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SOME DYNAMICAL PROPERTIES FOR LINEAR OPERATORS

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ABSTRACT. In this article, some dynamical properties for continuous linear operators are studied. We investigate that neither normal operators nor compact operators can be Li–Yorke chaotic. In addition, we show that a small compact perturbation of the unit operator could be distributionally chaotic.

1. Introduction and preliminaries

A discrete dynamical system is simply a continuous mapping $f : X \to X$ where X is a complete separable metric space. For $x \in X$, the orbit of x under f is $\mathrm{Orb}(f,x) = \{x, f(x), f^2(x), \dots\}$ where $f^n = f \circ f \circ \cdots \circ f$ is the nth iterate of f obtained by composing f with n times.

In 1975, Li and Yorke [8] observed complicated dynamical behavior for the class of interval maps with period 3. This phenomena is currently known under the name of chaos in the sense of Li and Yorke.

DEFINITION 1. $\{x, y\} \subseteq X$ is said to be a Li–Yorke chaotic pair, if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \qquad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Furthermore, f is called Li–Yorke chaotic, if there exists an uncountable subset $\Gamma \subseteq X$ such that each pair of two distinct points in Γ is a Li–Yorke chaotic pair.

In Schweizer and Smítal's paper [13], distributional chaos was defined, which is a kind of chaos stronger than Li–Yorke chaos.

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For any pair $\{x,y\} \subset X$ and any $n \in \mathbb{N}$, define distributional function $F^n_{xy} : \mathbb{R} \to [0,1]$:

$$F^n_{xy}(\tau) = \frac{1}{n}\#\{0 \le i \le n-1 : d(f^i(x), f^i(y)) < \tau\}.$$

Furthermore, define

$$F_{xy}(\tau) = \liminf_{n\to\infty} F^n_{xy}(\tau),$$
$$F^*_{xy}(\tau) = \limsup_{n\to\infty} F^n_{xy}(\tau).$$

Both F_{xy} and F^*_{xy} are nondecreasing functions and may be viewed as cumulative probability distributional functions satisfying $F_{xy}(\tau) = F^*_{xy}(\tau) = 0$ for $\tau < 0$.

Definition 2. $\{x,y\} \subset X$ is said to be a distributionally chaotic pair, if

$$\forall \tau > 0, \quad F^*_{xy}(\tau) \equiv 1 \quad \text{and} \quad \exists \varepsilon > 0, \quad F_{xy}(\varepsilon) = 0.$$

Furthermore, f is called distributionally chaotic, if there exists an uncountable subset $\Lambda \subseteq X$ such that each pair of two distinct points in Λ is a distributionally chaotic pair. Moreover, Λ is called a distributionally ε-scrambled set.

Distributional chaos always implies Li–Yorke chaos, as it requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic. The converse implication is not true in general. However, in practice, even in the simple case of Li–Yorke chaos, it might be quite difficult to prove chaotic behavior from the very definition. Such attempts have been made in the context of linear operators (see [2], [3]). Further results of [2] were extended in [11] to distributional chaos for the annihilation operator of a quantum harmonic oscillator. More about distributional chaos, we refer to [1], [9], [10], [15], [16].

We are interested in the dynamical systems induced by continuous linear operators on Banach spaces. From Rolewicz's article [12], hypercyclicity is widely studied (Grosse–Erdmann's and Shapiro's articles [5], [14] are good surveys). In fact, it coincides with a dynamical property "transitivity." With regard to distributional chaos, Martínez-Giménez et al. gave a discussion for shift operators in [4]. In a recent article [7] of the first author, one introduces a new dynamical property for linear operators called norm-unimodality which implies distributional chaos.

Definition 3. Let X be a Banach space and let $T \in \mathcal{L}(X)$. T is called norm-unimodal, if we have a constant $r > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X$ satisfying

$$\lim_{k\to\infty} \|T^k x_m\| = 0, \quad \text{and} \quad \|T^i x_m\| \ge r^i \|x_m\|, \quad i = 1, 2, \ldots, m.$$

Furthermore, such r is said to be a norm-unimodal constant for the norm-unimodal operator T.

THEOREM 4 (Distributionally chaotic criterion [7]). *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If T is norm-unimodal, then T is distributionally chaotic.*

More generally, is the following theorem.

