Illinois Journal of Mathematics
Volume 53, Number 4, Winter 2009, Pages 1051–1059
S 0019-2082

ON THE PROJECTIVE EMBEDDINGS OF GORENSTEIN TORIC DEL PEZZO SURFACES

T. KIKUCHI AND T. NAKANO

ABSTRACT. We study the projective embeddings of complete Gorenstein toric del Pezzo surfaces by ample complete linear systems, especially of minimal degree and dimension. Complete Gorenstein toric del Pezzo surfaces are in one-to-one correspondence with the 2-dimensional reflexive integral convex polytopes, which are classified into 16 types up to isomorphisms of lattices. Our main result shows that the minimal dimension and the minimal degree of all the ample complete linear systems on such a surface are attained by the primitive anti-canonical class except one case. From this, we determine the projective embeddings of these surfaces which are global complete intersections. We also show that the minimal free resolution of the defining ideal of the image under the anti-canonical embedding of these surfaces is given by an Eagon–Northcott complex.

1. Introduction

The purpose of this note is to study the embeddings of complete Gorenstein toric del Pezzo surfaces (abbreviated as GTDP surfaces in the following) into projective spaces by ample complete linear systems, especially of minimal dimension and minimal degree. The GTDP surfaces are in one-to-one correspondence with the 2-dimensional reflexive integral convex polytopes via toric geometry, which are classified into 16 types up to the isomorphisms of lattices ([4], [5]). They are labeled R_i $(1 \leq i \leq 16)$ in this note (Figure 1). Though these 16 polytopes contain all the geometrical information of these surfaces, we would like to describe them as a variety in projective spaces. Then the projective embeddings of minimal dimension and minimal degree are the most favorable and we will study them in this note. We also note

Received March 7, 2009; received in final form November 5, 2009.
2000 *Mathematics Subject Classification.* 14J25, 14M25.

©2010 University of Illinois

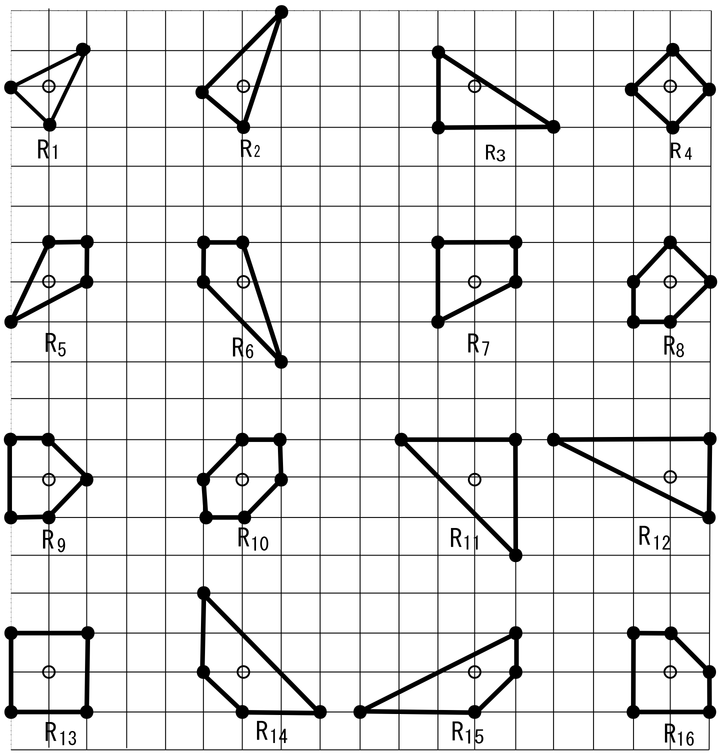

FIGURE 1. The 16 2-dimensional reflexive integral convex polytopes.

that these Gorenstein surfaces are actually locally complete intersection since they have at worst A_n-singularities. Thus, we have a natural question if these surfaces are *global* complete intersections in projective spaces or not.

