INTEGRAL REPRESENTATIONS ON NONSMOOTH DOMAINS

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ABSTRACT. We derive integral representations for $(0,q)$-forms, $q \geq 1$, on nonsmooth strictly pseudoconvex domains, the Henkin–Leiterer domains. A $(0,q)$-form, f is written in terms of integral operators acting on f, $\bar{\partial}f$, and $\bar{\partial}^* f$. The representation is applied to derive L^∞ estimates.

1. Introduction

Lieb and Range in [7] developed a powerful integral representation by which estimates in the theory of the $\bar{\partial}$-Neumann problem could be deduced. The main theorem was an integral representation of $(0,q)$-forms on $D \subset\subset X$ a smooth strictly pseudoconvex domain in a complex manifold X.

THEOREM 1.1 (Lieb–Range). *Let* $P_0 : L^2(D) \to \mathcal{O} \cap L^2(D)$ *be the Bergman projection. There exist integral operators* $T_q : L^2_{(0,q+1)}(D) \to L^2_{(0,q)}(D)$, $0 \leq q < n = \dim X$, *such that for* $f \in L^2_{(0,q)} \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ *one has*

$$f = P_0 f + T_0 \bar{\partial} f + error\ terms \quad for\ q = 0$$

and

$$(1.1) \qquad f = T_q \bar{\partial} f + T^*_{q-1} \bar{\partial}^* f + error\ terms \quad for\ q \geq 1.$$

In (1.1), the metric has to be carefully adapted to the boundary. The choice of the metric as the Levi metric as in Greiner and Stein [3] was essential in their "cancellation of singularities" argument, which allowed for treatment of terms in the representation as error terms.

We take up the problem here of establishing an integral representation in the manner of [7] relaxing the assumption that D be smooth. Let D have a

defining function, r. We allow for singularities in the boundary, ∂D of D by permitting the possibility that dr vanishes at points on ∂D. Such domains were first studied by Henkin and Leiterer in [5], and we therefore refer to them as Henkin–Leiterer domains.

We shall make the additional assumption that r is a Morse function. Let U be a neighborhood of ∂D. Then

$$U \cap D = \{x \in U : r(x) < 0\},$$

r with only nondegenerate critical points on U. We have

$$\partial D = \{x : r(x) = 0\},$$

and we can assume that there are finitely many critical points on bD, and none on $U \setminus bD$.

In [2], Lieb and the author studied the Bergman projection on Henkin–Leiterer domains in $\mathbb{C}^n$, and obtained weighted L^p estimates. We here concern ourselves with proving an analogue of (1.1) on Henkin–Leiterer domains. The domain D has an exhaustion of smooth strictly pseudoconvex domains $\{D_\epsilon\}_\epsilon$ on each of which the analysis of Lieb and Range applies. One immediate problem one runs into with this approach is that forms which are $\mathrm{Dom}(\bar{\partial}^*)$ on D may not be in $\mathrm{Dom}(\bar{\partial}_\epsilon^*)$ on D_ϵ. We deal with this problem by using a density lemma of Henkin and Iordan [4] which provides for forms f_ϵ which are in $L^2(D_\epsilon) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}_\epsilon^*)$ and which approximate a given $f \in L^2(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$. Our approach therefore is to obtain an integral representation valid on each domain D_ϵ and in the end let $\epsilon \to 0$. In this approach, we need to multiply our operators by factors of $|dr|$ so that convergence of the representation as $\epsilon \to 0$ is obtained. Let $\gamma = |\partial r|$. The analogue of Theorem 1.1 we establish here is the following.

THEOREM 1.2. *Let D be a Henkin–Leiterer domain with a defining function which is Morse. There exist integral operators $\tilde{T}_q : L^2_{(0,q+1)}(D) \to L^2_{(0,q)}(D)$, $0 \le q < n = \dim X$, such that for $f \in L^2_{(0,q)} \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ one has*

$$\gamma^3 f = \tilde{T}_q \,\bar{\partial} f + \tilde{T}_{q-1}^* \,\bar{\partial}^* f + error\ terms \quad for\ q \ge 1.$$

In a separate paper, [1], we build off the integral representation established here, and in particular we look at the mapping properties of the integral operators under differentiation so as to establish C^k estimates.

2. Admissible operators

With local coordinates denoted by $\zeta_1, \ldots, \zeta_n$, we define a Levi metric in a neighborhood of ∂D by

$$ds^2 = \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j \, \partial \bar{\zeta}_k}(\zeta) \, d\zeta_j \, d\bar{\zeta}_k.$$

A Levi metric on X is a Hermitian metric which is a Levi metric in a neighborhood of ∂D.

We thus equip X with a Levi metric and we take $\rho(x, y)$ to be a symmetric, smooth function on $X \times X$ which coincides with the geodesic distance in a neighborhood of the diagonal, Λ, and is positive outside of Λ.

For ease of notation, in what follows we will always work with local coordinates, ζ and z.

Since D is strictly pseudoconvex and r is a Morse function, we can take $r_\epsilon = r + \epsilon$ for epsilon small enough. Then r_ϵ will be defining functions for smooth, strictly pseudoconvex D_ϵ. For such r_ϵ, we have that all derivatives of r_ϵ are independent of ϵ. In particular, $\gamma_\epsilon(\zeta) = \gamma(\zeta)$ and $\rho_\epsilon(\zeta, z) = \rho(\zeta, z)$.

Let F be the Levi polynomial for D_ϵ:

$$F(\zeta, z) = \sum_{j=1}^{n} \frac{\partial r_\epsilon}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r_\epsilon}{\partial \zeta_j \, \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k).$$

We note that $F(\zeta, z)$ is independent of ϵ since derivatives of r_ϵ are.

For ϵ small enough, we can choose $\delta > 0$ and $\varepsilon > 0$ and a patching function $\varphi(\zeta, z)$, independent of ϵ, on $\mathbb{C}^n \times \mathbb{C}^n$ such that

$$\varphi(\zeta, z) = \begin{cases} 1 & \text{for } \rho^2(\zeta, z) \leq \frac{\varepsilon}{2}, \\ 0 & \text{for } \rho^2(\zeta, z) \geq \frac{3}{4}\varepsilon, \end{cases}$$

and defining $S_\delta = \{\zeta : |r(\zeta)| < \delta\}$, $D_{-\delta} = \{\zeta : r(\zeta) < \delta\}$, and

$$\phi_\epsilon(\zeta, z) = \varphi(\zeta, z)\big(F(\zeta, z) - r_\epsilon(\zeta)\big) + \big(1 - \varphi(\zeta, z)\big)\rho^2(\zeta, z),$$

we have the following lemma.

LEMMA 2.1. *On $D_\epsilon \times D_\epsilon \cap S_\delta \times D_{-\delta}$,*

$$|\phi_\epsilon| \gtrsim |\langle \partial r_\epsilon(z), \zeta - z \rangle| + \rho^2(\zeta, z),$$

where the constants in the inequalities are independent of ϵ.

Proof. From a Taylor series expansion,

$$(2.1) \qquad |\phi_\epsilon| \gtrsim -r_\epsilon(\zeta) - r_\epsilon(z) + \rho^2(\zeta, z) + |\operatorname{Im}\phi_\epsilon|.$$

On $D_\epsilon \times D_\epsilon$, $-r_\epsilon(\zeta) - r_\epsilon(z) \geq |r_\epsilon(\zeta) - r_\epsilon(z)|$. We combine this with

$$|\operatorname{Im}\phi_\epsilon| + \rho^2(\zeta, z) \gtrsim |\operatorname{Im}\langle \partial r_\epsilon(\zeta), \zeta - z \rangle|,$$

and we therefore write

$$|\phi_\epsilon| \gtrsim |r_\epsilon(\zeta) - r_\epsilon(z)| + |\operatorname{Im}\langle \partial r_\epsilon(\zeta), \zeta - z \rangle| + \rho^2(\zeta, z)$$
$$\gtrsim |\langle \partial r_\epsilon(z), \zeta - z \rangle| + \rho^2(\zeta, z),$$

where the last inequality follows from

$$|\langle \partial r_\epsilon(z), \zeta - z \rangle| + |\langle \partial r_\epsilon(\zeta), \zeta - z \rangle| + \rho^2(\zeta, z)$$
$$\approx |r_\epsilon(\zeta) - r_\epsilon(z)| + |\operatorname{Im}\langle \partial r_\epsilon(\zeta), \zeta - z \rangle| + \rho^2(\zeta, z),$$

which itself is an easy consequence of a Taylor expansion.

All inequality signs have constants which are independent of ϵ since $r_\epsilon \xrightarrow{C^2} r$.

$\square$

We at times have to be precise and keep track of factors of γ which occur in our integral kernels. We shall write $\mathscr{E}_{j,k}(\zeta, z)$, $k \leq 0$, for those double forms on open sets $U \subset D \times D$ such that $\mathscr{E}_{j,k}$ is smooth on U and satisfies

$$(2.2) \qquad \mathscr{E}_{j,k}(\zeta, z) \lesssim \gamma^l (\gamma^*)^m |\zeta - z|^j, \quad l + m = k,$$

where $l, m \leq 0$, with the property that

$$\Lambda \mathscr{E}_{j,k} = \mathscr{E}_{j-1,k} + \mathscr{E}_{j,k-1},$$

where Λ is a first order differential operator in either the ζ or z variable. Here and below $\gamma^* = \gamma(z)$, the $*$ having a similar meaning for other functions of one variable. We write $\xi_k(\zeta)$ for a function with the property

$$|\gamma^\alpha D_\zeta^\alpha \xi_k(\zeta)| \lesssim \gamma^k,$$

where D_ζ^α is a differential operator of order α.

We shall write $\mathscr{E}_j$ for those double forms on $X \times X$ such that $\mathscr{E}_j$ is smooth and satisfies

$$\mathscr{E}_j(x, y) \lesssim \rho^j(x, y).$$

Here

$$P_\epsilon(\zeta, z) = \rho^2(\zeta, z) + 2 \frac{r_\epsilon(\zeta)}{\gamma(\zeta)} \frac{r_\epsilon(z)}{\gamma(z)}.$$

Definition 2.2. A double differential form $\mathscr{A}^\epsilon(\zeta, z)$ on $\overline{D}_\epsilon \times \overline{D}_\epsilon$ is an *admissible* kernel, if it has the following properties:

(i) $\mathscr{A}^\epsilon$ is smooth on $\overline{D}_\epsilon \times \overline{D}_\epsilon - \Lambda_\epsilon$.

(ii) For each point $(\zeta_0, \zeta_0) \in \Lambda_\epsilon$, there is a neighborhood $U \times U$ of (ζ_0, ζ_0) on which $\mathscr{A}^\epsilon$ or $\overline{\mathscr{A}}^\epsilon$ has the representation

$$(2.3) \qquad \xi_N \xi_M^* \mathscr{E}_{j,\alpha} P_\epsilon^{-t_0} \phi_\epsilon^{t_1} \overline{\phi}_\epsilon^{t_2} \phi_\epsilon^{*t_3} \overline{\phi}_\epsilon^{*t_4} r_\epsilon^l r_\epsilon^{*m}$$

with $N, M, \alpha, j, t_0, \ldots, m$ integers and $j, k, t_0, l, m \geq 0$, $-t = t_1 + \cdots + t_4 \leq 0$, $N, M \geq 0$, and $N + M + \alpha \geq 0$.

The above representation is of *smooth type s* for

$$s = 2n + j + \min\{2, t - l - m\} - 2(t_0 + t - l - m).$$

We define the *type* of $\mathscr{A}^\epsilon(\zeta, z)$ to be

$$\tau = s - \max\{0, 2 - N - M - \alpha\}.$$

$\mathscr{A}^\epsilon$ has *smooth type $\geq s$* if at each point (ζ_0, ζ_0) there is a representation (2.3) of smooth type $\geq s$. $\mathscr{A}^\epsilon$ has *type $\geq \tau$* if at each point (ζ_0, ζ_0) there is a representation (2.3) of type $\geq \tau$. We shall also refer to the *double type* of an operator (τ, s) if the operator is of type τ and of smooth type s.

The definition of smooth type above is taken from [6].

Let $\mathscr{A}_j^\epsilon$ be kernels of type j. We denote by $\mathscr{A}_j$ the pointwise limit as $\epsilon \to 0$ of $\mathscr{A}_j^\epsilon$ and define the double type of $\mathscr{A}_j$ to be the double type of the $\mathscr{A}_j^\epsilon$ of which it is a limit. We also denote by A_j^ϵ to be operators with kernels of the form $\mathscr{A}_j^\epsilon$. A_j will denote the operators with kernels $\mathscr{A}_j$. We use the notation $\mathscr{A}_{(j,k)}^\epsilon$ (resp. $\mathscr{A}_{(j,k)}$) to denote kernels of double type (j,k).