THEOREM 5 (Weakly distributionally chaotic criterion [7]). *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Suppose that C_m is a sequence of positive numbers increasing to $+\infty$. If there exist $\{x_m\}_{m=1}^{\infty}$ in X and a sequence of positive integers $\{N_m\}_{m=1}^{\infty}$ increasing to $+\infty$, satisfying*

$$\text{(WNU1)} \qquad \lim_{k\to\infty} \|T^k x_m\| = 0;$$

$$\text{(WNU2)} \qquad \lim_{m\to\infty} \frac{\#\{0 \le i \le N_m - 1; \|T^i x_m\| \ge C_m \|x_m\|\}}{N_m} = 1.$$

Then T is distributionally chaotic.

2. Normal operators and compact operators

In this section, we will discuss some dynamical properties of normal operators and compact operators. Chaos for small compact perturbations of the unit operator are also considered.

Recall that the ω-limit set of a point $x \in X$ in a dynamical system (X, f), is defined by $\omega(x) = \{y \in X; \exists\{n_i\} \uparrow +\infty \text{ s.t. } \lim_{i\to\infty} f^{n_i}(x) = y\}$. Denote by $\mathbb{D}$ the unit open disk on complex plane. Moreover, $\mathbb{D}^-$ denoted as its closure and $\partial\mathbb{D}$ denoted as its boundary.

THEOREM 6. *Let N be a normal operator on separable complex Hilbert space. Then N is impossible to be Li–Yorke chaotic.*

Proof. Since N is normal, there exist a finite positive regular Borel measure μ and a Borel function $\eta \in L^\infty(\sigma(N), \mu)$ such that N and M_η are unitarily equivalent. M_η is multiplication by η on $L^2(\sigma(N), \mu)$. To see M_η being not Li–Yorke chaotic, it is sufficient to prove $\lim_{m\to\infty} \|M_\eta^m(f)\| = 0$ if $0 \in \omega(f)$.

Let

$$\begin{aligned}
\Delta_1 &= \{z \in \sigma(N); |\eta(z)| \ge 1\}, \\
\Delta_2 &= \{z \in \sigma(N); |\eta(z)| < 1\}, \\
\Delta_3 &= \{z \in \sigma(N); f(z) = 0 \text{ a.e. } [\mu]\}, \\
\Delta_4 &= \{z \in \sigma(N); f(z) \ne 0 \text{ a.e. } [\mu]\}.
\end{aligned}$$

Since $0 \in \omega(f)$, there exist $\{m_k\}_{k=1}^{\infty}$ such that $\lim_{m_k\to\infty} \|M_\eta^{m_k}(f)\| = 0$. Then

$$
\begin{aligned}
\|M_\eta^{m_k}(f)\|^2 &= \int_{\sigma(N)} |\eta^{m_k} f|^2 \, d\mu \\
&= \int_{\Delta_1\cap\Delta_4} |\eta^{m_k} f|^2 \, d\mu + \int_{\Delta_2\cap\Delta_4} |\eta^{m_k} f|^2 \, d\mu \\
&\geq \int_{\Delta_1\cap\Delta_4} |f|^2 \, d\mu + \int_{\Delta_2\cap\Delta_4} |\eta^{m_k} f|^2 \, d\mu,
\end{aligned}
$$

and hence $\mu(\Delta_1 \cap \Delta_4) = 0$. For any $m \in \mathbb{N}$, there exists k such that $m_k \leq m < m_{k+1}$. Consequently,

$$
\begin{aligned}
\|M_\eta^{m}(f)\|^2 &= \int_{\Delta_2\cap\Delta_4} |\eta^{m} f|^2 \, d\mu \\
&= \int_{\Delta_2\cap\Delta_4} |\eta^{m_k} f|^2 |\eta^{m-m_k}|^2 \, d\mu \\
&\leq \int_{\Delta_2\cap\Delta_4} |\eta^{m_k} f|^2 \, d\mu \\
&= \|M_\eta^{m_k}(f)\|^2.
\end{aligned}
$$

Therefore, $\lim_{m\to\infty} \|M_\eta^m(f)\| = 0$. □

THEOREM 7. *Let K be a compact operator on complex Hilbert space. Then K is impossible to be Li–Yorke chaotic.*

Proof. According to Riesz's decomposition theorem, we have

$$
K = \begin{bmatrix} K_1 & \\ & K_2 \end{bmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix},
$$

where $\sigma(K_1) = \sigma(K) \cap \mathbb{D}$ and $\sigma(K_2) = \sigma(K) \setminus \sigma(K_1)$.