Our three main results are summarized as follows. In Theorem 2.1, we show that the projective embedding of minimal dimension and minimal degree by ample complete linear systems on GTDP surfaces is given simultaneously by the primitive anti-canonical class except one case. We next determine when GTDP surfaces are complete intersections in projective spaces in Theorem 2.2. Finally, in Proposition 2.4, we show that the minimal free resolution of the defining ideal of the image under the anti-canonical embedding of GTDP surfaces is given by an Eagon–Northcott complex.

For toric varieties, we follow the notations and terminology of Fulton [2]. We also assume that all the varieties and morphisms are defined over a fixed algebraically closed field k of characteristic 0. We finally note that part of this note is based on the master thesis of the first author [3].

2. Main results

Let R_i $(1 \leq i \leq 16)$ be a 2-dimensional reflexive integral convex polytope. Let S_i be the GTDP surface corresponding to R_i and $\mathrm{AC}(S_i)$ the set of all the ample line bundles on S_i (the ample cone of S_i). We note that any ample line bundle on S_i is generated by global sections and very ample ([2, Exercise on p. 70]).

Let $-K_S$ be the anti-canonical bundle of $S = S_i$ and t the index of S (t is the maximal natural number such that $\frac{K_S}{t} \in \mathrm{Pic}(S)$). Then we call $-K'_S := -\frac{1}{t}K_S \in \mathrm{Pic}(S)$ *the primitive anti-canonical bundle of* S temporarily in this note.

Now, we state our first main result.

THEOREM 2.1. *Let $S = S_i$ be a GTDP surface. Consider two functions $h^0, \deg\colon \mathrm{AC}(S) \to \{1, 2, \dots\}$ defined by $h^0(L) := \dim H^0(S, L)$ and $\deg L := L^2$. Then*

(i) *If $S \neq S_{14}$, then the minimal values of h^0 and* deg *are taken simultaneously by and only by the primitive anti-canonical bundle of S.*

(ii) *If $S = S_{14}$ (the blowing-up of $\mathbf{P}^2$ at a point), then the minimal values of h^0 and* deg *are taken simultaneously by and only by $\pi^*(\mathcal{O}_{\mathbf{P}^2}(2)) + \mathcal{O}_{S_{14}}(-E)$, where $\pi : S_{14} \to \mathbf{P}^2$ is the blowing-up of a point of $\mathbf{P}^2$ and $E \subset S_{14}$ is the exceptional divisor.*

Proof. We consider the $S = S_8$ case as a typical case. Let R_8 be the 2-dimensional reflexive integral convex polytope and R_8° the polar of R_8. The fan $\Delta_8 := \Delta_{R_8}$ of S_8 is given by the set of cones generated by the faces of R_8° and the origin (Figure 2).

(i) We first describe $\mathrm{Pic}(S)$ and K_S explicitly. Let D be a T-Cartier divisor on S. Then D is given by (and identified with) a collection $\psi = \psi_D$ of 5 linear integral functions $\{\psi_i | 1 \leq i \leq 5\}$, where $\psi_i = a_i x + b_i y$ $(a_i, b_i \in$

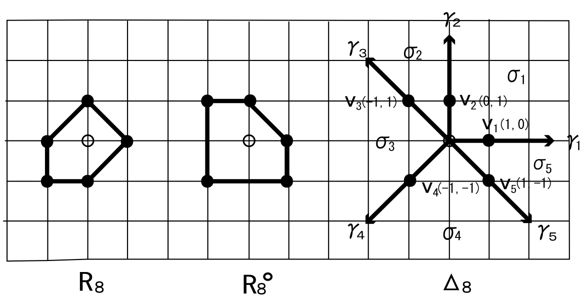

FIGURE 2. R_8, R_8° and Δ_8.

$\mathbf{Z}, 1 \leq i \leq 5)$ is a linear integral form on σ_i. We think of ψ as an element $\psi = {}^t(a_1, b_1, a_2, b_2, \ldots, a_5, b_5) \in \mathbf{Z}^{10}$.

