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THE EXTENSIONS OF C^*-ALGEBRAS WITH TRACIAL TOPOLOGICAL RANK NO MORE THAN ONE

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Abstract. Let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of C^*-algebras with A unital. Suppose that I has tracial topological rank no more than one and A/I is TAI (in particular, if A/I is simple and has tracial topological rank no more than one). It will be proved that A has tracial topological rank no more than one if the extension is quasidiagonal, and A has the property (P_1) if the extension is tracially quasidiagonal.

1. Introduction

Recently, much progress has been made in the classification of C^*-algebras (see [1], [3], [4], [5], [6], [7], [15], [20], [24], [26], [29]). The notion of tracial topological rank (denoted by TR) was first introduced by Lin (see [13], [19]). The purpose to introduce this notion was motivated by Elliott's program of classification of nuclear C^*-algebras. There are two previously known noncommutative topological ranks which are widely used in the theory of C^*-algebras, namely the real rank and the stable rank. Tracial topological rank is another analogy of the topological rank.

Simple C^*-algebras with tracial topological rank zero, also called TAF (Tracially Approximate Finite) C^*-algebras, have real rank zero, stable rank one, weakly unperforated ordered K_0-groups with the Rieze interpolation property and are quasidiagonal (see [2], [11], [17], [18], [22], [28], [31]). The classification theorem for unital nuclear separable simple C^*-algebras with tracial topological rank zero which satisfy the UCT was given in the paper [21] few years ago. (The C^*-algebras classified by Lin in [21] turn out to be all in the class of Elliott–Gong in [8].) The simple C^*-algebras with tracial topological rank

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no more than one have also stable rank one, weakly unperforated ordered K_0-groups with the Rieze interpolation property and are also quasidiagonal. Recently, the classification theorem for unital nuclear separable simple C^*-algebras with tracial topological rank no more than one which satisfy the UCT has also been obtained by Lin in [23]. (The C^*-algebras classified by Lin in [23] are also all in the class of Elloitt–Gong–Li in [9] and [12].) So it is very interesting to find which C^*-algebras are tracial topological rank zero or tracial topological rank no more than one, and along this line, much effort has been made until now (see [10], [25], [30]).

Let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of separable C^*-algebras with A unital. In the paper [14], it is proved that A has tracial topological rank zero if both I and A/I have tracial topological rank zero, and if the extension is tracially quasidiagonal (see Definition 4.1). Inspired by this result, we are interested in considering the C^*-algebras with tracial topological rank no more than one in a short exact sequence. Let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of C^*-algebras with A unital. Suppose that I has tracial topological rank no more than one and A/I is TAI (in particular if A/I is simple and has tracial topological rank no more than one). In this paper, it will be proved that A has tracial topological rank no more than one if the extension is quasidiagonal, and A has the property (P_1) (see Definition 4.2) if the extension is tracially quasidiagonal.

This paper is organized as follows. In Section 2, we list some preliminaries which will be used in the sections later. In Section 3, we consider the quasidiagonal extension $0 \to I \to A \to A/I \to 0$ with A unital. After several lemmas are introduced, it will be proved that A has tracial topological rank no more than one if I has tracial topological rank no more than one and A/I is TAI (in particular if A/I is simple and has tracial topological rank no more than one). In Section 4, we consider the tracially quasidiagonal extension $0 \to I \to A \to A/I \to 0$ with A unital. First, we introduce several definitions and lemmas. Then it will be shown that A has the property (P_1) if I has tracial topological rank no more than one and A/I is TAI (in particular, if A/I is simple and has tracial topological rank no more than one).

2. Preliminaries

In this paper, we assume that A is a unital C^*-algebra and I is a σ-unital closed ideal of A and π: $A \to A/I$ is the quotient map. Therefore, we have the following short exact sequence:

$$(\star) \qquad 0 \longrightarrow I \longrightarrow A \xrightarrow{\ \pi\ } A/I \longrightarrow 0.$$

We denote the extension $(\star)$ by the pair (A, I). In addition, we will use the following conventions:

(1*) For $a, b \in A$, we write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$.

(2^*) For a C^*-subalgebra C of A, we write $a \in_\varepsilon C$ if there is $b \in C$ such that $\|a-b\| < \varepsilon$.

(3^*) Denote by $\mathcal{I}^{(0)}$ the class of all finite dimensional C^*-algebras, and denote by $\mathcal{I}^{(k)}$ the class of all the C^*-algebras which are unital hereditary C^*-subalgebras of C^*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and F is a finite dimensional C^*-algebra.