Furthermore,

$$
K = \begin{bmatrix} K_1 & * \\ & \widetilde{K_2} \end{bmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix}
$$

and $\sigma(\widetilde{K_2}) = \sigma(K_2) = \{\mu_1, \mu_2, \ldots, \mu_l\}$.

Since K is a compact operator, one can see

(1) there exist $0 < \rho < 1$ and $N \in \mathbb{N}$ such that $\|K^n(x)\| \leq \rho^n \|x\|$, for any $x \in H_1$ and any $n \geq N$.

(2) $\widetilde{K_2}$ is similar to Jordan model $J = \bigoplus_{i=1}^{l} \{\bigoplus_{j=1}^{k_i} J_{n_j^i}(\mu_i)\}$, where

$$
J_n(\mu) = \begin{bmatrix} \mu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}_{(n\times n)}.
$$

We have

$$K \sim T = \begin{bmatrix} K_1 & * \\ & J \end{bmatrix} \begin{array}{l} H_1 \\ H_1^{\perp} \end{array}.$$

Hence, T and K are simultaneously Li–Yorke chaotic or not. At present, it suffices to consider the condition of only one Jordan block $J = J_n(\mu)$.

If $|\mu| > 1$, then $r_1(J) \triangleq \inf_{\lambda \in \sigma(J)} |\lambda| = |\mu| > 1$. According to spectral mapping theorem and spectral radius formula,

$$r_1(J)^{-1} = r(J^{-1}) = \lim_{k \to \infty} \|J^{-k}\|^{\frac{1}{k}}.$$

One can choose $\varepsilon > 0$ such that $r_1(J)^{-1} + \varepsilon < 1$. Then there exists $M \in \mathbb{N}$ such that for $k \geq M$,

$$\frac{1}{\|J^{-k}\|} \geq \left(\frac{1}{r_1(J)^{-1} + \varepsilon} \right)^k > 1.$$

Thus, for each nontrivial point x,

$$\|J^k x\| > \|J^{-k}\| \cdot \|J^k x\| \geq \|x\| \quad \text{for } k \geq M.$$

Consequently, J can not be Li–Yorke chaotic.

Now let $|\mu| = 1$. Since the dimension of $H_1^{\perp}$ is finite, then for each $y \in H_1^{\perp}$,

$$y = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $H_1^{\perp}$.

For each $z \in H$, there is a unique decomposition $z = x + y$ where $x \in H_1$ and $y \in H_1^{\perp}$. Claim that $y = 0$ if $0 \in \omega(z)$. Suppose $y \neq 0$. There exists a positive integer i such that $y_i \neq 0$ and $y_{i+1} = y_{i+2} = \cdots = y_n = 0$. Then

$$\begin{aligned}
\|T^m(z)\|^2 \geq \|J^m(y)\|^2 &= \left\| \begin{bmatrix} C_m^0 \mu^m & C_m^1 \mu^{m-1} & \cdots & C_m^{n-1} \mu^{m-n+1} \\ & C_m^0 \mu^m & \cdots & C_m^{n-2} \mu^{m-n+2} \\ & & \ddots & \vdots \\ & & & C_m^0 \mu^m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\|^2 \\
&= |C_m^0 \mu^m y_1 + C_m^1 \mu^{m-1} y_2 + \cdots + C_m^{n-1} \mu^{m-n+1} y_n|^2 \\
&\quad + |C_m^0 \mu^m y_2 + C_m^1 \mu^{m-1} y_3 + \cdots + C_m^{n-2} \mu^{m-n+2} y_n|^2 + \cdots \\
&\quad + |C_m^0 \mu^m y_n|^2 \\
&\geq |C_m^0 \mu^m y_i + C_m^1 \mu^{m-1} y_{i+1} + \cdots + C_m^{n-i} \mu^{m-n+i} y_n|^2 \\
&= |y_i|^2.
\end{aligned}$$

It is a contradiction to $0 \in \omega(z)$.