By the compatibility of ψ_i's on the edges γ_i, we have $\psi_1(v_2) = \psi_2(v_2)$ etc. Thus, we have five equations:

$$b_1 = b_2, \qquad -a_2 + b_2 = -a_3 + b_3, \qquad -a_3 - b_3 = -a_4 - b_4,$$
$$a_4 - b_4 = a_5 - b_5, \qquad a_5 = a_1.$$

Hence, by setting

$$A := \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

we have $\text{T-CDiv}(S) \simeq \ker(A : \mathbf{Z}^{10} \to \mathbf{Z}^5)$, where $\text{T-CDiv}(S)$ is the group of T-Cartier divisors. By the elementary divisor theory, we have $B = PAQ$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the Smith normal form (the elementary divisor matrix) of A,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{GL}(5, \mathbf{Z})$$

and

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & -1 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(10, \mathbf{Z}).$$

Thus, $\text{T-CDiv}(S) \simeq \ker A = Q(\ker B)$ is a free submodule generated by the ith columns $(6 \leq i \leq 10)$ of Q. Set $f_i := (i+5)$th column of Q $(1 \leq i \leq 5)$ so that $\text{T-CDiv}(S) \simeq \bigoplus_{i=1}^{5} \mathbf{Z} f_i$.

Furthermore, the principal T-Cartier divisors of S are given by a global integral linear form $ax+by$ on $\mathbf{R}^2$ $(a,b\in\mathbf{Z})$. Thus, the subgroup of the principal T-Cartier divisors is generated by $r_1 := {}^t(1,0,1,0,1,0,1,0,1,0)$ and $r_2 := {}^t(0,1,0,1,0,1,0,1,0,1)\in\mathbf{Z}^{10}$. Set $r_1=\sum_{j=1}^5\alpha_j f_j$ and $r_2=\sum_{j=1}^5\beta_j f_j$. Then we have $(\alpha_1,\ldots,\alpha_5)=(0,0,0,1,0)$ and $(\beta_1,\ldots,\beta_5)=(1,1,1,0,1)$. Thus, if we set

$$C := {}^t\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 1\end{pmatrix},$$

then $\mathrm{Pic}(S)\simeq\mathrm{coker}(C:\mathbf{Z}^2\to\mathbf{Z}^5)$.

Let

$$H = {}^t\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\end{pmatrix}$$

be the Smith normal form of C so that $H=RCU$, where

$$R=\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 1\end{pmatrix}\in\mathbf{GL}(5,\mathbf{Z}) \quad\text{and}\quad U=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\in\mathbf{GL}(2,\mathbf{Z}).$$

Then $\mathrm{Pic}(S)\simeq\mathrm{coker}\,C=R^{-1}(\mathrm{coker}\,H)$. Since $\mathrm{coker}\,H$ is generated by ${}^t(0,0,1,0,0),{}^t(0,0,0,1,0),{}^t(0,0,0,0,1)$ and

$$R^{-1}=\begin{pmatrix} 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\end{pmatrix},$$

we know $\mathrm{Pic}(S)$ is generated by ${}^t(0,0,1,0,0),{}^t(1,0,0,0,0),{}^t(0,0,0,0,1)$ in $\mathbf{Z}^5$. Thus, we conclude that $\mathrm{Pic}(S)=\mathbf{Z}\mathcal{O}_S(f_3)\oplus\mathbf{Z}\mathcal{O}_S(f_1)\oplus\mathbf{Z}\mathcal{O}_S(f_5)\simeq\mathbf{Z}^3$, where $\mathcal{O}_S(f_i)$ is the line bundle corresponding to f_i.

The canonical bundle $K_S\in\mathrm{Pic}(S)$ is represented as

$$\psi_{K_S}=\{(x+y),(y),(-x),(-y),(x)\}={}^t(1,1,0,1,-1,0,0,-1,1,0)\in\mathbf{Z}^{10}$$

as an integral piecewise linear function on $\mathbf{R}^2$ since it is characterized by $\psi_{K_S}(v_i)=1$ $(1\le i\le 5)$. Since $K_S=\mathcal{O}_S(f_2-f_3)=-\mathcal{O}_S(f_1)-2\mathcal{O}_S(f_3)-\mathcal{O}_S(f_5)\in\mathrm{Pic}(S)$, K_S is not divisible in $\mathrm{Pic}\,S$ and the index of $S=S_8$ is 1.