(4^*) Let $0 < \sigma_2 < \sigma_1 < 1$. Define $f_{\sigma_2}^{\sigma_1}$ by

$$f_{\sigma_2}^{\sigma_1}(t) = \begin{cases} 1, & t \geq \sigma_1, \\ \dfrac{t-\sigma_2}{\sigma_1-\sigma_2}, & \sigma_2 < t < \sigma_1, \\ 0, & 0 \leq t \leq \sigma_2. \end{cases}$$

(5^*) Suppose that A is a C^*-algebra and $a \in A$. We denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra of A generated by a.

(6^*) Let a, b be two positive elements of a C^*-algebra A. We write $[a] \leq [b]$ if there is $x \in A$ such that $x^*x = a, xx^* \in \mathrm{Her}(b)$. We write $n[a] \leq [b]$ if there are $x_1, \ldots, x_n \in A$ such that ${x_i}^* x_i = a, x_i {x_i}^* \in \mathrm{Her}(b)$ and $x_i {x_i}^*$ $(1 \leq i \leq n)$ are mutually orthogonal.

(7^*) $\{pt\}$ is the set of a single point.

Definition 2.1. Let A be a unital C^*-algebra. A is said to have tracial topological rank no more than k (denoted by $\mathrm{TR}(A) \leq k$) if for any $\varepsilon > 0$, any finite subset F of A containing a nonzero positive element a, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, and any integer $n > 0$, there exist a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{I}^{(k)}$ of A with $1_C = p$ such that

$$\|xp - px\| < \varepsilon \quad \text{for all } x \in F; \tag{1}$$

$$pxp \in_\varepsilon C \quad \text{for all } x \in F; \tag{2}$$

$$n\left[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\right] \leq [f_{\sigma_4}^{\sigma_3}(pap)]. \tag{3}$$

If $\mathrm{TR}(A) \leq k$ but $\mathrm{TR}(A) \nleq k-1$, we write $\mathrm{TR}(A) = k$. If A has no unit, we define $\mathrm{TR}(A) = \mathrm{TR}(A^+)$, where A^+ is the unitization of A.

Definition 2.2. Let I and A be as in ($\star$). An approximate unit $\{u_n\}_{n=1}^\infty$ of I is called quasicentral if $\lim_{n\to\infty} \|u_n x - x u_n\| = 0$ for any $x \in A$, and the extension (A, I) is said to be quasidiagonal if there is a quasicentral approximate unit $\{r_n\}_{n=1}^\infty$ of I consisting of projections.

Definition 2.3. A unital C^*-algebra A is TAI if for any $\varepsilon > 0$, any finite subset F of A containing a nonzero positive element $a \in A$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, and any integer $n > 0$, there exist a projection $p \in A$ and a C^*-subalgebra $C \cong \bigoplus_{k=1}^{l} M_{m_k} C(Y_k)$ of A with $1_C = p$, where $Y_k = \{pt\}$ or $[0,1]$, such that

$$\|px - xp\| < \varepsilon \quad \text{for all } x \in F; \tag{1}$$

$$pxp \in_\varepsilon C \quad \text{for all } x \in F; \tag{2}$$

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(pap)]. \tag{3}$$

Lemma 2.4 ([14], Lemma 2.5). *Let a, b be two positive elements in a C^*-algebra A and p be a projection in A.*

(1) *If $a \le \lambda b$ for some $\lambda > 0$, then $[a] \le [b]$;*

(2) *If there is $x \in A$ such that $a = x^*x$ and $b = xx^*$, then $[a] = [b]$ and $[f_{\sigma_2}^{\sigma_1}(a)] = [f_{\sigma_2}^{\sigma_1}(b)]$ for any $0 < \sigma_2 < \sigma_1 < \|a\|$;*

(3) *$[a] = [a^2]$;*

(4) *If $\|a - b\| < \delta_2$, then $[f_{\delta_2}^{\delta_1}(a)] \le [b]$ for any $0 < \delta_2 < \delta_1 < 1$;*

(5) *Suppose that $\|a\| \le 1, \|b\| \le 1$. Then for any $0 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < 1$, there is $\delta = \delta(\delta_3, \delta_4) > 0$ such that $\|a - b\| < \delta$ implies that $[f_{\delta_2}^{\delta_1}(a)] \le [f_{\delta_4}^{\delta_3}(b)]$;*

(6) *If $0 \le a \le b$, then $[f_{\delta_2}^{\delta_1}(a)] \le [f_{\delta_4}^{\delta_3}(b)]$ for any $0 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < 1$.*