Consequently, if $0 \in \omega(z)$, we have $\lim_{m \to \infty} \|T^m(z)\| = \lim_{m \to \infty} \|K_1^m(x)\| = 0$. Therefore, T is impossible to be Li–Yorke chaotic, and so is K. $\square$

In the research of hypercyclicity, Herrero and Wang [6] gave a surprising result.

Proposition 8 ([6]). *For any* $\varepsilon > 0$, *there is a small compact operator* $\|K_\varepsilon\| < \varepsilon$ *such that* $I + K_\varepsilon$ *is hypercyclic.*

Correspondingly, we obtain a similar result for distributional chaos. Although $I + K$ cannot be norm-unimodal, it may hold weakly distributionally chaotic criterion.

Theorem 9. *For any* $\varepsilon > 0$, *there is a small compact operator* $\|K_\varepsilon\| < \varepsilon$ *such that* $I + K_\varepsilon$ *is distributionally chaotic.*

Proof. Without losses, assume $\mathcal{H}$ is a separable complex Hilbert space. Given any $\varepsilon > 0$. Let C_i be a sequence of positive numbers increasing to $+\infty$. For each $i \in \mathbb{N}$, set $\varepsilon_i = 4^{-i}\varepsilon$. Then we can select L_i to satisfy $(1+\varepsilon_i)^{L_i} \geq \sqrt{2}C_i$. Moreover, choose m_i such that $\frac{L_i}{m_i} < \frac{1}{i}$.

Write $n_i = 2m_i$. We can obtain a orthogonal decomposition of Hilbert space $\mathcal{H} = \bigoplus_{i=1}^{\infty} H_i$, where H_i is an n_i-dimensional subspace. Define operators on each H_i as follows,

$$S_i = \begin{bmatrix} 0 & 2\varepsilon_i & & \\ & \ddots & \ddots & \\ & & \ddots & 2\varepsilon_i \\ & & & 0 \end{bmatrix}_{(n_i \times n_i)}, \qquad K_i = \begin{bmatrix} -\varepsilon_i & 2\varepsilon_i & & \\ & \ddots & \ddots & \\ & & \ddots & 2\varepsilon_i \\ & & & -\varepsilon_i \end{bmatrix}_{(n_i \times n_i)}.$$

Then

$$I_i + K_i = \begin{bmatrix} 1-\varepsilon_i & 2\varepsilon_i & & \\ & \ddots & \ddots & \\ & & \ddots & 2\varepsilon_i \\ & & & 1-\varepsilon_i \end{bmatrix}_{(n_i \times n_i)} = (1-\varepsilon_i)I_i + S_i.$$

Let $x_i = (1, 1, \ldots, 1) \in H_i$. We have for $1 \leq n \leq m_i$,

$$\begin{aligned}
&\|(I_i + K_i)^n(x_i)\| \\
&\quad = \left\|\big((1-\varepsilon_i)I_i + S_i\big)^n(x_i)\right\| \\
&\quad = \left\|\left(\sum_{k=0}^{n} C_n^k (1-\varepsilon_i)^k S_i^{\,n-k}\right)x_i\right\| \\
&\quad \geq \Bigg\|\Bigg(\underbrace{\sum_{k=0}^{n} C_n^k (1-\varepsilon_i)^k (2\varepsilon_i)^{n-k}, \ldots, \sum_{k=0}^{n} C_n^k (1-\varepsilon_i)^k (2\varepsilon_i)^{n-k}}_{m_i}, 0, \ldots, 0\Bigg)\Bigg\| \\
&\quad = \sqrt{m_i}(1+\varepsilon_i)^n \\
&\quad = \frac{(1+\varepsilon_i)^n}{\sqrt{2}}\|x_i\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{\#\{0\le k\le m_i-1; \|(I_i+K_i)^k x_i\|\ge C_i\|x_i\|\}}{m_i}\\ &\quad\ge \frac{\#\{L_i, L_i+1,\dots,m_i-1\}}{m_i}\\ &\quad= 1-\frac{L_i}{m_i}.\end{aligned}$$

Since K_i is of finite rank and $\|K_i\|\le 4^{1-i}\varepsilon$, $K_\varepsilon=\bigoplus_{i=1}^\infty K_i$ is a compact operator on $\mathcal{H}$ and $\|K_\varepsilon\|<\varepsilon$. Notice $I+K_\varepsilon=\bigoplus_{i=1}^\infty(I_i+K_i)$. In addition, the previous x_i could be seemed as a point in $\mathcal{H}$, thus

(WNU1) $\lim_{k\to\infty}\|(I+K_\varepsilon)^k x_i\|=0$ since $r(I+K_\varepsilon)<1$.