We begin with estimates on the kernels of a certain type.

PROPOSITION 2.3. *Let $\mathscr{A}_j^\epsilon$ be of type j, and*

$$1 \le \lambda < \frac{2n+2}{2n+2-j}.$$

Then

$$\text{(2.4)} \qquad \int_{D_\epsilon} |\mathscr{A}_j^\epsilon(\zeta, z)|^\lambda \, dV(\zeta) < C$$

and, similarly,

$$\text{(2.5)} \qquad \int_{D_\epsilon} |\mathscr{A}_j^\epsilon(\zeta, z)|^\lambda \, dV(z) < C$$

for $C < \infty$ a constant independent of ϵ, z or ζ.

Proof. That (2.4) and (2.5) hold for a fixed $\epsilon > 0$ and a constant C which may depend on ϵ follows from the results on smooth strictly pseudoconvex domains (see [6]). We will perform the calculations in the limit $\epsilon \to 0$ so that standard uniform boundedness principles apply to provide bounds uniform in ϵ.

We handle the estimates case by case depending on the kernel's double type. For the various cases, we now describe the coordinate system with which we work. Fix z such that $\gamma(z) \ne 0$. We define the complex tangent space at z:

$$T_z^c = \{\zeta : \langle \partial r(z), \zeta - z \rangle = 0\}.$$

We define the orthonormal system of coordinates, s_1, s_2, $t_1, \ldots, t_{2n-2}$ such that

$$s_1 = \operatorname{Re}\left\langle \frac{\partial r(z)}{\gamma(z)}, \zeta - z \right\rangle,$$

$$s_2 = \operatorname{Im}\left\langle \frac{\partial r(z)}{\gamma(z)}, \zeta - z \right\rangle,$$

and such that $t_1, \ldots, t_{2n-2}$ span T_z^c. Let also

$$s = \sqrt{s_1^2 + s_2^2},$$

$$t = \sqrt{t_1^2 + \cdots + t_{2n-2}^2}.$$

From Lemma 2.1, we have

$$|\phi| \gtrsim |\langle \partial r(z), \zeta - z \rangle| + \rho^2,$$

which in the above coordinates reads

$$|\phi| \gtrsim \gamma(z)s + s^2 + t^2.$$

Case (a). $\mathscr{A}_j$ is of double type (j, j).

For kernels of double type (j, j), we can use the relation $|\gamma(\zeta) - \gamma(z)| = O(|\zeta - z|)$ along with estimates for kernels of double types $(j, j+1)$ and $(j, j+2)$, to reduce the different subcases we need to consider to

(i) $|\mathscr{A}_j| \lesssim \dfrac{\gamma(z)^2}{P^{n-j/2}},$

(ii) $|\mathscr{A}_j| \lesssim \dfrac{\gamma(z)^2}{P^{n-(j+1)/2}|\phi|},$

(iii) $|\mathscr{A}_j| \lesssim \dfrac{\gamma(z)^2}{P^{n-j/2-\mu}|\phi|^{\mu+1}}, \quad \mu \geq 1.$

We will consider the last two subcases, since the first is easier to handle, and can be covered by case (c) below.

Subcase (ii). We choose $\alpha < 2$ such that

$$\lambda < \frac{2n - 2 + 2\alpha}{2n + 1 - j},$$

and let $\beta = \min(\alpha, \lambda)$. We have

$$(2.6) \qquad \int_D \gamma(z)^{2\lambda} \frac{1}{|\phi|^\lambda P^{\lambda(n-(j+1)/2)}} \, dV(\zeta)$$

$$\lesssim \gamma(z)^{2\lambda} \int_V \frac{st^{2n-3}}{(\gamma(z)s + s^2 + t^2)^\lambda (s^2 + t^2)^{\lambda(n-(j+1)/2)}} \, ds \, dt$$

$$\lesssim \gamma(z)^{2\lambda - \beta} \int_V s^{1-\beta} \frac{t^{2n-3}}{(s+t)^{2\lambda n + \lambda(1-j) - 2\beta}} \, ds \, dt,$$

where V is a bounded subset of $\mathbb{R}^2$. In the case $\beta = \alpha$, we can estimate the integral in (2.6) by

$$\gamma(z)^{2\lambda - \alpha} \int_V s^{1-\alpha} \frac{t^{2n-3}}{t^{2\lambda n + \lambda(1-j) - 2\alpha}} \, ds \, dt \lesssim 1,$$

where the inequality follows from our choice of α.

In the case that $\beta = \lambda$, we choose a σ such that

$$\sigma < 2 - \lambda,$$

$$\lambda < \frac{2n - 2 + \sigma}{2n - 1 - j},$$

and we have

$$\gamma(z)^\lambda \int_V s^{1-\lambda} \frac{t^{2n-3}}{(s+t)^{\lambda(2n-1-j)}}\, ds\, dt$$

$$\lesssim \gamma(z)^\lambda \int_V s^{1-\lambda-\sigma} \frac{t^{2n-3}}{t^{\lambda(2n-1-j)-\sigma}}\, ds\, dt$$

$$\lesssim 1,$$

where the last inequality follows from our choice of σ.

(2.5) holds in a similar manner by switching ζ and z.

Subcase (iii). In this case to prove (2.4), we choose α so that $\alpha < 2$ and

$$\lambda < \frac{2n-2+2\alpha}{2n+2-j},$$

and estimate

$$\gamma(z)^{2\lambda} \int_D \frac{1}{|\phi|^{\lambda(\mu+1)} P^{\lambda(n-\mu-j/2)}}\, dV(\zeta)$$

$$\lesssim \gamma(z)^{2\lambda} \int_V \frac{st^{2n-3}}{(\gamma(z)s+s^2+t^2)^{\lambda(\mu+1)}(s^2+t^2)^{\lambda(n-\mu-j/2)}}\, ds\, dt$$

$$\lesssim \gamma(z)^{2\lambda-\alpha} \int_V s^{1-\alpha} \frac{t^{2n-3}}{t^{\lambda(2n+2-j)-2\alpha}}\, dt\, ds$$

$$\lesssim 1,$$

where V is a bounded subset of $\mathbb{R}^2$.

Again, (2.5) holds in a similar manner.

Case (b). $\mathscr{A}_j$ is of double type $(j, j+1)$.

The different subcases we need to consider are

$$\text{(i)} \quad |\mathscr{A}_j| \lesssim \frac{\gamma(z)}{P^{n-(j+1)/2}},$$

$$\text{(ii)} \quad |\mathscr{A}_j| \lesssim \frac{\gamma(z)}{P^{n-(j+2)/2}|\phi|},$$

$$\text{(iii)} \quad |\mathscr{A}_j| \lesssim \frac{\gamma(z)}{P^{n-(j+1)/2-\mu}|\phi|^{\mu+1}}, \quad \mu \geq 1.$$

Subcases (i) and (ii) can be handled by the estimate in case (c) below. The more difficult estimate is that of subcase (iii), for which we choose an $\alpha < 1/2$ which satisfies

$$\lambda < \frac{2n+2\alpha}{2n+1-j}$$

and estimate

$$\gamma(z)^\lambda \int_D \frac{1}{|\phi|^{\lambda(\mu+1)} P^{\lambda(n-(j+1)/2-\mu)}}\, dV(\zeta)$$

$$\lesssim \gamma(z)^\lambda \int_V \frac{st^{2n-3}}{(\gamma(z)s+s^2+t^2)^{\lambda(\mu+1)}(s^2+t^2)^{\lambda(n-(j+1)/2-\mu)}}\, ds\, dt$$

$$\lesssim \gamma(z)^{\lambda-1} \int_V \frac{t^{2n-3}}{(s^2+t^2)^{\lambda(n-(j-1)/2)-1}} \, dt \, ds$$

$$\lesssim \int_V s^{-2\alpha} t^{2n-1-\lambda(2n+1-j)+2\alpha} \, dt \, ds$$

$$\lesssim 1,$$

where V is a bounded subset of $\mathbb{R}^2$.

Case (c). $\mathscr{A}_j$ is of double type $(j, j+2)$.

Using the coordinates of cases (a) and (b), we can estimate all the subcases for kernels of double type $(j, j+2)$ by

$$\int_D |\mathscr{A}_j(\zeta, z)|^\lambda \, dV(\zeta) \lesssim \int_V \frac{s t^{2n-3}}{(s^2+t^2)^{\lambda(n-j/2)}} \, ds \, dt$$

$$\lesssim \int_0^M r^{2n-1-\lambda(2n-j)} \, dr$$

$$\lesssim 1,$$

where V is a bounded subset of $\mathbb{R}^2$, $M > 0$ is a bounded constant, and $r = \sqrt{s^2+t^2}$.

The same estimates hold for (2.5). $\qquad\square$

As a consequence of Proposition 2.3 and a generalization of Young's inequality [8] is the following.

COROLLARY 2.4. *Let A_j be an operator of type $j \geq 1$. Then*

$$A_j : L^p(D) \to L^s(D), \quad \frac{1}{s} > \frac{1}{p} - \frac{j}{2n+2}.$$

We let $\mathscr{E}^i_{j-2n}(\zeta, z)$ be a kernel of the form

$$\mathscr{E}^i_{j-2n}(\zeta, z) = \frac{\mathscr{E}_{m,0}(\zeta, z)}{\rho^{2k}(\zeta, z)}, \quad j \geq 1,$$

where $m - 2k \geq j - 2n$. We denote by E_{j-2n} the corresponding isotropic operator. The following theorem follows from [6] (see Theorem VII.4.1).

THEOREM 2.5. *Then we have the following properties*:

$$E_{1-2n} : L^p(D) \to L^s(D)$$

for any $1 \leq p \leq s \leq \infty$ with $1/s > 1/p - 1/2n$.

3. Basic integral representation

In this section, we present the basic integral representation for forms on bounded smooth strictly pseudoconvex domains as worked out by Lieb and Range [7].

We will need the following lemma.

LEMMA 3.1.
$$\frac{r_\epsilon}{\gamma} \in C^1(D_\epsilon)$$

with C^1-estimates independent of ϵ.

Proof. Since $r_\epsilon \xrightarrow{C^1} r$, we show that
$$\frac{r}{\gamma} \in C^1(D).$$

Outside of a neighborhood of any critical point of r, the result is obvious. We denote the critical points of r by $p_1, \ldots, p_k$, and take ε small enough so that in each
$$U_{2\varepsilon}(p_j) = \{\zeta : D \cap |\zeta - p_j| < 2\varepsilon\}$$

for $j = 1, \ldots, k$, there are coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ such that

(3.1) $$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2$$

with $u_{j_\alpha}(p_j) = v_{j_\beta}(p_j) = 0$ for all $1 \le \alpha \le m$ and $m + 1 \le \beta \le 2n$, from the Morse lemma.

In these coordinates
$$\frac{r(\zeta)}{\gamma(\zeta)} = -\frac{u_j^2 - v_j^2}{\sqrt{u_j^2 + v_j^2}},$$

where $u_j^2 = u_{j_1}^2 + \cdots + u_{j_m}^2$ and $v_j^2 = v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2$.