Lemma 2.5 ([19], Theorem 5.3 and Theorem 5.8). *Let A be a unital C^*-algebra with $\mathrm{TR}(A) \le k$. Then $\mathrm{TR}(M_n(A)) \le k$ for $n \ge 1$ and $\mathrm{TR}(C) \le k$ for any unital hereditary C^*-subalgebra C of A.*

Lemma 2.6 ([14], Corollary 2.7). *Let $a \in A$ with $0 \le a \le 1$ and p be a nonzero projection of A. Then for any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there is $\delta = \delta(\sigma_3, \sigma_4) > 0$ such that $\|ap - pa\| < \delta$ implies*

$$[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_4}^{\sigma_3}(pap)] + \big[f_{\sigma_4}^{\sigma_3}\big((1-p)a(1-p)\big)\big]; \tag{1}$$

$$[f_{\sigma_2}^{\sigma_1}(pap)] + \big[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(a)]. \tag{2}$$

Lemma 2.7 ([14], Lemma 2.12). *Let $0 < \sigma_8 < \sigma_7 < \cdots < \sigma_2 < \sigma_1 < 1$ and n be a positive integer. There is a $\delta = \delta(n, \sigma_1, \ldots, \sigma_8) > 0$ satisfying the following: Suppose that A is a C^*-algebra and $a, b, x_i \in A$ $(i = 1, \ldots, n)$ with $0 \le a \le 1$ such that $x_i{}^*x_i = f_{\sigma_4}^{\sigma_3}(a)$, $x_i x_i{}^* \in \mathrm{Her}(f_{\sigma_6}^{\sigma_5}(b))$, and $x_i x_i{}^*$ $(1 \le i \le n)$ are mutually orthogonal. If there is a projection $p \in A$ such that $\|py - yp\| < \delta$ for $y \in \{a, b, x_i, x_i{}^* | i = 1, \ldots, n\}$, then*

$$n[f_{\sigma_2}^{\sigma_1}(pap)] \le [f_{\sigma_8}^{\sigma_7}(pbp)].$$

3. Quasidiagonal extensions of C^*-algebras

The following two lemmas are taken from [14].

Lemma 3.1 ([14], Corollary 3.3). *Let I and A be as in $(\star)$ and $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$. Suppose the extension (A, I) is quasidiagonal. If $a \in A$ with $0 \le a \le 1$ and*

$$n\big[f_{\delta_2}^{\delta_1}\big(\big(1-\pi(p)\big)\pi(a)\big(1-\pi(p)\big)\big)\big] \le [f_{\delta_4}^{\delta_3}(\pi(p)\pi(a)\pi(p))]$$

for some projection $p \in A$ and any integer $n > 0$, then there is a projection $r \in (1-p)I(1-p)$ such that

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-p-r)a(1-p-r)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(pap)].$$

Moreover, for any finite subset $F \subset A$ *and* $\varepsilon > 0$*, if* $\|px - xp\| < \varepsilon$ *for all* $x \in F \cup \{a\}$*, we can require that* $\|rx - xr\| < 3\varepsilon$ *for all* $x \in F \cup \{a\}$.

The following lemma is from Lemma 3.1 in [14] with a little change, and their proofs are the same.

LEMMA 3.2. *Let* I *and* A *be as in* $(\star)$*. If the extension* (A, I) *is quasidiagonal, then for any finite dimensional* C^**-subalgebra* $\bar{C}$ *of* A/I*, there is a finite dimensional* C^**-subalgebra* C *of* A *such that* $C \cong \bar{C}$ *and* $\pi(C) = \bar{C}$*. Moreover, there exists a quasicentral approximate unit* $\{r_n\}_{n=1}^{\infty}$ *of* I *consisting of projections such that* $r_n x = x r_n, \forall x \in C, n \geq 1$.

LEMMA 3.3. *Let* I *and* A *be as in* $(\star)$*. Suppose the extension* (A, I) *is quasidiagonal. If* D *is a* C^**-subalgebra of* A/I *which is isomorphic to* $C[0,1] \otimes M_n$*, then there is a* C^**-subalgebra* C *of* A *such that* $\pi|_C$ *is the isomorphism from* C *onto* D.