(WNU2) The sequence of positive integers m_i increasing to $+\infty$ satisfies

$$\begin{aligned}&\lim_{i\to\infty}\frac{\#\{0\le k\le m_i-1; \|(I+K_\varepsilon)^k x_i\|\ge C_i\|x_i\|\}}{m_i}\\ &\quad=\lim_{i\to\infty}\frac{\#\{0\le k\le m_i-1; \|(I_i+K_i)^k x_i\|\ge C_i\|x_i\|\}}{m_i}\\ &\quad=\lim_{i\to\infty}1-\frac{L_i}{m_i}=1.\end{aligned}$$

Therefore, $I+K_\varepsilon$ is distributionally chaotic by Theorem 5. □

Remark 1. From the construction above, we can see that distributional chaos is not preserved under compact perturbations for bounded linear operators. The previous operator $I+K_\varepsilon$ is an example. In fact, $(I+K_\varepsilon)-\widetilde{K_i}$ is not distributionally chaotic, where $\widetilde{K_i}=(\bigoplus_{j=1}^i 0_{H_j})\oplus(\bigoplus_{j=i+1}^\infty K_j)$ is a compact operator with norm less than $4^{-i}\varepsilon$.

References

[1] F. Balibrea, J. Smítal and M. Štefánková, *The three versions of distributional chaos*, Chaos, Solitons & Fractals **23** (2005), 1581–1583. MR 2101573

[2] J. Duan, X. C. Fu, P. D. Liu and A. Manning, *A linear chaotic quantum harmonic oscillator*, Appl. Math. Lett. **12** (1999), 15–19. MR 1663405

[3] X. C. Fu and J. Duan, *Infinite-dimensional linear dynamical systems with chaoticity*, J. Nonlinear Sci. **9** (1999), 197–211. MR 1674380

[4] F. Martínez-Giménez, P. Oprocha and A. Peris, *Distributional chaos for backward shifts*, J. Math. Anal. Appl. **351** (2009), 607–615. MR 2473967

[5] K. G. Grosse-Erdmann, *Recent developments in hypercyclicity*, Rev. R. Acad. Cien. Serie A Mat., RACSAM **97** (2003), 273–286. MR 2068180

[6] D. Herrero and Z. Wang, *Compact perturbations of hypercyclic and supercyclic operators*, Indiana Univ. Math. J. **39** (1990), 819–829. MR 1078739

[7] B. Hou, P. Cui and Y. Cao, *Chaos for Cowen-Douglas operators*, Proc. Amer. Math. Soc. **138** (2010), 929–976.

[8] T. Y. Li and J. A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992. MR 0385028

[9] G. Liao, L. Wang and X. Duan, *A chaotic function with a distributively scrambled set of full Lebesgue measure*, Nonlinear Anal. **66** (2007), 2274–2280. MR 2311031

[10] G. Liao, Z. Chu and Q. Fan, *Relations between mixing and distributional chaos*, Chaos, Solitons & Fractals **41** (2009), 1994–2000. MR 2537785

[11] P. Oprocha, *A quantum harmonic oscillator and strong chaos*, J. Phys. A **39** (2006), 14559–14565. MR 2277068

[12] S. Rolewicz, *On orbits of elements*, Studia Math. **32** (1969), 17–22. MR 0241956

[13] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. **344** (1994), 737–754. MR 1227094

[14] J. H. Shapiro, *Notes on dynamics of linear operators*, unpublished Lecture Notes, available at `http://www.math.msu.edu/~shapiro`.

[15] J. Smítal and M. Štefánková, *Distributional chaos for triangular maps*, Chaos, Solitons & Fractals **21** (2004), 1125–1128. MR 2047330

[16] H. Wang, G. Liao and Q. Fan, *Substitution systems and the three versions of distributional chaos*, Topology Appl. **156** (2008), 262–267. MR 2475113

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