It is then easy to see $r(\zeta)/\gamma(\zeta)$ is in C^1 by differentiating with respect to the given coordinates. □

We start with the differential forms
$$\beta(\zeta, z) = \frac{\partial_\zeta \rho^2(\zeta, z)}{\rho^2(\zeta, z)},$$

$$\alpha_\epsilon(\zeta, z) = \xi(\zeta) \frac{\partial r_\epsilon(\zeta)}{\phi_\epsilon(\zeta, z)},$$

where $\xi(\zeta)$ is a smooth patching function which is equivalently 1 for $|r(\zeta)| < \delta$ and 0 for $|r(\zeta)| > \frac{3}{2}\delta$, and $\delta > 0$ is sufficiently small. We define
$$C_q^\epsilon = C_q(\alpha_\epsilon, \beta) = \sum_{\mu=0}^{n-q-2} \sum_{\nu=0}^{q} a_{q\mu\nu} C_{q\mu\nu}(\alpha_\epsilon, \beta),$$

where
$$a_{q\mu\nu} = \left(\frac{1}{2\pi i}\right)^n \binom{\mu + \nu}{\mu} \binom{n - 2 - \mu - \nu}{q - \mu}$$

and
$$C_{q\mu\nu}(\alpha_\epsilon, \beta) = \alpha_\epsilon \wedge \beta \wedge (\bar{\partial}_\zeta \alpha_\epsilon)^\mu \wedge (\bar{\partial}_\zeta \beta)^{n-q-\mu-2} \wedge (\bar{\partial}_z \alpha_\epsilon)^\nu \wedge (\bar{\partial}_z \beta)^{q-\nu}.$$

Denoting the Hodge $*$-operator by $*$, we then define

$$\mathcal{L}_q^\epsilon(\zeta, z) = (-1)^{q+1} *_\zeta \overline{C_q^\epsilon(\zeta, z)}.$$

We also write

$$K_q^\epsilon(\zeta, z) = (-1)^{q(q-1)/2} \binom{n-1}{q} \frac{1}{(2\pi i)^n} \alpha_\epsilon \wedge (\bar{\partial}_\zeta \alpha_\epsilon)^{n-q-1} \wedge (\bar{\partial}_z \alpha_\epsilon)^q$$

and

$$\Gamma_{0,q}^\epsilon(\zeta, z) = \frac{(n-2)!}{2\pi^n} \frac{1}{\rho^{2n-2}} (\bar{\partial}_\zeta \, \partial_z \rho^2)^q.$$

The kernels in our integral representation are defined through the following for $q \geq 1$:

$$\begin{aligned}
\mathcal{T}_q^{a\epsilon}(\zeta, z) &= \vartheta_\zeta \mathcal{L}_q^\epsilon(\zeta, z) - \partial_z \mathcal{L}_{q-1}^\epsilon(\zeta, z), \\
\mathcal{T}_q^{i\epsilon}(\zeta, z) &= \bar{\partial}_\zeta \Gamma_{0,q}^\epsilon(\zeta, z), \\
\mathcal{T}_q^\epsilon(\zeta, z) &= \mathcal{T}_q^{a\epsilon}(\zeta, z) + \mathcal{T}_q^{i\epsilon}(\zeta, z), \\
\mathcal{P}_q^\epsilon(\zeta, z) &= \mathcal{Q}_q^\epsilon(\zeta, z) - \mathcal{Q}_q^{\epsilon*}(\zeta, z) \\
&= \vartheta_\zeta \, \partial_z \mathcal{L}_{q-1}^\epsilon(\zeta, z) - (\vartheta_\zeta \, \partial_z \mathcal{L}_{q-1}^\epsilon(\zeta, z))^*, \\
\mathcal{Q}_q^\epsilon(\zeta, z) &= \vartheta_\zeta \, \partial_z \mathcal{L}_{q-1}^\epsilon(\zeta, z).
\end{aligned}$$

We denote the operators with kernels $\mathcal{T}_q^\epsilon$ and $\mathcal{P}_q^\epsilon$ by $\mathbf{T}_q^\epsilon$ and $\mathbf{P}_q^\epsilon$, respectively.

As mentioned above, our goal is to establish C^k-estimates on the Henkin–Leiterer domain, D, by exhausting D by smooth strictly pseudoconvex domains, $\{D_\epsilon\}_\epsilon$ and using the analysis of Lieb and Range [7] on the smooth domains D_ϵ. It is therefore necessary to be able to approximate a given form $f \in \mathrm{Dom}(\bar{\partial}^*) \cap \mathrm{Dom}(\bar{\partial})$ by forms f_ϵ such that

$$f_\epsilon \xrightarrow{L^2} f,$$

$$\bar{\partial} f_\epsilon \xrightarrow{L^2} \bar{\partial} f,$$

$$\bar{\partial}_\epsilon^* f_\epsilon \xrightarrow{L^2} \bar{\partial}^* f.$$

For this purpose, we define the graph norm on D

$$\|u\|_G^2 = \|u\|^2 + \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2.$$

With $H_\epsilon = \bar{\partial}\bar{\partial}_\epsilon^* + \bar{\partial}_\epsilon^* \bar{\partial} + I$,

$$\mathrm{Dom}(H_\epsilon) = \{f \in \mathrm{Dom}(\bar{\partial}_\epsilon^*) \cap \mathrm{Dom}(\bar{\partial}) | \bar{\partial} f \in \mathrm{Dom}(\bar{\partial}_\epsilon^*), \bar{\partial}_\epsilon^* f \in \mathrm{Dom}(\bar{\partial})\},$$

and $\square_\epsilon$ defined by

$$\square_\epsilon = \bar{\partial}\bar{\partial}_\epsilon^* + \bar{\partial}_\epsilon^* \bar{\partial},$$

we make the following.

Definition 3.2. We say f is in the space $\mathcal{M}_{(p,q)}(D)$, $f \in \mathcal{M}_{(p,q)}(D)$, if f is the limit in $L^2_{(p,q)}(D; \mathrm{loc})$ of $f_\epsilon \in \mathrm{Dom}\, H_\epsilon$ such that $\sup_\epsilon \{\|f_\epsilon\|_{G,\epsilon}, \|\square_\epsilon f_\epsilon\|_\epsilon\} < \infty$.

From Lemma 1 in [4], we have the following.

PROPOSITION 3.3. $\mathcal{M}_{(p,q)}(D)$ *is dense in* $\mathrm{Dom}(\bar{\partial}^*) \cap \mathrm{Dom}(\bar{\partial})$ *for the graph norm.*

Let $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\bar{\partial}^*) \cap \mathrm{Dom}(\bar{\partial})$. We take a sequence $\{f_\epsilon\}_\epsilon$ such that $f_\epsilon \in \mathrm{Dom}\, H_\epsilon$ and $f_\epsilon \to f$ in the graph norm.

For each f_ϵ, we apply the analysis of [7] on D_ϵ, taking into account factors of γ, and obtain the integral representation.

THEOREM 3.4.

$$f_\epsilon(z) = \mathbf{T}^\epsilon_q \bar{\partial} f_\epsilon + (\mathbf{T}^\epsilon_{q-1})^* \bar{\partial}^*_\epsilon f_\epsilon + \mathbf{P}^\epsilon_q f_\epsilon$$

$$+ \left(A^\epsilon_{(0,2)} + E_{2-2n} \right) \bar{\partial} f_\epsilon + E_{2-2n} \bar{\partial}^*_\epsilon f_\epsilon + \left(\frac{1}{\gamma^*} A^\epsilon_{(-1,1)} + E_{1-2n} \right) f_\epsilon.$$

The proof follows as in [6] (see the proof of Theorem IV.5.52), but since the factors of γ are of particular importance here, we sketch the proof including this new detail. For $N \geq 0$, we let R_N denote an N-fold product, or a sum of such products, of first derivatives of $r(z)$, with the notation $R_0 = 1$.

Sketch of proof of Theorem 3.4. Our starting point is the Bochner–Martinelli–Koppelman (BMK) formula for $f \in C^1_{0,q}(\overline{D}_\epsilon)$. Let B_q be defined by

$$B_q = \Omega_q(\beta) = (-1)^{q(q-1)/2} \binom{n-1}{q} \frac{1}{(2\pi i)^n} \beta \wedge (\bar{\partial}_\zeta \beta)^{n-q-1} \wedge (\bar{\partial}_z \beta)^q.$$

Then for $z \in D_\epsilon$

$$(3.2) \qquad f(z) = \int_{\partial D_\epsilon} f(\zeta) \wedge B_q(\zeta, z) - \int_{D_\epsilon} \bar{\partial} f(\zeta) \wedge B_q(\zeta, z)$$

$$- \bar{\partial}_z \int_{D_\epsilon} f(\zeta) \wedge B_{q-1}(\zeta, z)$$

$$+ (f(\zeta), \mathscr{E}_{1-2n}(\zeta, z)) + (\bar{\partial} f(\zeta), \mathscr{E}_{2-2n}(\zeta, z)).$$

Define the kernels $K^\epsilon_q(\zeta, z)$ by $K^\epsilon_q(\zeta, z) = \Omega_q(\alpha_\epsilon)$. We then proceed to replace the boundary integral in the BMK formula by

$$\int_{\partial D_\epsilon} f \wedge K^\epsilon_q.$$

Let $\zeta_0 \in \partial D_\epsilon$ be a fixed point and U a sufficiently small neighborhood of ζ_0. $F(\zeta, z)$ vanishes on the diagonal of $\overline{U} \times \overline{U}$, so Hefer's theorem applies to give us

$$F(\zeta, z) = \sum_{j=1}^n h_j(\zeta, z)(\zeta_j - z_j).$$

We set

$$\alpha^0(\zeta, z) = \frac{\sum_{j=1}^n h_j(\zeta, z)\, d\zeta_j}{F}.$$

With the metric given by

$$ds^2 = \sum g_{jk}(\zeta)\,dz_j\,d\overline{z}_k$$

(recall the Levi metric is independent of ϵ), we define

$$b^0(\zeta, z) = \sum_{j,k=1}^{n} g_{jk}(\overline{\zeta}_k - \overline{z}_k)\,d\zeta_j,$$

$$R^2(\zeta, z) = \sum_{j,k=1}^{n} g_{jk}(\zeta_j - z_j)(\overline{\zeta}_k - \overline{z}_k),$$

$$\beta^0(\zeta, z) = \frac{b^0(\zeta, z)}{R^2(\zeta, z)}.$$

With use of the transition kernels C_q defined above, we have via Koppelman's homotopy formula

$$\Omega_q(\beta^0) = \Omega_q(\alpha^0) + (-1)^{q+1}\,\bar{\partial}_\zeta C_q(\alpha^0, \beta^0) + \bar{\partial}_z C_{q-1}(\alpha^0, \beta^0).$$

On $(\partial D \cap U) \times U$ we have

$$\Omega_q(\alpha^\epsilon) = \Omega_q(\alpha_0) + \mathscr{E}_\infty,$$

$$\Omega_0(\alpha^\epsilon) = \Omega_0(\alpha_0) + \frac{\mathscr{E}_1(\zeta, z)}{\phi_\epsilon(\zeta, z)^n}$$

and on $U \times U$ we have

$$\Omega_q(\beta) = \frac{R^{2n}}{\rho^{2n}}\Omega_q\beta^0 + \mathscr{E}_{2-2n}.$$

Thus, we write

$$\Omega_q(\beta) = \frac{R^{2n}}{\rho^{2n}}\Omega_q(\beta^0) + \mathscr{E}_{2-2n}$$

$$= \frac{R^{2n}}{\rho^{2n}}\left(\Omega_q(\beta^0) - \Omega_q(\alpha^0)\right) + \Omega_q(\alpha^0) + \frac{\mathscr{E}_{2n+1}}{\rho^{2n}}\Omega_q(\alpha^0) + \mathscr{E}_{2-2n}$$

by which it then follows from the homotopy formula and the relations between b^0 and ρ^2 and R^2, exactly as it was obtained in [6], that we have

$$\begin{aligned}
(3.3) \qquad \Omega_q(\beta) = {} & \Omega_q(\alpha_\epsilon) + (-1)^{q+1}\,\bar{\partial}_\zeta C_q^\epsilon + \bar{\partial}_z C_{q-1}^\epsilon \\
& + \left(\Omega_q(\alpha^0) - \Omega_q(\alpha_\epsilon)\right) + \frac{\mathscr{E}_{2n+1}}{\rho^{2n}}\Omega_q(\alpha^0) \\
& + \mathscr{E}_{2-2n} + \bar{\partial}_\zeta\left[\left(C_q\left(\alpha^0, \frac{\partial\rho^2 + \mathscr{E}_2}{\rho^2}\right) - C_q(\alpha_\epsilon, \beta)\right)\right. \\
& \left. + \sum_{\mu,\nu} \frac{\mathscr{E}_{3+2\mu+2\nu}}{(\rho^2)^{1+\mu+\nu}} C_{q\mu\nu}\left(\alpha^0, \frac{\partial\rho^2 + \mathscr{E}_2}{\rho^2}\right)\right]
\end{aligned}$$

$$+ \sum_{\mu,\nu} \frac{\mathscr{E}_{4+2\mu+2\nu}}{(\rho^2)^{2+\mu+\nu}} C_{q\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right)$$

$$+ \sum_{\mu,\nu} \frac{\mathscr{E}_{4+2\mu+2\nu}}{(\rho^2)^{2+\mu+\nu}} C_{(q-1)\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right)$$

$$+ \bar{\partial}_z \left[\left(C_{q-1}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) - C_{q-1}(\alpha_\epsilon, \beta) \right) \right.$$

$$\left. + \sum_{\mu,\nu} \frac{\mathscr{E}_{3+2\mu+2\nu}}{(\rho^2)^{1+\mu+\nu}} C_{(q-1)\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) \right].$$

We now work with $\zeta \in \partial D_\epsilon$ so that $F = \phi_\epsilon$.

