theorem 6, M^n is homeomorphic to the sphere S^n. □

Proof of Theorem 4. Let $f : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion with unit normal bundle UN_f and shape operator A_ξ with respect to ξ, where $(p, \xi) \in UN_f$. Bearing in mind (4), we deduce that

$$\|\mathrm{Ric}_g - kg\|^2(p) = \|(\operatorname{tr} A_\xi)A_\xi - A_\xi^2 - kI\|^2, \quad (p, \xi) \in UN_f,$$

where I denotes the identity transformation. In view of our assumption and appealing to Proposition 11, we have

$$\|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}(p) \geq (c_3(n))^{\frac{n}{4}} |\det A_\xi| \tag{16}$$

for all $(p, \xi) \in U^i N_f$ where

$$U^i N_f = \{(p, \xi) \in UN_f : A_\xi \text{ has exactly } i \text{ negative eigenvalues}\}$$

and $1 \leq i \leq n-1$. Integrating (16) over UN_f, we obtain

$$\int_{UN_f} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, d\Sigma \geq (c_3(n))^{\frac{n}{4}} \sum_{i=1}^{n-1} \int_{U^i N_f} |\det A_\xi|\, d\Sigma.$$

Bearing in mind (3) and the definition of the total curvature of index i, we get

$$\int_{UN_f} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, d\Sigma \geq (c_3(n))^{\frac{n}{4}} \operatorname{Vol}(S^n) \sum_{i=1}^{n-1} \tau_i.$$

On the other hand, observe that

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, dM_g = \frac{1}{2} \int_{UN_f} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, d\Sigma.$$

Then by virtue of (1), we finally have

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, dM_g \geq c(n) \sum_{i=1}^{n-1} \tau_i \geq c(n) \sum_{i=1}^{n-1} \beta_i(M^n; \mathcal{F}), \tag{17}$$

where the constant $c(n)$ is given by $c(n) := \frac{1}{2}(c_3(n))^{\frac{n}{4}} \,\mathrm{Vol}(S^n)$. This completes the proof of part (i) of the theorem.

Now suppose that

$$\int_{M^n} \|\mathrm{Ric}_g - kg\|^{\frac{n}{2}}\, dM_g < c(n).$$

Then in view of (17), we conclude that $\sum_{i=1}^{n-1} \tau_i < 1$. Thus, there exists $u \in S^n$ such that the height function $h_u : M^n \longrightarrow \mathbb{R}$ is a Morse function whose number of critical points of index i satisfies $\mu_i(u) = 0$ for any $1 \leq i \leq n-1$. This means that the index of each critical point of h_u is either zero or n. Appealing to [1, Lemma 4.11(2)], we deduce that h_u has at most one critical point of index zero, i.e., $\mu_0(u) \leq 1$. Moreover, from the weak Morse inequalities we have $\mu_0(u) \geq \beta_0(M^n; \mathcal{F})$ and $\mu_n(u) \geq \beta_n(M^n; \mathcal{F})$. Since M^n is connected and oriented, we infer that $\mu_0(u) = \mu_n(u) = 1$. Therefore, the Morse function h_u has only two critical points. According to Reeb's theorem, M^n is homeomorphic to S^n. $\square$

Proof of Theorem 5. Let $f : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion with unit normal bundle UN_f and shape operator A_ξ with respect to ξ, where $(p, \xi) \in UN_f$. Bearing in mind (4) and (5), we deduce that the squared length of the traceless Ricci tensor of (M^n, g) is given by

$$\left\|\mathrm{Ric}_g - \frac{s_g}{n} g\right\|^2 (p) = \left\|(\mathrm{tr}\, A_\xi) A_\xi - A_\xi^2 - \frac{2}{n}\sigma_2(A_\xi) I\right\|^2, \quad (p, \xi) \in UN_f,$$

where I denotes the identity transformation. In view of our assumption and appealing to Proposition 9, we have

$$\left\|\mathrm{Ric}_g - \frac{s_g}{n} g\right\|^{\frac{n}{2}} (p) \geq (c_2(n))^{\frac{n}{4}} |\det A_\xi| \tag{18}$$

for all $(p, \xi) \in U^i N_f$, $1 \leq i \leq n-1$. The rest of the proof is almost verbatim the same as the proof of Theorem 4, apart from the fact that the constant $b(n)$ is given by $b(n) := \frac{1}{2}(c_2(n))^{\frac{n}{4}} \,\mathrm{Vol}(S^n)$. $\square$

Now Theorem 1 follows from Theorems 3 and 4, where $\varepsilon(n, k) = 3a(n)$ if $k = 0$ and $\varepsilon(n, k) = c(n)$ if $k > 0$, while Theorem 2 is part of Theorem 5.

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