Proof. Let φ be the isomorphism from $C[0,1] \otimes M_n$ onto D and $D_1 = \varphi(1_{C[0,1]} \otimes M_n)$, then D_1 is isomorphic to M_n. By Lemma 3.2, there is a C^*-subalgebra C_1 of A such that $\pi|_{C_1}$ is the isomorphism from C_1 onto D_1. Let $C_0 = \{x \in 1_{C_1} A 1_{C_1} | \pi(x) \in D\}$, then $1_{C_0} = 1_{C_1}, C_1 \subseteq C_0$ and $\pi(C_0) = D$. Let $\{e_{ij}\}_{1 \leq i,j \leq n}, \{c_{ij}\}_{1 \leq i,j \leq n}, \{d_{ij}\}_{1 \leq i,j \leq n}$ be the matrix units of M_n, C_1, D_1 respectively such that $\pi(c_{ij}) = d_{ij} = \varphi(1 \otimes e_{ij})$. Let $C_{0,e} = c_{11} C_0 c_{11}$ and $D_e = d_{11} D d_{11}$, then $\varphi(C[0,1] \otimes e_{11}) = D_e$. Since $\pi(C_0) = D$, $\pi|_{C_{0,e}}$ is surjective from $C_{0,e}$ onto D_e.

By Theorem 6.1.2 and Remark 6.1.3 in [27], there are canonical isomorphisms $\alpha_e : C_{0,e} \otimes M_n \to C_0$ and $\alpha_d : D_e \otimes M_n \to D$ such that the following diagram

$$\begin{tikzcd}
C_0 \arrow[r, "\pi"] & D & C[0,1] \otimes M_n \arrow[l, "\varphi"'] \\
C_{0,e} \otimes M_n \arrow[u, "\alpha_e"] \arrow[r, "\pi|_{C_{0,e}} \otimes id"] & D_e \otimes M_n \arrow[u, "\alpha_d"] & (C[0,1] \otimes e_{11}) \otimes M_n \arrow[u, "s \otimes id"'] \arrow[l, "\varphi|_{C[0,1] \otimes e_{11}} \otimes id"']
\end{tikzcd}$$

commutes, where s is the canonical isomorphism from $C[0,1] \otimes e_{11}$ onto $C[0,1]$ with $s(f \otimes e_{11}) = f$ for any $f \in C[0,1]$, $\alpha_e(c \otimes e_{ij}) = c_{i1} c c_{1j}$ for any $c \in C_{0,e}$, and $\alpha_d(d \otimes e_{ij}) = d_{i1} d d_{1j}$ for any $d \in D_e$, $1 \leq i,j \leq n$.

Let $\psi = s \circ (\varphi|_{C[0,1] \otimes e_{11}})^{-1} \circ \pi|_{C_{0,e}}$, then ψ is the unital surjective homomorphism from $C_{0,e}$ onto $C[0,1]$. Let g be the identity function on $[0,1]$ and h be any element in $C_{0,e}$ such that $0 \leq h \leq c_{11}$ and $\psi(h) = g$, then $c_{11} \notin C^*(h)$, where $C^*(h)$ is the C^*-algebra generated by h. Let $C_e = C^*(h, c_{11}) \subseteq C_{0,e}$, then $1_{C_e} = c_{11} = 1_{C_{0,e}}$. So ψ is the isomorphism from C_e onto $C[0,1]$ and $\psi \otimes id$ is the isomorphism from $C_e \otimes M_n$ onto $C[0,1] \otimes M_n$. From the commutative diagram above, we have $\pi(h) = \pi \circ \alpha_e(h \otimes c_{11}) = \varphi(g \otimes e_{11})$. Let

$C = \alpha_e(C_e \otimes M_n) \subseteq C_0$, we have $1_C = 1_{C_0} = 1_{C_1}$ and $\pi|_C$ is the isomorphism from C onto D. □

From the proof of Lemma 3.3, we have the following corollary.

Corollary 3.4. *Let I and A be as in $(\star)$. Suppose the extension (A, I) is quasidiagonal. If φ is an isomorphism from $C[0,1] \otimes M_n$ onto a C^*-subalgebra D of A/I and C_1 is a C^*-subalgebra of A satisfying that $\pi|_{C_1}$ is an isomorphism from C_1 onto $\varphi(1_{C[0,1]} \otimes M_n)$, then we can find a C^*-subalgebra C of A such that $1_C = 1_{C_1}$ and $\pi|_C$ is the isomorphism from C onto D.*