For $q > 0$ we have $\Omega_q(\alpha_\epsilon) = \Omega_q(\alpha^0) = 0$ near the boundary diagonal. Furthermore,

$$C_{q\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) = R_1 \frac{\mathscr{E}_1}{\phi_\epsilon^{1+\mu+\nu}(\rho^2)^{n-1-\mu-\nu}},$$

and thus

$$\bar{\partial}_\zeta \left(\sum_{\mu,\nu} \frac{\mathscr{E}_{3+2\mu+2\nu}}{(\rho^2)^{1+\mu+\nu}} C_{q\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) \right)$$

$$= \bar{\partial}_\zeta \left(R_1 \frac{\mathscr{E}_{4+2\mu+2\nu}}{\phi_\epsilon^{1+\mu+\nu} \rho^{2n}} \right)$$

$$= \frac{\mathscr{E}_{4+2\mu+2\nu}}{\phi_\epsilon^{1+\mu+\nu} \rho^{2n}} + R_1 \frac{\mathscr{E}_{3+2\mu+2\nu}}{\phi_\epsilon^{1+\mu+\nu} \rho^{2n}} + R_1 \frac{\mathscr{E}_{5+2\mu+2\nu}}{\phi_\epsilon^{1+\mu+\nu}(\rho^2)^{n+1}}$$

$$+ R_1 \frac{\mathscr{E}_{5+2\mu+2\nu} + \mathscr{E}_{4+2\mu+2\nu} \wedge \bar{\partial}_\zeta r_\epsilon}{\phi_\epsilon^{2+\mu+\nu} \rho^{2n}}.$$

And a similar formula holds for the

$$\bar{\partial}_z \left(\sum_{\mu,\nu} \frac{\mathscr{E}_{3+2\mu+2\nu}}{(\rho^2)^{1+\mu+\nu}} C_{(q-1)\mu\nu}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) \right)$$

term.

For $\nu > 0$, we have $C_{q\mu\nu}(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}) = C_{q\mu\nu}(\alpha_\epsilon, \beta) = 0$ near the boundary diagonal, and for $\nu = 0$ we have

$$(3.4) \qquad C_{q\mu 0}\left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) - C_{q\mu 0}(\alpha_\epsilon, \beta) = \frac{\mathscr{E}_2}{\phi_\epsilon^{1+\mu}(\rho^2)^{n-1-\mu}},$$

and thus

$$\bar{\partial}_\zeta \left(C_{q\mu 0}\left(\alpha_\epsilon^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2}\right) - C_{q\mu 0}(\alpha_\epsilon, \beta) \right)$$

$$= \frac{\mathscr{E}_1}{\phi_\epsilon^{1+\mu}(\rho^2)^{n-1-\mu}} + \frac{\mathscr{E}_3 + \mathscr{E}_2 \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{2+\mu}(\rho^2)^{n-1-\mu}} + \frac{\mathscr{E}_3}{\phi_\epsilon^{1+\mu}(\rho^2)^{n-\mu}}.$$

An analogous formula holds for

$$\bar{\partial}_z \left(C_{q\mu 0} \left(\alpha^0, \frac{\partial \rho^2 + \mathscr{E}_2}{\rho^2} \right) - C_{q\mu 0}(\alpha_\epsilon, \beta) \right)$$

Equation (3.3) can thus be written

$$\Omega_q(\beta) = \Omega_q(\alpha_\epsilon) + (-1)^{q+1} \bar{\partial}_\zeta C_q(\alpha_\epsilon, \beta) + \bar{\partial}_z C_{q-1}(\alpha_\epsilon, \beta) + \mathscr{E}_{2-2n}$$

$$+ \sum_{\tau=0}^n \frac{\mathscr{E}_{3+2\tau}}{\phi_\epsilon^{1+\tau} \rho^{2n}} + \frac{\mathscr{E}_{4+2\tau} \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{2+\tau} \rho^{2n}} + R_1 \frac{\mathscr{E}_{5+2\tau}}{\phi_\epsilon^{1+\tau} (\rho^2)^{n+1}}.$$

Thus, after integrating by parts we obtain

$$\int_{\partial D_\epsilon} f \wedge B_q^\epsilon$$

$$= \int_{\partial D_\epsilon} f \wedge \Omega_q(\alpha_\epsilon) + \int_{\partial D_\epsilon} \bar{\partial} f \wedge C_q^\epsilon + \bar{\partial}_z \int_{\partial D_\epsilon} f \wedge C_{q-1}^\epsilon + \int_{\partial D_\epsilon} f \wedge \mathscr{E}_{2-2n}$$

$$+ \sum_{\tau=0}^n \int_{\partial D_\epsilon} f \wedge \left(\frac{\mathscr{E}_{3+2\tau}}{\phi_\epsilon^{1+\tau} \rho^{2n}} + R_1 \frac{\mathscr{E}_{5+2\tau}}{\phi_\epsilon^{1+\tau} (\rho^2)^{n+1}} \right).$$

We now replace all occurrences of ρ^2 in the denominators by P_ϵ, since the two are equal on ∂D_ϵ, and then we change the boundary integrals to volume integrals by Stoke's theorem. In the calculation, we use the property

$$\bar{\partial} P_\epsilon = \mathscr{E}_1 + \frac{r_\epsilon^*}{\gamma^*} \xi_0(\zeta),$$

which follows from Lemma 3.1 and the definition of P_ϵ. We have

$$\int_{\partial D_\epsilon} f \wedge B_q^\epsilon$$

$$= \int_{D_\epsilon} \bar{\partial} f \wedge \Omega_q(\alpha_\epsilon) + (-1)^q \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_q(\alpha_\epsilon) + (-1)^{q+1} \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C_q^\epsilon$$

$$+ \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_z C_{q-1}^\epsilon + (-1)^q \bar{\partial}_z \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta C_{q-1}^\epsilon + \int_{D_\epsilon} \bar{\partial} f \wedge \mathscr{E}_{2-2n}$$

$$+ \int_{D_\epsilon} f \wedge \mathscr{E}_{1-2n} + \sum_{\tau=0}^n \int_{D_\epsilon} \bar{\partial} f \wedge \left(\frac{\mathscr{E}_{3+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^n} + R_1 \frac{\mathscr{E}_{5+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^{n+1}} \right)$$

$$+ \sum_{\tau=0}^n \int_{D_\epsilon} f \wedge \left(\frac{\mathscr{E}_{2+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^n} + R_1 \frac{\mathscr{E}_{3+2\tau}}{\phi_\epsilon^{2+\tau} P_\epsilon^n} + \frac{\mathscr{E}_{4+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^{n+1}} + \frac{r_\epsilon^*}{\gamma^*} \frac{\mathscr{E}_{3+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^{n+1}} \right.$$

$$\left. + R_2 \frac{\mathscr{E}_{5+2\tau}}{\phi_\epsilon^{2+\tau} P_\epsilon^{n+1}} + R_1 \frac{\mathscr{E}_{6+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^{n+2}} + R_1 \frac{r_\epsilon^*}{\gamma^*} \frac{\mathscr{E}_{5+2\tau}}{\phi_\epsilon^{1+\tau} P_\epsilon^{n+2}} \right).$$

Inserting this expression of the boundary integral into (3.2), and using our notation of operators of a certain type, we can write

$$f(z) = (-1)^q \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_q(\alpha_\epsilon) + \int_{D_\epsilon} \bar{\partial} f \wedge \Omega_q(\alpha_\epsilon) + (-1)^{q+1} \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C_q^\epsilon$$

$$- \int_{D_\epsilon} \bar{\partial} f \wedge B_q^\epsilon + \bar{\partial}_z \left((-1)^q \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta C_{q-1}^\epsilon - \int_{D_\epsilon} f \wedge B_{q-1}^\epsilon \right)$$

$$+ (f, \mathscr{E}_{1-2n}) + \left(f, \frac{1}{\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon \right) + (\bar{\partial} f, \mathscr{E}_{2-2n}) + (\bar{\partial} f, \mathscr{A}_{(0,2)}^\epsilon)$$

The rest of the proof follows as in [6] to obtain a rearrangement of the terms, and we arrive at the form of the representation in our theorem. $\quad\square$

4. Cancellation of singularities

THEOREM 4.1.
$$\mathbf{T}_q^\epsilon = E_{1-2n} + A_1^\epsilon.$$

Proof. The proof is a direct result of the operators which make up $\mathbf{T}_q^\epsilon$. In particular, the kernels $\mathcal{L}_q^\epsilon$ and $\mathcal{L}_{q-1}^\epsilon$ are sums of terms of the form

$$(4.1) \qquad\qquad R_1(\zeta) \frac{\mathscr{E}_1}{\bar{\phi}_\epsilon^{\mu+1} P_\epsilon^{n-\mu-1}}, \quad \mu \geq 0,$$

and if X is an arbitrary vector field in either ζ or z, we use

$$X P_\epsilon = \mathscr{E}_{1,0} + \xi_1(\zeta)\xi_0(z) + \xi_1(z)\xi_0(\zeta)$$

to calculate derivatives of the kernels (4.1). $\quad\square$

The proof of the next theorem will take up the bulk of this section. If one calculates the type of the operators associated with the operator $\mathbf{P}_q^\epsilon$ as we did in Theorem 4.1 just by looking at two vector fields operating on the kernels $\mathcal{L}_{q-1}^\epsilon$, the conclusion would be that $\mathbf{P}_q^\epsilon$ is an operator of double type $(-1,0)$. However, the combination of the two terms involved in $\mathbf{P}_q^\epsilon$ cancels one order of singularity in the kernels and thus leads to better mapping properties. We shall prove the following theorem.

THEOREM 4.2.
$$\mathbf{P}_q^\epsilon = \frac{1}{\gamma} A_{(-1,1)}^\epsilon + \frac{1}{\gamma^*} A_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*} A_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*} A_{(1,3)}^\epsilon.$$

The following lemma follows as in the smooth case (see Proposition 3.12 of [7]).

LEMMA 4.3.
$$\phi_\epsilon - \phi_\epsilon^* = \mathscr{E}_3.$$

For all ϵ sufficiently small, we work in coordinate patch near a boundary point of D and define orthogonal frame of $(1,0)$-forms on a neighborhood $U \cap D_\epsilon$ with $\omega_\epsilon^1, \ldots, \omega_\epsilon^n$ where $\partial r_\epsilon = \gamma \omega_\epsilon^n$ as the orthogonal frame, and $L_1^\epsilon, \ldots, L_n^\epsilon$ comprising the dual frame. These operators refer to the variable ζ. When they are to refer to the variable z, they will be denoted by Θ_ϵ^j and Λ_j^ϵ, respectively.

PROPOSITION 4.4.

(i) $\gamma \Lambda_n^\epsilon P_\epsilon = -2\overline{\phi}_\epsilon + \gamma \xi_{-1}^* (P_\epsilon + \mathscr{E}_2) + \xi_1 \xi_0^* \mathscr{E}_{2,-1} + \xi_0 \mathscr{E}_2 + \xi_0 \xi_0^* \mathscr{E}_{3,-1}$,

(ii) $\gamma^* L_n^\epsilon P_\epsilon = -2\phi_\epsilon^* + \gamma^* \xi_{-1}(P_\epsilon + \mathscr{E}_2) + \xi_0 \xi_1^* \mathscr{E}_{2,-1} + \xi_0^* \mathscr{E}_2 + \xi_0 \xi_0^* \mathscr{E}_{3,-1}$.

Proof. We follow the proof of Proposition 2.18 in [6]. We prove (i) since (ii) is a consequence of (i).

We have

$$\Lambda_n^\epsilon P_\epsilon = \Lambda_n \rho^2 + 2\frac{r_\epsilon}{\gamma} - \frac{1}{\gamma^*} \xi_0(z) \frac{r_\epsilon r_\epsilon^*}{\gamma \gamma^*}.$$

We fix the point ζ and choose local coordinates z^ϵ such that

$$dz_j^\epsilon(\zeta) = \Theta_j^\epsilon(\zeta).$$

Working in a neighborhood of a singularity in the boundary and using the coordinates in (3.1), we see $\Lambda_n - \frac{\partial}{\partial z_n^\epsilon}$ is a sum of terms of the form

$$(4.2) \qquad \left(\frac{a(z)}{\gamma(z)} - \frac{a(\zeta)}{\gamma(\zeta)} \right) \Lambda^\epsilon = (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1) \Lambda^\epsilon,$$

where a is a smooth function, $|a(\zeta)| \lesssim \gamma$, and Λ^ϵ denotes a first order differential operator. (4.2) follows from

$$\begin{aligned}
\frac{a(z)}{\gamma(z)} - \frac{a(\zeta)}{\gamma(\zeta)} &= a(z) \frac{\gamma(\zeta) - \gamma(z)}{\gamma(\zeta)\gamma(z)} + \frac{a(z) - a(\zeta)}{\gamma(\zeta)} \\
&= \xi_0^* \frac{\gamma^2(\zeta) - \gamma^2(z)}{\gamma(\zeta)(\gamma(\zeta) + \gamma(z))} + \xi_{-1} \mathscr{E}_1 \\
&= \xi_0^* \frac{\xi_1(\zeta) \mathscr{E}_1 + \mathscr{E}_2}{\gamma(\zeta)(\gamma(\zeta) + \gamma(z))} + \xi_{-1} \mathscr{E}_1 \\
&= \xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1.
\end{aligned}$$

By symmetry, we also have

$$(4.3) \qquad \Lambda_n - \frac{\partial}{\partial z_n^\epsilon} = (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{2,-1} + \xi_{-1}^* \mathscr{E}_1) \Lambda^\epsilon.$$

We note that $\Theta_j^\epsilon = \Theta_j|_{D_\epsilon}$, and therefore we will suppress the ϵ superscript in the variable z_n^ϵ as well as in the differential operators denoted by Λ. We have

$$(4.4) \qquad\qquad\qquad \rho^2 = R^2 + \mathscr{E}_3$$

and

$$\Lambda_n \rho^2 = \frac{\partial}{\partial z_n} R^2 + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \mathscr{E}_2$$
$$= -2(\overline{\zeta}_n - \overline{z}_n) + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \mathscr{E}_2,$$

where the last line follows from $g_{jk} = 2\delta_{jk}$ due to the orthogonality of the Θ_j.