(ii) Let D_i be the prime T-Weil divisor corresponding to the edge $\gamma_i=\mathbf{R}_{\ge 0}v_i$ $(1\le i\le 5)$. Then the Weil divisor $E_1=[f_3]$ corresponding to the T-Cartier divisor f_3 is calculated as $E_1=\sum_{j=1}^5-\langle f_3,v_j\rangle D_j=2D_1+2D_5$. Similarly, we have $E_2:=[f_1]=-2D_1+2D_4-2D_5$ and $E_3:=[f_5]=-D_1$.

Now let us determine the intersection product on the Weil divisor class group over $\mathbf{Q}$, $A_1(S)_{\mathbf{Q}}=A_1(S)\otimes_{\mathbf{Z}}\mathbf{Q}$. In the case $i\neq j$, the intersection number $(D_i, D_j) \in \mathbf{Q}$ is calculated as

$$(D_i, D_j) = \begin{cases} 0, & \text{if } \gamma_{\mathrm{i}} \text{ and } \gamma_{\mathrm{j}} \text{ does not generate a cone } \sigma \in \Delta, \\ \frac{1}{|\det(v_i, v_j)|}, & \text{if } \gamma_{\mathrm{i}} \text{ and } \gamma_{\mathrm{j}} \text{ generate a cone } \sigma \in \Delta. \end{cases}$$

The principal T-Cartier divisor $\phi = \{-x\}$ defines a T-Weil divisor $[\phi] = \sum_{j=1}^{5} -\phi(v_j)D_j = 1D_1 + 0D_2 + (-1)d_3 + (-1)D_4 + 1D_5 = D_1 - D_3 - D_4 + D_5$, and the corresponding line bundle $\mathcal{O}_S(\phi) = 0$ in $\mathrm{Pic}(S)$. Similarly, the principal T-Cartier divisor $\psi = \{-y\}$ defines a T-Weil divisor $[\psi] = \sum_{j=1}^{5} -\psi(v_j) \times D_j = D_2 + D_3 - D_4 - D_5$, and $\mathcal{O}_S(\psi) = 0$ in $\mathrm{Pic}(S)$.

From these, we get the intersection numbers of $\{D_j\}$ as $(D_1, D_2) = (D_2, D_3) = (D_1, D_5) = 1, (D_3, D_4) = (D_4, D_5) = \frac{1}{2}$, $(D_i, D_j) = 0$ for other $i \neq j$, and ${D_1}^2 = {D_2}^2 = -1, {D_3}^2 = {D_5}^2 = -\frac{1}{2}, {D_4}^2 = 0$.

Thus, we get the intersection matrix

$$I = ((E_i, E_j)) = \begin{pmatrix} -2 & 4 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For the calculation of $h^0(L) = \dim H^0(S, L)$ and $\deg L = L^2$ for $L \in \mathrm{Pic}(S)$, it is convenient to take an orthogonal basis of $\mathrm{Pic}(S)$. Set $F_1 := E_1 + 2E_3, F_2 := E_2, F_3 := -E_1 - E_3$. Then we have the intersection matrix

$$J = ((F_i, F_j)) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then noting that $K_S = F_1 - F_2 + 3F_3$, from Riemann–Roch and the vanishing theorem ([2, Corollary on p. 74]), we have for any $aF_1 + bF_2 + cF_3$ in the ample cone $\mathrm{AC}(S)$,

$$\begin{aligned} & h^0(aF_1 + bF_2 + cF_3) \\ & \quad = 1 + \frac{1}{2}(aF_1 + bF_2 + cF_3)(-F_1 + F_2 - 3F_3) + \frac{1}{2}(aF_1 + bF_2 + cF_3)^2 \\ & \quad = 1 + a - b - \frac{3}{2}c - a^2 - b^2 + \frac{c^2}{2}. \end{aligned}$$