Theorem 3.5. *Let I and A be as in $(\star)$. Suppose that (A, I) is quasidiagonal, then for any C^*-subalgebra D of A/I and $D \cong \bigoplus_{k=1}^{l} M_{n_k}(C(Y_k))$, where Y_k is $\{pt\}$ or $[0,1]$, there is a C^*-subalgebra C of A such that $\pi|_C$ is an isomorphism from C onto D. Moreover, we can find a quasicentral approximate unit $\{r_n\}_{n=1}^{\infty}$ of I consisting of projections such that $r_n x = x r_n$ for any $x \in C$ and any $n \geq 1$.*

Proof. Let φ be the isomorphism from $\bigoplus_{k=1}^{l} C(Y_k) \otimes M_{n_k}$ onto D. Set $D_1 = \varphi(\bigoplus_{k=1}^{l} 1_{C(Y_k)} \otimes M_{n_k})$, then D_1 is a finite dimensional C^*-subalgebra of D. By Lemma 3.2, there exist a finite dimensional C^*-subalgebra C_1 of A such that $\pi|_{C_1}$ is the isomorphism from C_1 onto D_1 and a quasicentral approximate unit $\{r_n\}_{n=1}^{\infty}$ of I consisting of projections such that

$$r_n x = x r_n \quad \text{for any } x \in C_1 \text{ and any } n \geq 1.$$

Without loss of generality, we may assume there exists an integer $l_0 > 0$ such that $Y_k = \{pt\}$ if $l_0 < k \leq l$ and $Y_k = [0,1]$ if $1 \leq k \leq l_0$. Then $(\pi|_{C_1})^{-1} \circ \varphi$ is the isomorphism from $\bigoplus_{k=1}^{l} 1_{C(Y_k)} \otimes M_{n_k}$ onto C_1. Let e_k be the unit of M_{n_k}, $q_k = \varphi(1_{C(Y_k)} \otimes e_k)$ and $p_k = (\pi|_{C_1})^{-1}(q_k)$, $1 \leq k \leq l$. Setting $C_{1k} = p_k C_1 p_k$, we have $C_1 = (\bigoplus_{k=1}^{l} p_k) C_1 (\bigoplus_{k=1}^{l} p_k) = \bigoplus_{k=1}^{l} C_{1k}$. Let φ_k be the isomorphism from $C(Y_k) \otimes M_{n_k}$ onto $q_k D q_k$ which is defined by φ, then $\pi|_{C_{1k}}$ is the isomorphism from C_{1k} onto $\varphi_k(1_{C(Y_k)} \otimes M_{n_k})$.

In the case $Y_k = \{pt\}$ $(l_0 < k \leq l)$, then we have $C(Y_k) \otimes M_{n_k} = 1_{C(Y_k)} \otimes M_{n_k}$. Therefore, $(\pi|_{C_1})^{-1} \circ \varphi$ is the isomorphism from

$$\bigoplus_{k=l_0+1}^{l} C(Y_k) \otimes M_{n_k} \quad \text{onto} \left(\bigoplus_{k=l_0+1}^{l} p_k \right) C_1 \left(\bigoplus_{k=l_0+1}^{l} p_k \right) = \bigoplus_{k=l_0+1}^{l} C_{1k}.$$

In the case $Y_k = [0,1]$ $(1 \leq k \leq l_0)$, let $\{c_{k,ij}\}_{1 \leq i,j \leq n_k}$ and $\{e_{k,ij}\}_{1 \leq i,j \leq n_k}$ be the matrix units in C_{1k} and M_{n_k}, respectively. From the proof of Lemma 3.3, we have:

(1) Let $C_{0k} = \{x \in 1_{C_{1k}} A 1_{C_{1k}} = p_k A p_k | \pi(x) \in q_k D q_k\}$, then $C_{0k} \supseteq C_{1k}$ with $1_{C_{0k}} = 1_{C_{1k}} = p_k$ and $\pi(C_{0k}) = q_k D q_k$;

(2) Let $C_{0k,e} = c_{k,11} C_{0k} c_{k,11}$, there is an isomorphism $\alpha_{k,e}$ from $C_{0k,e} \otimes M_{n_k}$ onto C_{0k} with $\alpha_{k,e}(c \otimes e_{ij}) = c_{k,i1} c c_{k,1j}$ for any $c \in C_{0k,e}$ $1 \leq i, j \leq n_k$;

(3) For any $h_k \in C_{0k,e}$ with $0 \leq h_k \leq c_{k,11}$ satisfying $\pi(h_k) = \varphi_k(g \otimes e_{k,11})$, where g is the identity function on $[0,1]$. Let $\widetilde{C}_{k,e} = C^*(h_k, c_{k,11}) \subseteq C_{0k,e}$ and $\widetilde{C}_k = \alpha_{k,e}(\widetilde{C}_{k,e} \otimes M_{n_k})$, then $\pi|_{\widetilde{C}_k}$ is the isomorphism from $\widetilde{C}_k$ onto $q_k D q_k = \varphi_k(C(Y_k) \otimes M_{n_k})$. Moreover, there exists $h_k \in C_{0k,e}$ with the properties above.