Finally, this gives

$$(4.5) \qquad \Lambda_n P_\epsilon = -2(\overline{\zeta}_n - \overline{z}_n) + 2\frac{r_\epsilon}{\gamma} - \frac{1}{\gamma^*} \xi_0^* \frac{r_\epsilon r_\epsilon^*}{\gamma\gamma^*} + \xi_0 \xi_0^* \mathscr{E}_{2,-1}$$
$$+ \xi_{-1} \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \mathscr{E}_2.$$
$$= -2(\overline{\zeta}_n - \overline{z}_n) + 2\frac{r_\epsilon}{\gamma} - \xi_{-1}^*(P_\epsilon + \mathscr{E}_2) + \xi_0 \xi_0^* \mathscr{E}_{2,-1}$$
$$+ \xi_{-1} \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \mathscr{E}_2.$$

We compare (4.5) to $\overline{\phi}_\epsilon$ by calculating the Levi polynomial, $F_\epsilon(\zeta, z)$ in the above coordinates:

$$(4.6) \qquad \overline{\phi}_\epsilon(\zeta, z) = \overline{F}_\epsilon(\zeta, z) - r_\epsilon(\zeta) + \mathscr{E}_2$$
$$= \gamma(\zeta)(\overline{\zeta}_n - \overline{z}_n) - r_\epsilon(\zeta) + \mathscr{E}_2.$$

(i) then easily follows. $\qquad \qquad \square$

PROPOSITION 4.5.

$$(4.7) \qquad \gamma\gamma^*\left(2P_\epsilon - \sum_{j<n} |L_j \rho_\epsilon^2|^2\right)$$
$$= 4|\phi_\epsilon|^2 + r_\epsilon(\zeta)\mathscr{E}_2 + \xi_1 \xi_1^* \mathscr{E}_{3,-1} + \gamma^* \xi_0 \mathscr{E}_3$$
$$+ \xi_0 \xi_1^* \mathscr{E}_{4,-1} + r\xi_1 \mathscr{E}_{2,-1} + r\mathscr{E}_{3,-1} + \mathscr{E}_2 \mathscr{E}_{3,-1} + \gamma\mathscr{E}_{4,-1}.$$

Proof. We use coordinates as in the proof of Proposition 4.4. In particular, we write

$$L_j = \frac{\partial}{\partial \zeta_j} + (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1)\Lambda.$$

Thus,

$$|L_j \rho^2|^2 = (L_j \rho^2)\left(\frac{\partial}{\partial \overline{\zeta}_j} + (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1)\Lambda\right)\rho^2$$
$$= (L_j \rho^2)\frac{\partial \rho^2}{\partial \overline{\zeta}_j} + \xi_0 \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{4,-1} + \xi_{-1} \mathscr{E}_3$$
$$= \left|\frac{\partial}{\partial \zeta_j} \rho^2\right|^2 + \xi_0 \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{4,-1} + \xi_{-1} \mathscr{E}_3$$
$$= 4|\zeta_j - z_j|^2 + \xi_0 \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{4,-1} + \xi_{-1} \mathscr{E}_3.$$

We can then write

$$2P_\epsilon - \sum_{j<n} |L_j \rho^2|^2 = 4|\zeta_n - z_n|^2 + 4\frac{r_\epsilon r_\epsilon^*}{\gamma \gamma^*} + \xi_0 \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1}\xi_0^* \mathscr{E}_{4,-1} + \xi_{-1}\mathscr{E}_3.$$

Furthermore, from (4.6), we have

$$\begin{aligned}
\phi_\epsilon \overline{\phi}_\epsilon &= \big(\gamma(\zeta_n - z_n) - r_\epsilon(\zeta) + \mathscr{E}_2\big)\overline{\phi}_\epsilon \\
&= \gamma(\zeta_n - z_n)[\gamma(\overline{\zeta}_n - \overline{z}_n) - r_\epsilon(\zeta) + \mathscr{E}_2] - r_\epsilon(\zeta)\overline{\phi}_\epsilon + \mathscr{E}_2 \overline{\phi}_\epsilon \\
&= \gamma\gamma^*|\zeta_n - z_n|^2 - r_\epsilon(\zeta)[\gamma(\zeta_n - z_n) + \overline{\phi}_\epsilon] + r_\epsilon(\zeta)\mathscr{E}_2 + \xi_2\mathscr{E}_{3,-1} + \gamma\mathscr{E}_{4,-1},
\end{aligned}$$

where we use $\gamma(\zeta) = \gamma(z) + \xi_1 \mathscr{E}_{1,-1} + \mathscr{E}_{2,-1}$ in the last step.

From Lemma 4.3, we have

$$\begin{aligned}
\gamma(\zeta_n - z_n) + \overline{\phi}_\epsilon &= \gamma(\zeta_n - z_n) + \overline{\phi}_\epsilon^* + \mathscr{E}_3 \\
&= \gamma(\zeta_n - z_n) + \gamma^*(z_n - \zeta_n) - r_\epsilon(z) + \mathscr{E}_2 \\
&= -r_\epsilon(z) + \mathscr{E}_2 + \xi_1 \mathscr{E}_{2,-1} + \mathscr{E}_{3,-1},
\end{aligned}$$

and so we can write

$$\phi_\epsilon \overline{\phi}_\epsilon = \gamma\gamma^*|\zeta_n - z_n|^2 + r_\epsilon r_\epsilon^* + r_\epsilon \mathscr{E}_2 + r\xi_1 \mathscr{E}_{2,-1} + r\mathscr{E}_{3,-1} + \xi_2 \mathscr{E}_{3,-1} + \gamma\mathscr{E}_{4,-1}.$$

(4.7) now easily follows. $\square$

From

$$\begin{aligned}
(4.8)\quad A_{q\mu 0}^\epsilon &= \frac{1}{\phi_\epsilon^{\mu+1} P_\epsilon^{n-\mu-1}} \\
&\quad \times \partial r_\epsilon \wedge \partial \rho^2 \wedge (\bar{\partial}_\zeta \partial_\zeta r_\epsilon)^\mu \wedge (\bar{\partial}_\zeta \partial_\zeta \rho^2)^{n-q-\mu-2} \wedge (\bar{\partial}_z \partial_\zeta \rho^2)^q + \mathscr{E}_0,
\end{aligned}$$

we have

$$\mathcal{L}_q = \sum_{\mu=0}^{n-q-2} \left(g_{q\mu} C_{q\mu}^\epsilon + \frac{R_1(\zeta)\mathscr{E}_2}{\phi_\epsilon^{\mu+1} P_\epsilon^{n-\mu-1}} \right) + \mathscr{E}_0,$$

where

$$(4.9)\qquad C_{q\mu}^\epsilon = \sum_{\substack{|Q|=q \\ j<n}} \frac{\overline{L}_j \rho^2}{\overline{\phi}_\epsilon^{\mu+1} P_\epsilon^{n-\mu-1}} \overline{\omega}^{njQ} \wedge \Theta^Q$$

and $g_{q\mu}(\zeta)$ is a real valued function of the form $R_1(\zeta)\sigma_{q\mu}(\zeta)$ for a real valued function $\sigma_{q\mu}$. It follows that

$$\mathcal{P}_{q+1}^\epsilon = \sum_{\mu=0}^{n-q-2} [g_{q\mu}(\zeta)\vartheta_\zeta \partial_z C_{q\mu}^\epsilon - g_{q\mu}(z)(\vartheta_\zeta \partial_z C_{q\mu}^\epsilon)^*] + \frac{1}{\gamma^*}\mathscr{A}_1^\epsilon.$$

PROPOSITION 4.6. *Let $C_{q\mu}^\epsilon$ be given by* (4.9). *Then*

$$\begin{aligned}
g_{q\mu}(\zeta)\vartheta_\zeta \partial_z C_{q\mu}^\epsilon &- g_{q\mu}(z)(\vartheta_\zeta \partial_z C_{q\mu}^\epsilon)^* \\
&= \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(1,3)}^\epsilon.
\end{aligned}$$

We write

$$\vartheta_\zeta \, \partial_z C_{q\mu}^\epsilon = \sum_{\substack{|K|=q+1 \\ |L|=q+1}} A_{KL}^{\epsilon\mu} \overline{\omega}^K \wedge \Theta^L$$

and

$$(\vartheta_\zeta \, \partial_z C_{q\mu}^\epsilon)^* = \sum_{\substack{|K|=q+1 \\ |L|=q+1}} A_{LK}^{\epsilon\mu\,*} \overline{\omega}^K \wedge \Theta^L.$$

To prove Proposition 4.6, we show

$$\gamma(\zeta) A_{KL}^{\epsilon\mu} - \gamma(z) A_{LK}^{\epsilon\mu\,*} = \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(1,3)}^\epsilon.$$

With

$$\mathcal{M}_{kj}^{\epsilon\mu} = \frac{1}{\overline{\phi}_\epsilon^{\mu+1}} \Lambda_k \left(\frac{\overline{L}_j \rho^2}{P_\epsilon^{n-\mu-1}} \right),$$

and using $\Lambda_k \overline{\phi}_\epsilon = \mathscr{E}_{1,0}$, we have

$$A_{KL}^{\epsilon\mu} = - \sum_{\substack{|Q|=q \\ j<n \\ k,m}} \varepsilon_{kQ}^L \varepsilon_{mK}^{njQ} L_m \mathcal{M}_{kj}^{\epsilon\mu} + \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(0,2)}^\epsilon,$$

where K, L, and Q are multi-indices, and the symbol ε_{kQ}^L is defined by

$$\varepsilon_{kQ}^L = \begin{cases} 0 & \text{if } \{k\} \cup Q \neq L, \\ 1 & \text{if } kQ \text{ differs from } L \text{ by an even permutation,} \\ -1 & \text{if } kQ \text{ differs from } L \text{ by an odd permutation.} \end{cases}$$

LEMMA 4.7.

(i) $\mathcal{M}_{kj}^{\epsilon\mu} = \dfrac{1}{\overline{\phi}_\epsilon^{\mu+1}} \left[\dfrac{-2\delta_{kj}}{P_\epsilon^{n-\mu-1}} + \dfrac{n-\mu-1}{P_\epsilon^{n-\mu}}(L_k\rho^2)(\overline{L}_j\rho^2) \right] + error, \quad k<n,$

(ii) $\mathcal{M}_{nj}^{\epsilon\mu} = \dfrac{1}{\gamma(\zeta)} \dfrac{2(n-\mu-1)\overline{L}_j\rho^2}{\overline{\phi}_\epsilon^\mu P_\epsilon^{n-\mu}} + error, \quad j<n,$

where "error" refers to error terms with the property that any derivative with respect to the ζ variable leads to kernels of the form

$$\frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^3\gamma^*}\mathscr{A}_{(2,4)}^\epsilon.$$

Proof. We will prove (ii), the proof of (i) following similar arguments, and being easier to prove.