We also have

$$\deg(aF_1 + bF_2 + cF_3) = (aF_1 + bF_2 + cF_3)^2 = -2a^2 - 2b^2 + c^2.$$

(iii) We determine the ample cone $\mathrm{AC}(S)$. Note that $A_1(S)$ is generated by D_i $(1 \leq i \leq 5)$. Then by Kleiman's ampleness criterion, $L \in \mathrm{Pic}(S)$ is ample if and only if $(L, D_i) > 0$ $(1 \leq i \leq 5)$. Since $(aF_1 + bF_2 + cF_3, D_1) = 2a - c, (aF_1 + bF_2 + cF_3, D_2) = -2b - c, (aF_1 + bF_2 + cF_3, D_3) = b, (aF_1 + bF_2 + cF_3, D_4) = a - b - c, (aF_1 + bF_2 + cF_3, D_5) = -a$, we have that $aF_1 + bF_2 + cF_3$ is ample if and only if $2a - c > 0, -2b - c > 0, b > 0, -a > 0$ (note that $a - b - c > 0$

is unnecessary). Hence, $\mathrm{AC}(S) = \{(a,b,c) \in \mathbf{Z}^3 | 2a - c > 0, -2b - c > 0, b > 0, -a > 0\}$.

So our minimization problem reduces to an elementary problem of minimizing $h^0(aF_1 + bF_2 + cF_3)$ and $\deg(aF_1 + bF_2 + cF_3)$ on $\mathrm{AC}(S)$. We use Mathematica [8] to solve this. If we type:

$$\begin{aligned}&\mathrm{Minimize}[\{-a^2 - b^2 + c^2/2 + a - b - (3/2)c + 1,\\ &\quad 2a - c \geqq 1, -2b - c \geqq 1, b \geqq 1, -a \geqq 1\}, \{a,b,c\}],\end{aligned}$$

then the answer is:

$$\longrightarrow \big\{5, \{a = -1, b = 1, c = -3\}\big\}.$$

Similarly, if we type:

$$\mathrm{Minimize}[\{-2a^2 - 2b^2 + c^2, 2a - c \geqq 1, -2b - c \geqq 1, b \geqq 1, -a \geqq 1\}, \{a,b,c\}],$$

then we have:

$$\longrightarrow \big\{6, \{a = -1, b = 1, c = -3\}\big\}.$$

We note that Mathematica minimizes h^0 and deg in the closed continuous domain $\{(a,b,c) \in \mathbf{R}^3 | 2a - c \geq 1, -2b - c \geq 1, b \geq 1, -a \geq 1\}$, but this causes no problem. Thus, the minimum values of h^0 and deg on $\mathrm{AC}(S)$ are taken simultaneously at $-F_1 + F_2 - 3F_3 = -K_S$.

The other cases are computed similarly. The key computational data of the other 15 surfaces can be seen in the Appendix, which is put in our web page (URL: http://www.r.dendai.ac.jp/~nakano/research.html). In this Appendix, the defining equations of the image under these minimal projective embeddings are also computed. □

Next, we determine if S_i is a complete intersection in a projective space or not.

THEOREM 2.2. *Let $f : S_i \hookrightarrow \mathbf{P}^N$ be an embedding of S_i by a* (*not necessarily complete*) *linear system δ and suppose that the image $f(S_i)$ is not contained in a hyperplane and is a global complete intersection in $\mathbf{P}^N$. Then there is at most* 1 *such δ on each S_i. More precisely*:

(i) *If $i = 1, 2, 4, 5, 11, 12, 13$, then δ is the complete primitive anti-canonical system $|-K'_{S_i}|$.*