Now let us consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & M(I) & \xrightarrow{\pi_1} & Q(I) & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\pi} & A/I & \longrightarrow & 0, \end{array}$$

where ρ is the C^*-homomorphism with $\rho|_I = id_I$ defined by the extension (A, I), and τ is the Busby invariant of the extension (A, I). Since the approximate unit $\{r_n\}_{n=1}^{\infty}$ of I is quasicentral, then

$$\lim_{n\to\infty} \|r_n h_k - h_k r_n\| = 0, \quad 1 \leq k \leq l_0.$$

So $\lim_{n\to\infty} \|r_n \rho(h_k) - \rho(h_k) r_n\| = 0$ since $\rho|_I = id_I$. For any $\varepsilon > 0$, we can find a subsequence $\{r_{n_i}\}_{i=1}^{\infty}$ of $\{r_n\}_{n=1}^{\infty}$ such that for all h_k $(1 \leq k \leq l_0)$,

$$\|(r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}}) - (r_{n_i} - r_{n_{i-1}})\rho(h_k)\| < \varepsilon/2^i,$$

where $r_{n_0} = 0$.

It is easy to see that $\sum_{i=1}^{n} [(r_{n_i} - r_{n_{i-1}})\rho(h_k) - (r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}})]$ is convergent in the norm topology as $n \to \infty$. Let $a_k = \sum_{i=1}^{\infty} [(r_{n_i} - r_{n_{i-1}})\rho(h_k) - (r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}})]$, then $a_k \in I$.

Since $\sum_{i=1}^{\infty} (r_{n_i} - r_{n_{i-1}})\rho(h_k)$ is convergent to $\rho(h_k)$ in the strict topology in $M(I)$, then $\sum_{i=1}^{\infty} (r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}})$ is convergent to $\rho(h_k) - a_k$ in the strict topology. Let $\rho(h_k)' = \sum_{i=1}^{\infty} (r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}})$, then $\rho(h_k) = \rho(h_k)' + a_k$. It is clear that $r_{n_i}\rho(h_k)' = \rho(h_k)' r_{n_i}$. Since $h_k \leq c_{k,11}$, then $\rho(h_k) \leq \rho(c_{k,11})$. So we have

$$(r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}}) \leq (r_{n_i} - r_{n_{i-1}})\rho(c_{k,11})(r_{n_i} - r_{n_{i-1}}).$$

Since $r_{n_i} c_{k,11} = c_{k,11} r_{n_i}$ and $\rho(r_n) = r_n$ $(\forall n)$, then $(r_{n_i} - r_{n_{i-1}})\rho(c_{k,11})(r_{n_i} - r_{n_{i-1}}) = \rho(c_{k,11})(r_{n_i} - r_{n_{i-1}})$. Therefore, we have

$$\begin{aligned} \rho(h_k)' &= \sum_{i=1}^{\infty} (r_{n_i} - r_{n_{i-1}})\rho(h_k)(r_{n_i} - r_{n_{i-1}}) \\ &\leq \sum_{i=1}^{\infty} (r_{n_i} - r_{n_{i-1}})\rho(c_{k,11})(r_{n_i} - r_{n_{i-1}}) \\ &= \sum_{i=1}^{\infty} \rho(c_{k,11})(r_{n_i} - r_{n_{i-1}}). \end{aligned}$$

Since $\sum_{i=1}^{\infty}\rho(c_{k,11})(r_{n_i}-r_{n_{i-1}})$ is convergent to $\rho(c_{k,11})$ in the strict topology, then we have

$$\rho(h_k)' \leq \rho(c_{k,11}).$$

It is known that the pullback $E(\tau)=\{x\oplus b\in M(I)\oplus A/I|\pi_1(x)=\tau(b)\}$ is isomorphic to A. We denote this isomorphism by γ, then $\gamma(a)=\rho(a)\oplus\pi(a)$ for any $a\in A$. Since $\pi_1(\rho(h_k)')=\pi_1(\rho(h_k))=\tau(\pi(h_k))$, then we have $\rho(h_k)'\oplus\pi(h_k)\in E(\tau)$. Let