With the aid of coordinates chosen as in Proposition 4.5, we have

$$\overline{L}_j \rho^2 = 2(\zeta_j - z_j) + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1}\xi_0^* \mathscr{E}_{3,-1} + \xi_{-1}\mathscr{E}_2.$$

When it is not necessary to refer to the special coordinates in Proposition 4.5, we can also write $\overline{L}_j\rho^2 = \mathscr{E}_1$. We will also refer to the calculation

$$
\begin{aligned}
\Lambda_n \overline{L}_j \rho^2 &= \Lambda_n \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1) \Lambda \right) R^2 \right] + \mathscr{E}_2 \\
&= \Lambda_n \left(2(\zeta_j - z_j) + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{3,-1} + \xi_{-1} \mathscr{E}_2 \right) + \mathscr{E}_2 \\
&= 2 \left(\frac{\partial}{\partial z_n} + (\xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1) \Lambda \right) (\zeta_j - z_j) \\
&\quad + \xi_0 \xi_{-1}^* \mathscr{E}_{2,-1} + \xi_0 \xi_0^* \mathscr{E}_{2,-2} + \xi_{-1} \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} \\
&\quad + \xi_{-1} \xi_0^* \mathscr{E}_{3,-2} + \xi_{-1} \mathscr{E}_1 \\
&= \xi_0 \xi_0^* \mathscr{E}_{1,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \mathscr{E}_1 + \xi_0 \xi_{-1}^* \mathscr{E}_{2,-1} + \xi_0 \xi_0^* \mathscr{E}_{2,-2} \\
&\quad + \xi_{-1} \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1} \xi_0^* \mathscr{E}_{3,-2}
\end{aligned}
$$

for $j < n$, below.

We have

$$
\begin{aligned}
(4.10) \quad \mathcal{M}_{nj}^{\epsilon\mu} &= \frac{1}{\overline{\phi}_\epsilon^{\mu+1}} \Lambda_n \left(\frac{\overline{L}_j \rho^2}{P_\epsilon^{n-\mu-1}} \right) \\
&= -(n - \mu - 1) \frac{\overline{L}_j \rho^2}{\overline{\phi}_\epsilon^{\mu+1} P_\epsilon^{n-\mu}} \Lambda_n P_\epsilon + \frac{1}{\gamma} \mathscr{A}_{(0,2)}^\epsilon \\
&\quad + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^2 \gamma^*} \mathscr{A}_{(2,4)}^\epsilon \\
&= -\frac{1}{\gamma} \frac{(n - \mu - 1) \overline{L}_j \rho^2}{\overline{\phi}_\epsilon^{\mu+1} P_\epsilon^{n-\mu}} \\
&\quad \times \left(-2\overline{\phi}_\epsilon + \gamma \xi_{-1}^* (P_\epsilon + \mathscr{E}_2) + \xi_1 \xi_0^* \mathscr{E}_{2,-1} + \xi_0 \mathscr{E}_2 + \xi_0 \xi_0^* \mathscr{E}_{3,-1} \right) \\
&\quad + \frac{1}{\gamma} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^2 \gamma^*} \mathscr{A}_{(2,4)}^\epsilon \\
&= 2 \frac{1}{\gamma} \frac{(n - \mu - 1) \overline{L}_j \rho^2}{\overline{\phi}_\epsilon^\mu P_\epsilon^{n-\mu}} + \frac{1}{\gamma} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^*} \mathscr{A}_{(0,2)}^\epsilon \\
&\quad + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^2 \gamma^*} \mathscr{A}_{(2,4)}^\epsilon,
\end{aligned}
$$

from which (ii) easily follows. We used Proposition 4.4 in the third equality above.

Applying a differential operator with respect to ζ to the last three terms in (4.10) and noting the types of kernels which arise finishes the proof.   □

REMARK 4.8. We make a note here which is crucial in the proof of Proposition 4.6 that, by making use of the property (2.2), we can replace the last

term of the error terms in Lemma 4.7 by a term

$$\frac{1}{\gamma^2(\gamma^*)^2}\mathscr{A}^{\epsilon}_{(2,4)}.$$

We start with the case $\mu = 0$ and compute $L_m \mathcal{M}^{\epsilon 0}_{kj}$.

LEMMA 4.9. *Let* $m, k, j < n$. *Then*

(i) $L_m \mathcal{M}^{\epsilon 0}_{kj} = \dfrac{2(n-1)}{\overline{\phi}_\epsilon P^n_\epsilon}(\delta_{kj} L_m \rho^2 + \delta_{mj} L_k \rho^2)$

$\qquad\qquad - \dfrac{n(n-1)}{\overline{\phi}_\epsilon P^{n+1}_\epsilon}(L_m \rho^2)(L_k \rho^2)(\overline{L}_j \rho^2) + \dfrac{1}{\gamma}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma^*}\mathscr{A}^{\epsilon}_{(-1,1)}$

$\qquad\qquad + \dfrac{1}{\gamma^2}\mathscr{A}^{\epsilon}_{(0,2)} + \dfrac{1}{\gamma\gamma^*}\mathscr{A}^{\epsilon}_{(0,2)} + \dfrac{1}{\gamma^2\gamma^*}\mathscr{A}^{\epsilon}_{(1,3)} + \dfrac{1}{\gamma^3\gamma^*}\mathscr{A}^{\epsilon}_{(2,4)},$

(ii) $L_m \mathcal{M}^{\epsilon 0}_{nj} = \dfrac{2(n-1)}{\gamma}\left(\dfrac{2\delta_{mj}}{P^n} - \dfrac{n}{P^{n+1}}(L_m \rho^2)(\overline{L}_j \rho^2)\right)$

$\qquad\qquad + \dfrac{1}{\gamma^2}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma\gamma^*}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma^2\gamma^*}\mathscr{A}^{\epsilon}_{(0,2)} + \dfrac{1}{\gamma^3\gamma^*}\mathscr{A}^{\epsilon}_{(1,3)},$

(iii) $L_n \mathcal{M}^{\epsilon 0}_{kj} = -\gamma\dfrac{2\delta_{kj}}{\overline{\phi}^2_\epsilon P^{n-1}_\epsilon} + \dfrac{(n-1)(L_k \rho^2)(\overline{L}_j \rho^2)}{\overline{\phi}^2_\epsilon P^n_\epsilon}$

$\qquad\qquad + \dfrac{1}{\gamma^*}\left(\dfrac{2n(n-1)(L_k \rho^2)(\overline{L}_j \rho^2)}{P^{n+1}} - \dfrac{4(n-1)\delta_{kj}}{P^n}\right)\dfrac{\phi^*_\epsilon}{\overline{\phi}_\epsilon}$

$\qquad\qquad + \dfrac{1}{\gamma}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma^*}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma^2}\mathscr{A}^{\epsilon}_{(0,2)} + \dfrac{1}{\gamma\gamma^*}\mathscr{A}^{\epsilon}_{(0,2)}$

$\qquad\qquad + \dfrac{1}{\gamma^2\gamma^*}\mathscr{A}^{\epsilon}_{(1,3)} + \dfrac{1}{\gamma^3\gamma^*}\mathscr{A}^{\epsilon}_{(2,4)},$

(iv) $L_n \mathcal{M}^{\epsilon 0}_{nj} = \dfrac{4n(n-1)}{\gamma\gamma^*}\dfrac{\phi^*_\epsilon}{P^{n+1}_\epsilon}\overline{L}_j \rho^2 + \dfrac{1}{\gamma^2}\mathscr{A}^{\epsilon}_{(-1,1)} + \dfrac{1}{\gamma\gamma^*}\mathscr{A}^{\epsilon}_{(-1,1)}$

$\qquad\qquad + \dfrac{1}{\gamma^2\gamma^*}\mathscr{A}^{\epsilon}_{(0,2)} + \dfrac{1}{\gamma^3\gamma^*}\mathscr{A}^{\epsilon}_{(1,3)}.$

Proof. (i) We use

$$(4.11) \qquad L_m \frac{1}{\overline{\phi}_\epsilon P^{n-1}_\epsilon} = -\frac{1}{\overline{\phi}^2_\epsilon P^{n-1}_\epsilon} L_m \overline{\phi}_\epsilon - (n-1)\frac{1}{\overline{\phi}_\epsilon P^n_\epsilon} L_m P_\epsilon$$

$$= -\frac{(n-1)}{\overline{\phi}_\epsilon P^n_\epsilon} L_m \rho^2 + \frac{1}{\gamma}\mathscr{A}^{\epsilon}_{(-1,1)},$$

since for $m < n$ we have

$$L_m P_\epsilon = L_m \rho^2 + \frac{\xi_0}{\gamma}(P_\epsilon + \mathscr{E}_2),$$

and

(4.12)
$$L_m\left(\frac{1}{\phi_\epsilon P_\epsilon^n}(L_k\rho^2)(\overline{L}_j\rho^2)\right)$$

$$= \frac{1}{\overline{\phi}_\epsilon P_\epsilon^n}(L_mL_k\rho^2)(\overline{L}_j\rho^2) + \frac{1}{\overline{\phi}_\epsilon P_\epsilon^n}(L_k\rho^2)(L_m\overline{L}_j\rho^2)$$

$$+ \frac{\mathscr{E}_3}{\overline{\phi}_\epsilon^2 P_\epsilon^n} - \frac{n}{\overline{\phi}_\epsilon P_\epsilon^{n+1}}(L_k\rho^2)(\overline{L}_j\rho^2)(L_mP_\epsilon).$$

An easy calculation, using (4.3), in combination with the relation in (4.4), gives

(4.13)
$$L_mL_k\rho^2 = L_m\left(2(\overline{\zeta_k - z_k}) + \xi_0\xi_0^*\mathscr{E}_{2,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_{-1}^*\mathscr{E}_2\right)$$

$$= 2\left(\frac{\partial}{\partial\zeta_m} + (\xi_0\xi_0^*\mathscr{E}_{1,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{2,-1} + \xi_{-1}^*\mathscr{E}_1)\Lambda\right)(\overline{\zeta_k - z_k})$$

$$+ L_m(\xi_0\xi_0^*\mathscr{E}_{2,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_{-1}^*\mathscr{E}_2)$$

$$= \xi_0\xi_0^*\mathscr{E}_{1,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{2,-1} + \xi_{-1}^*\mathscr{E}_1 + \xi_{-1}\xi_0^*\mathscr{E}_{2,-1} + \xi_0\xi_0^*\mathscr{E}_{2,-2}$$

$$+ \xi_{-1}\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{2,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-2}.$$

Similarly, we can write

(4.14)
$$L_m\overline{L}_j\rho^2 = 2\delta_{mj} + \xi_0\xi_0^*\mathscr{E}_{1,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{2,-1} + \xi_{-1}^*\mathscr{E}_1 + \xi_{-1}\xi_0^*\mathscr{E}_{2,-1}$$

$$+ \xi_0\xi_0^*\mathscr{E}_{2,-2} + \xi_{-1}\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{2,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-2}.$$

Thus the right-hand side of (4.12) becomes

(4.15)
$$\frac{2\delta_{mj}}{\overline{\phi}_\epsilon P_\epsilon^n}(L_k\rho^2)$$

$$= \frac{n}{\overline{\phi}_\epsilon P_\epsilon^{n+1}}(L_k\rho^2)(\overline{L}_j\rho^2)(L_m\rho^2)$$

$$+ \frac{\xi_0\xi_0^*\mathscr{E}_{2,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_{-1}^*\mathscr{E}_2 + \xi_{-1}\xi_0^*\mathscr{E}_{3,-1}}{\overline{\phi}_\epsilon P_\epsilon^n}$$

$$+ \frac{\xi_0\xi_0^*\mathscr{E}_{3,-2} + \xi_{-1}\xi_{-1}^*\mathscr{E}_{4,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{3,-1} + \xi_0\xi_{-1}^*\mathscr{E}_{4,-2}}{\overline{\phi}_\epsilon P_\epsilon^n}$$

$$+ \frac{\xi_{-1}\mathscr{E}_2}{\overline{\phi}_\epsilon P_\epsilon^n} + \frac{\xi_{-1}\mathscr{E}_4}{\overline{\phi}_\epsilon P_\epsilon^{n+1}} + \frac{\mathscr{E}_3}{\overline{\phi}_\epsilon^2 P_\epsilon^n}$$

$$= \frac{2\delta_{mj}}{\overline{\phi}_\epsilon P_\epsilon^n}(L_k\rho^2) - \frac{n}{\overline{\phi}_\epsilon P_\epsilon^{n+1}}(L_k\rho^2)(\overline{L}_j\rho^2)(L_m\rho^2)$$

$$+ \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(1,3)}^\epsilon.$$

(4.11), and (4.15) together with the form of $\mathcal{M}_{kj}^{\epsilon 0}$ from Lemma 4.7 prove part (i).