(ii) *If $i = 3, 6, 7, 8, 9, 10, 14, 15, 16$, then there is no such δ.*

Proof. Let δ' be the complete linear system containing δ and $f' : S = S_i \hookrightarrow \mathbf{P}^{N+k}$ be the corresponding embedding, where $k = \dim \delta' - \dim \delta$. We have a projection $h : \mathbf{P}^{N+k} \to \mathbf{P}^N$ such that $h \circ f' = f$. By assumption, $\mathrm{im}\, f$ is defined by $N-2$ homogeneous equations of degree d_i $(1 \leq i \leq N-2, d_i \geq 2)$ in $\mathbf{P}^N$. Hence, by adjunction formula for the canonical bundle, we have $K_S \simeq \mathcal{O}_S(-N - 1 + \sum_{i=1}^{N-2} d_i)$. Since $-K_S$ is ample, we have $0 > -N - 1 + \sum_{i=1}^{N-2} d_i \geq -N - 1 + 2(N-2) = N - 5$. Thus, we conclude $2 \leq N \leq 4$.

Proof of (i): We deal with $S = S_4$ case. The other cases (S_i for $i = 1,2,5,11,12,13$) can be dealt with similarly. We have $\mathrm{Pic}(S) = \mathbf{Z}\mathcal{O}_S(f_1) \oplus \mathbf{Z}\mathcal{O}_S(f_4)$, $K_S = 2\mathcal{O}_S(f_1) + \mathcal{O}_S(f_4)$, index $= 1$, $h^0(a\mathcal{O}_S(f_1) + b\mathcal{O}_S(f_4)) = 1 - a + 2ab - 2b^2$, $\deg(a\mathcal{O}_S(f_1) + b\mathcal{O}_S(f_4)) = 4ab - 4b^2$, the ample cone $\mathrm{AC}(S) = \{a\mathcal{O}_S(f_1) + b\mathcal{O}_S(f_4) | b > a, b < 0\}$ (see the Appendix for the details). Now set $\delta' = |x \times \mathcal{O}_S(f_1) + y\mathcal{O}_S(f_4)|$.

(a) Suppose $N = 4$. Then $\mathrm{im}(f : S \hookrightarrow \mathbf{P}^4)$ is defined by 2 equations of degree p, q $(p, q \geq 2)$. Since $K_S \simeq \mathcal{O}_S(-5 + p + q)$, we must have $p + q < 5$. Hence, $p = q = 2$ and we get $4 = \deg \mathrm{im}\, f = \deg \mathrm{im}\, f' = 4xy - 4y^2$ $(x, y \in \mathbf{Z}, y > x, y < 0)$. Thus, we have $(x, y) = (-2, -1)$ and $\delta' = |-K_S|$. Since $4 = \dim |-K_S| = \dim \delta$, we have $\delta = |-K_S| = |-K'_S|$ in this case. Conversely, it is easy to see that the image of the embedding associated to $|-K_S|$ is actually a complete intersection. Actually, this image is given by $\{x_1x_3 - {x_0}^2 = 0, x_2x_4 - {x_0}^2 = 0\} \subset \mathbf{P}^4$ (see the Appendix).

(b) Suppose $N = 3$. Then the $\mathrm{im}\, f$ is defined by 1 equation of degree $p < 4$. Thus, $p = 2, 3$. But we have 2 (or 3) $= \deg \mathrm{im}\, f = \deg \mathrm{im}\, f' = 4xy - 4y^2$, a contradiction.

(c) Suppose $N = 2$. Then $1 = \deg \mathrm{im}\, f = \deg \mathrm{im}\, f' = 4xy - 4y^2$, a contradiction.

Proof of (ii): Suppose that $S = S_i$ $(i = 3,6,7,8,9,10,15,16)$. Then, we know that the minimum value of $\deg L$ for $L \in \mathrm{AC}(S)$ is greater than or equal to 5 (see the Appendix). On the other hand, we have $\deg \mathrm{im}\, f = \deg \mathrm{im}\, f' \leq 4$ as in (i), a contradiction.

Suppose $S = S_{14}$. In this case, $S \simeq Q_P(\mathbf{P}^2)$, the blowing-up of $\mathbf{P}^2$ at a point P.