$$h_k' = \gamma^{-1}\big(\rho(h_k)'\oplus\pi(h_k)\big).$$

Since

$$\begin{aligned}\gamma(r_{n_i}h_k') &= \gamma(r_{n_i})\gamma(h_k') = (r_{n_i}\oplus 0)\big(\rho(h_k)'\oplus\pi(h_k)\big)\\ &= r_{n_i}\rho(h_k)'\oplus 0 = \rho(h_k)'r_{n_i}\oplus 0 = \big(\rho(h_k)'\oplus\pi(h_k)\big)(r_{n_i}\oplus 0)\\ &= \gamma(h_k')\gamma(r_{n_i}) = \gamma(h_k'r_{n_i}),\end{aligned}$$

and

$$\gamma(h_k') = \rho(h_k)'\oplus\pi(h_k) \leq \rho(c_{k,11})\oplus\pi(c_{k,11}) = \gamma(c_{k,11}),$$

we have

$$r_{n_i}h_k' = h_k'r_{n_i}, \quad h_k' \leq c_{k,11} \leq p_k.$$

Since $\pi(h_k')=\pi(\gamma^{-1}(\rho(h_k)\oplus\pi(h_k)))=\pi(h_k)=\varphi_k(g\otimes e_{k,11})$, where g is the identity function on $[0,1]$, then $h_k'\in C_{0k}$. Therefore, we have $h_k'\in c_{k,11}C_{0k}c_{k,11}=C_{0k,e}$.

Let $C_{k,e}=C^*(h_k',c_{k,11})$ and $C(k)=\alpha_{k,e}(C_{k,e}\otimes M_{n_k})$. Since r_{n_i} commutes with h_k' and $c_{k,11}$, we have $r_{n_i}x=xr_{n_i}$ for any $x\in C_{k,e}$. Since $\alpha_{k,e}(c\otimes e_{ij})=c_{k,i1}cc_{k,1j}$ for any $c\in C_{k,e}$ and r_{n_i} commutes with $c_{k,ij}$ and $C_{k,e}$, we have $r_{n_i}c=cr_{n_i}$ for any $c\in C(k)$. By (3), which is from the proof of Lemma 3.3, we have $1_{C(k)}=p_k$ and $\pi|_{C(k)}$ is an isomorphism from $C(k)$ onto q_kDq_k. Therefore, we have that $\pi|_{\bigoplus_{k=1}^{l_0}C(k)}$ is the isomorphism from $\bigoplus_{k=1}^{l_0}C(k)$ onto $(\bigoplus_{k=1}^{l_0}q_k)D(\bigoplus_{k=1}^{l_0}q_k)=\bigoplus_{k=1}^{l_0}q_kDq_k$. Setting $C=(\bigoplus_{k=1}^{l_0}C(k))\oplus(\bigoplus_{k=l_0+1}^{l}C_{1k})$, we have that $\pi|_C$ is the isomorphism from C onto D and r_{n_i} commutes with every element of C. Since $\{r_{n_i}\}_{i=1}^{\infty}$ is also a quasicentral approximate unit of I, it completes the proof. □

THEOREM 3.6. *Suppose the extension (A,I) is quasidiagonal. Then for any $\varepsilon>0$, any finite subset F of A containing a nonzero positive element a, any $0<\sigma_4<\sigma_3<\sigma_2<\sigma_1<1$, and any integer $n>0$, if there exist $\delta_i, i=1,2,3,4$, satisfying $0<\sigma_4<\sigma_3<\delta_4<\delta_3<\delta_2<\delta_1<\sigma_2<\sigma_1$, a projection $\bar{q}\in A/I$ and $\bar{C}\cong\bigoplus_{k=1}^{l}M_{n_k}(C(Y_k))$ with $1_{\bar{C}}=\bar{q}$, where Y_k is $\{pt\}$ or $[0,1]$, such that*

$$\|\bar{q}\pi(x)-\pi(x)\bar{q}\| < \varepsilon \quad \textit{for all } x\in F; \tag{1}$$

$$\bar{q}\pi(x)\bar{q} \in_{\varepsilon} \bar{C} \quad \textit{for all } x\in F; \tag{2}$$

$$n\big[f_{\delta_2}^{\delta_1}\big((1-\bar{q})\pi(a)(1-\bar{q})\big)\big] \leq [f_{\delta_4}^{\delta_3}(\bar{q}\pi(a)\bar{q})], \tag{3}$$