(ii) We have

$$
L_m\left(\frac{1}{\gamma(\zeta)}\frac{2(n-1)\overline{L}_j\rho^2}{P_\epsilon^n}\right)
$$

$$
=\frac{2(n-1)}{\gamma}\frac{L_m\overline{L}_j\rho^2}{P_\epsilon^n}-\frac{2n(n-1)}{\gamma}\frac{\overline{L}_j\rho^2}{P_\epsilon^{n+1}}L_mP_\epsilon+\frac{1}{\gamma^2}\frac{\xi_0\mathscr{E}_1}{P_\epsilon^n}
$$

$$
=\frac{4(n-1)}{\gamma}\frac{\delta_{mj}}{P_\epsilon^n}-\frac{2n(n-1)}{\gamma}\frac{\overline{L}_j\rho^2}{P_\epsilon^{n+1}}L_m\rho^2
$$

$$
+\frac{1}{\gamma^2}\mathscr{A}_{(-1,1)}^\epsilon+\frac{1}{\gamma\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon+\frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(0,2)}^\epsilon+\frac{1}{\gamma^3\gamma^*}\mathscr{A}_{(1,3)}^\epsilon
$$

as in the proof of (i). Taking into consideration the error terms from Lemma 4.7, we conclude (ii).

(iii)

$$
L_n\left(\frac{-2\delta_{kj}}{\overline{\phi}_\epsilon P_\epsilon^{n-1}}\right)
$$

$$
=\frac{2\delta_{kj}}{\overline{\phi}_\epsilon^2 P_\epsilon^{n-1}}L_n\overline{\phi}_\epsilon+2(n-1)\frac{\delta_{kj}}{\overline{\phi}_\epsilon P_\epsilon^n}L_nP_\epsilon
$$

$$
=\frac{2\delta_{kj}}{\overline{\phi}_\epsilon^2 P_\epsilon^{n-1}}\left(-\gamma(\zeta)+\mathscr{E}_1\right)
$$

$$
+2(n-1)\frac{\delta_{kj}}{\overline{\phi}_\epsilon P_\epsilon^n}
$$

$$
\times\left(-2\frac{\phi_\epsilon^*}{\gamma^*}+\xi_{-1}(P_\epsilon+\mathscr{E}_2)+\xi_0\xi_0^*\mathscr{E}_{2,-1}+\xi_{-1}^*\mathscr{E}_2+\xi_0\xi_{-1}^*\mathscr{E}_{3,-1}\right)
$$

$$
=-\gamma\frac{2\delta_{kj}}{\overline{\phi}_\epsilon^2 P_\epsilon^{n-1}}-\frac{1}{\gamma^*}\frac{4(n-1)\delta_{kj}}{P_\epsilon^n}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon}+\frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon+\frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon+\frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon.
$$

As in (4.13), we have

$$
L_nL_k\rho^2=\xi_0\xi_0^*\mathscr{E}_{1,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{2,-1}+\xi_{-1}^*\mathscr{E}_1+\xi_{-1}\xi_0^*\mathscr{E}_{2,-1}+\xi_0\xi_0^*\mathscr{E}_{2,-2}
$$

$$
+\xi_{-1}\xi_{-1}^*\mathscr{E}_{3,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{2,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{3,-2},
$$

$$
L_n\overline{L}_j\rho^2=\xi_0\xi_0^*\mathscr{E}_{1,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{2,-1}+\xi_{-1}^*\mathscr{E}_1+\xi_{-1}\xi_0^*\mathscr{E}_{2,-1}+\xi_0\xi_0^*\mathscr{E}_{2,-2}
$$

$$
+\xi_{-1}\xi_{-1}^*\mathscr{E}_{3,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{2,-1}+\xi_0\xi_{-1}^*\mathscr{E}_{3,-2}.
$$

Thus,

$$
L_n\left(\frac{n-1}{\overline{\phi}_\epsilon P_\epsilon^n}(L_k\rho^2)(\overline{L}_j\rho^2)\right)
$$

$$
=-\frac{n-1}{\overline{\phi}_\epsilon^2 P_\epsilon^n}(L_n\overline{\phi}_\epsilon)(L_k\rho^2)(\overline{L}_j\rho^2)-\frac{n(n-1)}{\overline{\phi}_\epsilon P_\epsilon^{n+1}}(L_nP_\epsilon)(L_k\rho^2)(\overline{L}_j\rho^2)
$$

$$+ \frac{n-1}{\overline{\phi}_\epsilon P_\epsilon^n}(L_n L_k \rho^2)(\overline{L}_j \rho^2) + \frac{n-1}{\overline{\phi}_\epsilon P_\epsilon^n}(L_k \rho^2)(L_n \overline{L}_j \rho^2)$$

$$= \gamma \frac{n-1}{\overline{\phi}_\epsilon^2 P_\epsilon^n}(L_k \rho^2)(\overline{L}_j \rho^2) + \frac{\mathscr{E}_3}{\overline{\phi}_\epsilon^2 P_\epsilon^n}$$

$$- \frac{n(n-1)}{\overline{\phi}_\epsilon P_\epsilon^{n+1}}(L_k \rho^2)(\overline{L}_j \rho^2)$$

$$\times \left(-2\frac{\phi_\epsilon^*}{\gamma^*} + \xi_{-1}(P_\epsilon + \mathscr{E}_2) + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1}^* \mathscr{E}_2 + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} \right)$$

$$+ \frac{\xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_{-1}^* \mathscr{E}_2 + \xi_{-1} \xi_0^* \mathscr{E}_{3,-1}}{\overline{\phi}_\epsilon P_\epsilon^n}$$

$$+ \frac{\xi_0 \xi_0^* \mathscr{E}_{3,-2} + \xi_{-1} \xi_{-1}^* \mathscr{E}_{4,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{4,-2}}{\overline{\phi}_\epsilon P_\epsilon^n}$$

$$= \gamma \frac{(n-1)(L_k \rho^2)(\overline{L}_j \rho^2)}{\overline{\phi}_\epsilon^2 P_\epsilon^n} + \frac{1}{\gamma^*}\frac{2n(n-1)(L_k \rho^2)(\overline{L}_j \rho^2)}{P_\epsilon^{n+1}}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon}$$

$$+ \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(1,3)}^\epsilon.$$

Putting these calculations together and including the error terms from Lemma 4.7, we can prove (iii).

(iv) To prove (iv), we calculate

$$L_n\left(\frac{1}{\gamma(\zeta)}\frac{2(n-1)\overline{L}_j \rho^2}{P_\epsilon^n} \right)$$

$$= -\frac{1}{\gamma^2}\frac{2(n-1)\overline{L}_j \rho^2}{P_\epsilon^n}L_n\gamma + \frac{1}{\gamma}\frac{2(n-1)L_n\overline{L}_j \rho^2}{P_\epsilon^n} - \frac{1}{\gamma}\frac{2(n-1)\overline{L}_j \rho^2}{P_\epsilon^{n+1}}L_n P_\epsilon$$

$$= -\frac{1}{\gamma}\frac{2(n-1)\overline{L}_j \rho^2}{P_\epsilon^{n+1}}$$

$$\times \left(-2\frac{\phi_\epsilon^*}{\gamma^*} + \xi_{-1}(P_\epsilon + \mathscr{E}_2) + \xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_{-1}^* \mathscr{E}_2 + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} \right)$$

$$+ \frac{1}{\gamma^2}\frac{\xi_0 \mathscr{E}_1}{P_\epsilon^n} + \frac{1}{\gamma}\frac{\xi_0 \xi_0^* \mathscr{E}_{2,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_{-1}^* \mathscr{E}_2 + \xi_{-1} \xi_0^* \mathscr{E}_{3,-1}}{P_\epsilon^n}$$

$$+ \frac{1}{\gamma}\frac{\xi_0 \xi_0^* \mathscr{E}_{3,-2} + \xi_{-1} \xi_{-1}^* \mathscr{E}_{4,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{3,-1} + \xi_0 \xi_{-1}^* \mathscr{E}_{4,-2}}{P_\epsilon^n}$$

$$= \frac{4n(n-1)}{\gamma\gamma^*}\frac{\phi_\epsilon^*}{P_\epsilon^{n+1}}\overline{L}_j \rho^2$$

$$+ \frac{1}{\gamma^2}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^3\gamma^*}\mathscr{A}_{(1,3)}^\epsilon.$$

The error terms from Lemma 4.7 may also be absorbed into the terms of the last calculation. □

REMARK 4.10. Again, we note that the last of the error terms in each part of Lemma 4.9 may be replaced by

$$\frac{1}{\gamma^2(\gamma^*)^2}\mathscr{A}^\epsilon_{(1,3)}.$$

We implicitly use this property below in each case composing the proof of Proposition 4.6, to handle errors in subtracting adjoint terms.

Proof of Proposition 4.6. To compute $A^{\epsilon 0}_{KL}$ we follow the proof of Theorem VI.1.29 in [6] and consider four cases:

Case 1. $n \in K$ and $n \in L$.

$$A^{\epsilon 0}_{KL} = \sum_{j,m<n} \varepsilon^{jL}_{mK} L_m \mathcal{M}^{\epsilon 0}_{nj},$$

$$A^{\epsilon 0\,*}_{KL} = \sum_{j,m<n} \varepsilon^{jL}_{mK} (L_j \mathcal{M}^{\epsilon 0}_{nm})^*.$$

By Lemma 4.9(ii),

$$\gamma(\zeta) L_m \mathcal{M}^{\epsilon 0}_{nj} - \gamma(z)(L_j \mathcal{M}^{\epsilon 0}_{nm})^*$$
$$= \frac{1}{\gamma}\mathscr{A}^\epsilon_{(-1,1)} + \frac{1}{\gamma^*}\mathscr{A}^\epsilon_{(-1,1)} + \frac{1}{\gamma\gamma^*}\mathscr{A}^\epsilon_{(0,2)} + \frac{1}{\gamma^2\gamma^*}\mathscr{A}^\epsilon_{(1,3)}.$$

Case 2. $n \notin K$ and $n \notin L$. From [7] (see also [6])

$$A^{\epsilon 0}_{KL} = -\sum_{\substack{j\in K\\k\in L}} \varepsilon^L_{kQ}\varepsilon^{jQ}_K L_n \mathcal{M}^{\epsilon 0}_{kj},$$

$$A^{\epsilon 0\,*}_{LK} = -\sum_{\substack{j\in K\\k\in L}} \varepsilon^K_{jQ}\varepsilon^{kQ}_L (L_n \mathcal{M}^{\epsilon 0}_{jk})^*.$$

We refer to Lemma 4.9(iii) to calculate

$$\gamma(\zeta) L_n \mathcal{M}^{\epsilon 0}_{kj} - \gamma(z)(L_n \mathcal{M}^{\epsilon 0}_{jk})^*.$$

We have

$$(4.16) \quad \frac{\gamma^2}{\overline{\phi}_\epsilon^2 P^{n-1}_\epsilon} - \frac{(\gamma^*)^2}{(\overline{\phi}^*_\epsilon)^2 P^{n-1}_\epsilon} = \frac{\gamma^2}{P^{n-1}_\epsilon}\left(\frac{1}{\overline{\phi}_\epsilon^2} - \frac{1}{(\overline{\phi}^*_\epsilon)^2}\right) + \frac{\xi_1\mathscr{E}_1 + \mathscr{E}_2}{\overline{\phi}_\epsilon^2 P^{n-1}_\epsilon}$$

$$= \frac{\gamma^2}{P^{n-1}_\epsilon}\frac{(\overline{\phi}_\epsilon + \overline{\phi}^*_\epsilon)\mathscr{E}_3}{\overline{\phi}_\epsilon^2(\overline{\phi}^*_\epsilon)^2} + \mathscr{A}^\epsilon_{(0,1)} + \mathscr{A}^\epsilon_{(0,2)}$$

$$= \mathscr{A}^\epsilon_{(0,1)} + \mathscr{A}^\epsilon_{(0,2)}.$$

Furthermore,

$$(4.17) \quad \gamma(\zeta)\frac{(L_k\rho^2)(\overline{L}_j\rho^2)}{\overline{\phi}_\epsilon^2 P_\epsilon^n} - \gamma(z)\left(\frac{(L_j\rho^2)(\overline{L}_k\rho^2)}{\overline{\phi}_\epsilon^2 P_\epsilon^n}\right)^*$$

$$= \gamma\left(\frac{(L_k\rho^2)(\overline{L}_j\rho^2)}{\overline{\phi}_\epsilon^2 P_\epsilon^n} - \left(\frac{(L_j\rho^2)(\overline{L}_k\rho^2)}{\overline{\phi}_\epsilon^2 P_\epsilon^n}\right)^*\right) + \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(0,2)}^\epsilon$$

$$= \frac{\xi_1\mathscr{E}_2}{P_\epsilon^n}\left(\frac{1}{\overline{\phi}_\epsilon^2} - \frac{1}{(\overline{\phi}_\epsilon^*)^2}\right) + \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(0,2)}^\epsilon$$

$$= \mathscr{A}_{(0,1)}^\epsilon + \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(0,2)}^\epsilon$$

$$= \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma}\mathscr{A}_{(0,2)}^\epsilon$$