(a) Suppose $N = 4$. Then S is a complete intersection of $(2,2)$-type in $\mathbf{P}^4$. Then by the adjunction formula, $-K_S = \mathcal{O}_S(1)$ and $K_S^2 = 4$. On the other hand, $-K_S = 3L - E$, where $L = \pi^*(\mathcal{O}_{\mathbf{P}^2}(1)), \pi : S \to \mathbf{P}^2$ is the blowing-up, and E is the line bundle associated to the exceptional divisor. Thus $K_S^2 = 8$, a contradiction.

(b) Suppose $N = 3$. Then S is a smooth surface of degree 2 or 3 in $\mathbf{P}^3$. Since $S \simeq Q_P(\mathbf{P}^2)$, this is a contradiction.

(c) Suppose $N = 2$. Then $S \simeq \mathbf{P}^2$, a contradiction. □

Thus, we have the following corollary.

Corollary 2.3. *If* $i = 3,6,7,8,9,10,14,15,16$, *then* S_i *can never be realized as a complete intersection in a projective space.*

We finally consider the defining equations and their syzygies of the image under the anti-canonical embedding (not necessarily minimal embedding as in Theorem 2.1) of our surfaces. The generators and syzygies of the defining ideal of toric varieties have been a subject of interest and studied by several authors ([6], [7, Chapter 13]). The following proposition is due to the referee.

PROPOSITION 2.4. *Let $S = S_i$ be a GTDP surface. Then the minimal free resolution of the ideal I_{-K_S} of the image under the anti-canonical embedding is given by an Eagon–Northcott complex.*

Proof. Let $r + 2 = h^0(-K_S)$. We use a hyperplane section argument as in [6]: Toric surfaces embedded by complete linear systems are projectively normal hence arithmetically Cohen–Macaulay. So taking a hyperplane section yields a smooth, projectively normal curve (a generic section is smooth since the singular locus is in codimension 2). Since reflexive polygons have a single interior point, this curve has genus 1 [2, p. 91] and degree K_S^2. Thus, the minimal graded free resolution is that of an elliptic normal curve, which is an Eagon–Northcott complex ([1, Theorem 6.26]—our choice of r above is made to coincide with the notation in [1]). □

Acknowledgments. We thank the referee very much for giving us useful suggestions and comments for the improvement of this note. Especially, we learn Proposition 2.4 and several important references from him.

REFERENCES

[1] D. Eisenbud, *The geometry of syzygies*, GTM, vol. 229, Springer, New York, 2005. MR 2103875

[2] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton, NJ, 1993. MR 1234037

[3] T. Kikuchi, *On toric Gorenstein del Pezzo surfaces*, Master Thesis, Tokyo Denki Univ., 2007.

[4] R. Koelman, *The number of moduli of families of curves on toric surfaces*, Thesis, Univ. Nijmegen, 1991.

[5] B. Poonen and F. Rodriguez-Villegas, *Lattice polygons and the number 12*, Amer. Math. Monthly **107** (2000), 238–250. MR 1742122

[6] H. Schenck, *Lattice polygons and Green's theorem*, Proc. Amer. Math. Soc. **132** (2004), 3509–3512. MR 2084071

[7] B. Sturmfels, *Gröbner bases and convex polytopes*, Amer. Math. Soc., Providence, RI, 1996. MR 1363949

[8] Wolfram Research, Inc., *Mathematica 6*, Wolfram Research, Inc., Champaign IL, 2007.

T. KIKUCHI, DEPARTMENT OF MATHEMATICAL SCIENCES, TOKYO DENKI UNIVERSITY, HIKI-GUN, HATOYAMA-MACHI, SAITAME-KEN, 350-0394, JAPAN

T. NAKANO, DEPARTMENT OF MATHEMATICAL SCIENCES, TOKYO DENKI UNIVERSITY, HIKI-GUN, HATOYAMA-MACHI, SAITAME-KEN, 350-0394, JAPAN

E-mail address: tnakano@mail.dendai.ac.jp