and

$$(4.18) \quad \frac{\gamma}{\gamma^*}\frac{(L_k\rho^2)(\overline{L}_j\rho^2)}{P_\epsilon^{n+1}}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon} - \frac{\gamma^*}{\gamma}\left(\frac{(L_j\rho^2)(\overline{L}_k\rho^2)}{P_\epsilon^{n+1}}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon}\right)^*$$

$$= \frac{\mathscr{E}_2}{P_\epsilon^{n+1}}\left(\frac{\gamma}{\gamma^*}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon} - \frac{\gamma^*}{\gamma}\frac{\phi_\epsilon}{\overline{\phi}_\epsilon^*}\right)$$

$$= \frac{\mathscr{E}_2}{P_\epsilon^{n+1}}\left(\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon} - \frac{\phi_\epsilon}{\overline{\phi}_\epsilon^*} + (\xi_1\xi_{-1}^*\mathscr{E}_{1,-1} + \xi_{-1}^*\mathscr{E}_{2,-1})\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon}\right.$$

$$\left. + (\xi_{-1}1\xi_1^*\mathscr{E}_{1,-1} + \xi_{-1}\mathscr{E}_{2,-1})\frac{\phi_\epsilon}{\overline{\phi}_\epsilon^*}\right)$$

$$= \frac{\mathscr{E}_2}{P_\epsilon^{n+1}}\left(\frac{\phi_\epsilon\mathscr{E}_3 + \overline{\phi}_\epsilon^*\mathscr{E}_3}{\overline{\phi}_\epsilon\overline{\phi}_\epsilon^*}\right) + \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon$$

$$= \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon.$$

Similarly,

$$(4.19) \quad \frac{\gamma}{\gamma^*}\frac{1}{P_\epsilon^n}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon} - \frac{\gamma^*}{\gamma}\left(\frac{1}{P_\epsilon^n}\frac{\phi_\epsilon^*}{\overline{\phi}_\epsilon}\right)^* = \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon.$$

From the last three terms in Lemma 4.9(iii) and (4.16), (4.17), (4.18), and (4.19), we conclude

$$\gamma(\zeta)L_n\mathcal{M}_{kj}^{\epsilon0} - \gamma(z)(L_n\mathcal{M}_{jk}^{\epsilon0})^*$$

$$= \frac{1}{\gamma}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*}\mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*}\mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*}\mathscr{A}_{(2,4)}^\epsilon.$$

Case 3. $n \notin K$ and $n \in L$.

$$A_{KL}^{\epsilon0} = -\varepsilon_{nQ}^L\sum_{j<n}\varepsilon_{nK}^{njQ}(L_n\mathcal{M}_{nj}^{\epsilon0})$$

$$(4.20) \qquad = -\varepsilon_{nQ}^{L} \sum_{j<n} \varepsilon_{K}^{jQ} \overline{L}_j \rho^2 \frac{4n(n-1)\phi_\epsilon^*}{\gamma\gamma^* P_\epsilon^{n+1}}$$

$$(4.21) \qquad + \frac{1}{\gamma^2} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(1,3)}^\epsilon$$

by Lemma 4.9(iv). We work out the rest of Case 3, after we do some calculations which also apply to Case 4.

Case 4. $n \in K$ and $n \notin L$. From [6] (see IV.2.57),

$$A_{KL}^{\epsilon 0} = \varepsilon_{nJ}^{K} \sum_{k \in L} \varepsilon_{kJ}^{L} \sum_{j<n} L_j \mathcal{M}_{kj}^{\epsilon 0} - \sum_{\substack{m,k \in L \\ j \in J}} \varepsilon_{kmM}^{L} \varepsilon_{njM}^{K} L_m \mathcal{M}_{kj}^{\epsilon 0}.$$

The second sum is $\frac{1}{\gamma} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(2,4)}^\epsilon$ because

$$L_k \mathcal{M}_{mj}^{\epsilon 0} = L_m \mathcal{M}_{kj}^{\epsilon 0} + \frac{1}{\gamma} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon$$

$$+ \frac{1}{\gamma^2} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(2,4)}^\epsilon$$

from Lemma 4.9(i). For the first sum, we set $m = j$ in Lemma 4.9(i), and use

$$\sum_{j<n} L_j \mathcal{M}_{kj}^{\epsilon 0} = \frac{2n(n-1)}{\overline{\phi}_\epsilon P_\epsilon^n} L_k \rho^2 - \frac{n(n-1)}{\overline{\phi}_\epsilon P_\epsilon^{n+1}} L_k \rho^2 \sum_{j<n} |L_j\rho^2|^2$$

$$+ \frac{1}{\gamma} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(0,2)}^\epsilon$$

$$+ \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(2,4)}^\epsilon,$$

the first two terms on the right of which can be written by Proposition 4.5 as

$$\frac{n(n-1)}{\overline{\phi}_\epsilon P_\epsilon^{n+1}} L_k \rho^2 \left(2P_\epsilon - \sum_{j<n} |L_j\rho^2|^2 \right) = \frac{4n(n-1)}{\gamma\gamma^*} \frac{|\phi_\epsilon|^2}{\overline{\phi}_\epsilon P_\epsilon^{n+1}} L_k \rho^2 + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon.$$

This gives

$$(4.22) \quad A_{KL}^{\epsilon 0} = \frac{4n(n-1)}{\gamma\gamma^*} \frac{\phi_\epsilon}{P_\epsilon^{n+1}} \varepsilon_{nJ}^{K} \sum_{k<n} \varepsilon_{kJ}^{L} L_k \rho^2$$

$$+ \frac{1}{\gamma\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(2,4)}^\epsilon.$$

Case 3. $n \notin K$ and $n \in L$. Comparing (4.20) and (4.22), we obtain

$$A_{KL}^{\epsilon 0} = -\varepsilon_{nQ}^{L} \varepsilon_{K}^{kQ} \frac{4n(n-1)}{\gamma\gamma^*} \frac{\phi_\epsilon^*}{P_\epsilon^{n+1}} \overline{L}_k \rho^2$$

$$+ \frac{1}{\gamma^2} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(-1,1)}^\epsilon + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(0,2)}^\epsilon + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(1,3)}^\epsilon,$$

$$A_{LK}^{\epsilon 0} = \varepsilon_{nJ}^{L} \varepsilon_{kJ}^{K} \frac{4n(n-1)}{\gamma\gamma^*} \frac{\phi_\epsilon}{P_\epsilon^{n+1}} L_k \rho^2$$
$$+ \frac{1}{\gamma\gamma^*} \mathscr{A}_{(-1,1)}^{\epsilon} + \frac{1}{\gamma^2} \mathscr{A}_{(0,2)}^{\epsilon} + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^{\epsilon} + \frac{1}{\gamma^3\gamma^*} \mathscr{A}_{(2,4)}^{\epsilon},$$

and

$$\gamma(\zeta)A_{KL}^{\epsilon 0} - \gamma(z)(A_{LK}^{\epsilon 0})^* = \frac{1}{\gamma} \mathscr{A}_{(-1,1)}^{\epsilon} + \frac{1}{\gamma^*} \mathscr{A}_{(-1,1)}^{\epsilon} + \frac{1}{\gamma\gamma^*} \mathscr{A}_{(0,2)}^{\epsilon} + \frac{1}{\gamma^2\gamma^*} \mathscr{A}_{(1,3)}^{\epsilon}.$$

Case 4. We can reduce it to Case 3 by

$$\gamma(\zeta)A_{KL}^{\epsilon 0} - \gamma(z)(A_{LK}^{\epsilon 0})^* = -\left(\gamma(\zeta)A_{LK}^{\epsilon 0} - \gamma(z)(A_{KL}^{\epsilon 0})^*\right)^*. \qquad \square$$

This also concludes the proof of Theorem 4.2. As an important corollary to Theorems 4.1 and 4.2 we see if we compose the operator $\mathbf{P}_q^\epsilon$ with $\gamma^2\gamma^*$, we obtain operators which are of type 1. This is the idea behind the next theorem which results from multiplying the basic integral representation Theorem 3.4 by an appropriate number of factors of γ and γ^*. We can then let $\epsilon \to 0$ to obtain a representation on the domain D.

For a given $f \in L_{0,q}^2(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, we take a sequence $\{f_\epsilon\}_\epsilon$ which approaches f in the graph norm by Proposition 3.3. With the use of Theorem 3.4, we define operators T_q^ϵ, S_q^ϵ, and P_q^ϵ so that we have the representation for each f_ϵ

$$(4.23) \qquad f_\epsilon(z) = T_q^\epsilon \bar{\partial} f_\epsilon + S_q^\epsilon \bar{\partial}_\epsilon^* f_\epsilon + P_q^\epsilon f_\epsilon.$$

We then define the operators T_q, S_q, and P_q to be such that $\gamma^* T_q \circ \gamma^2$, $\gamma^* S_q \circ \gamma^2$, and $\gamma^* P_q \circ \gamma^2$ are the limit operators, as $\epsilon \to 0$, of $\gamma^* T_q^\epsilon \circ \gamma^2$, $\gamma^* S_q^\epsilon \circ \gamma^2$, and $\gamma^* P_q^\epsilon \circ \gamma^2$, respectively, which exist by Proposition 2.3. We therefore obtain the

THEOREM 4.11. *For* $f \in L_{(0,q)}^2(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\gamma(z)^3 f(z) = \gamma^* T_q \bar{\partial}(\gamma^2 f) + \gamma^* S_q \bar{\partial}^*(\gamma^2 f) + \gamma^* P_q(\gamma^2 f).$$

We can see how Theorem 1.2 follows from Theorem 4.11. To obtain the theorem in the form of Theorem 1.2, we set $\tilde{T}_q = \gamma^* T_q \gamma^2$. Then, from Theorem 3.4, we see $\gamma^* S_q \gamma^2$ has the form

$$\gamma^* T_{q-1}^* \gamma^2 + A_2 = (\gamma^*)^2 T_{q-1}^* \gamma + A_2$$
$$= \tilde{T}_{q-1}^* + A_2.$$

The error terms in Theorem 1.2 then involve operators of type 2 acting on the derivatives of f, or those of type 1 acting on f.

5. Estimates

We define Z_1 operators to be those which take the form

$$Z_1 = A_{(1,1)} + E_{1-2n} \circ \gamma,$$

and we write Theorem 4.11 as

$$(5.1) \qquad \gamma^3 f = Z_1 \gamma^2 \, \bar{\partial} f + Z_1 \gamma^2 \, \bar{\partial}^* f + Z_1 f.$$

We define Z_j operators to be those operators of the form

$$Z_j = \overbrace{Z_1 \circ \cdots \circ Z_1}^{j \text{ times}}.$$

By applying Corollary 2.4 and Theorem 2.5 $n+2$ times, we have the property

$$Z_{n+2} : L^2(D) \to L^\infty(D).$$

We now iterate (5.1) to get

$$\gamma^{3j} f = \left(Z_1 \gamma^{3(j-1)+2} + Z_2 \gamma^{3(j-2)+2} + \cdots + Z_j \gamma^2 \right) \bar{\partial} f$$
$$+ \left(Z_1 \gamma^{3(j-1)+2} + Z_2 \gamma^{3(j-2)+2} + \cdots + Z_j \gamma^2 \right) \bar{\partial}^* f + Z_j f.$$

Then we can prove the following theorem.

THEOREM 5.1. *For* $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $q \geq 1$,

$$\left\| \gamma^{3(n+2)} f \right\|_{L^\infty} \lesssim \left\| \gamma^2 \, \bar{\partial} f \right\|_\infty + \left\| \gamma^2 \, \bar{\partial}^* f \right\|_\infty + \| f \|_2.$$

6. Note added in proof

In recent work of the author with I. Lieb, a new notation to deal with error terms is developed, as well as a more efficient definition of type. Furthermore, a more careful study of the mapping properties of operators involved shows that $A_2 \circ 1/\gamma$ is bounded from $L^2(D)$ to itself. These observations lead to slight improvements of Theorem 4.11 and Theorem 5.1. In particular, Theorem 4.11 can be written as follows.

THEOREM 6.1. *For* $f \in L^2_{(0,q)}(D) \cap \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$,

$$\gamma(z) f(z) = \gamma^* T_q \overline{\partial} f + \gamma^* S_q \overline{\partial}^* f + \gamma^* P_q f.$$

Theorem 5.1 can be written as follows.

THEOREM 6.2. *For* $f \in L^2_{0,q}(D) \cap \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, $q \geq 1$,

$$\| \gamma^{2n+1} f \|_{L^\infty} \lesssim \| \gamma^2 \overline{\partial} f \|_\infty + \| \gamma^2 \overline{\partial}^* f \|_\infty + \| f \|_2.$$

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