then there exist a projection $q \in A$ and a C^*-subalgebra $C \in \mathcal{I}^{(1)}$ of A with $1_C = q$, $\pi(q) = \bar{q}$ and a projection $r \in (1-q)I(1-q)$ such that

$$(1') \qquad \|qx - xq\| < 8\varepsilon \quad \text{and} \quad \|rx - xr\| < 24\varepsilon \quad \text{for all } x \in F;$$

$$(2') \qquad qxq \in_{2\varepsilon} C \quad \text{and} \quad (q+r)x(q+r) \in_{18\varepsilon} C + rIr \quad \text{for all } x \in F;$$

$$(3') \qquad n\big[f_{\sigma_2}^{\sigma_1}\big((1-q-r)a(1-q-r)\big)\big] \leq [f_{\sigma_4}^{\sigma_3}(qaq)].$$

Proof. Since the extension (A, I) is quasidiagonal, by Theorem 3.5, there exist a projection $\tilde{q} \in A$, a C^*-subalgebra $\tilde{C}$ of A with $1_{\tilde{C}} = \tilde{q}$ such that $\pi(\tilde{q}) = \bar{q}$, $\pi|_{\tilde{C}}$ is an isomorphism from $\tilde{C}$ onto $\bar{C}$, and an approximate unit $\{r_n\}_{n=1}^{\infty}$ of I consisting of projections which commutes with every element of $\tilde{C}$.

By (1) and (2), for any $x \in F$, we can find $a_x, b_x \in I$ and $c_x \in \tilde{C}$ which depend on x such that

$$\|x\tilde{q} - \tilde{q}x - a_x\| < \varepsilon \quad \text{and} \quad \|\tilde{q}x\tilde{q} - b_x - c_x\| < \varepsilon.$$

Let $G = \{a_x, b_x | x \in F\}$. Since G is finite, we can choose some r_n such that $\|(1-r_n)y\| < \varepsilon$ and $\|y(1-r_n)\| < \varepsilon$ for all $y \in G$. Let $q = \tilde{q}(1-r_n)\tilde{q}$ and $C = q\tilde{C}q = (1-r_n)\tilde{C}(1-r_n)$, then we have $\pi(q) = \bar{q}$ and $\pi(C) = \bar{C}$. Since $r_n \in I$ and commutes with $\tilde{C}$, we may define a map ψ from $\tilde{C}$ to $C = (1-r_n)\tilde{C}(1-r_n) = (1-r_n)\tilde{C}$ by $\psi(a) = (1-r_n)a$ $(\forall a \in \tilde{C})$. It is routine to check that ψ is a surjective homomorphism. Suppose there is $a \in \tilde{C}$ such that $(1-r_n)a = 0$, then $a = r_n a \in I$ and $\pi|_{\tilde{C}}(a) = \pi(a) = 0$. Since $\pi|_{\tilde{C}}$ is an isomorphism from $\tilde{C}$ onto $\bar{C}$, we have $a = 0$, that is, ψ is injective. Therefore, ψ is an isomorphism from $\tilde{C}$ onto C. Since $\tilde{C} \in \mathcal{I}^{(1)}$, then we have $C \in \mathcal{I}^{(1)}$.

For any $x \in F$, since

$$\begin{aligned}
qx - xq &= \tilde{q}(1-r_n)\tilde{q}x - x\tilde{q}(1-r_n)\tilde{q} \\
&\approx_{2\varepsilon} \tilde{q}(1-r_n)\tilde{q}(x\tilde{q} - a_x) - (\tilde{q}x + a_x)\tilde{q}(1-r_n)\tilde{q} \\
&= \tilde{q}(1-r_n)\tilde{q}x\tilde{q} - \tilde{q}(1-r_n)\tilde{q}a_x - \tilde{q}x\tilde{q}(1-r_n)\tilde{q} - a_x(1-r_n)\tilde{q} \\
&\approx_{2\varepsilon} \tilde{q}(1-r_n)(b_x + c_x) - \tilde{q}(1-r_n)a_x - (b_x + c_x)(1-r_n)\tilde{q} \\
&\qquad - a_x(1-r_n)\tilde{q} \\
&= \tilde{q}(1-r_n)b_x - \tilde{q}(1-r_n)a_x - b_x(1-r_n)\tilde{q} - a_x(1-r_n)\tilde{q}
\end{aligned}$$

then

$$\begin{aligned}
\|qx - xq\| &< 4\varepsilon + \|\tilde{q}(1-r_n)b_x\| + \|\tilde{q}(1-r_n)a_x\| \\
&\qquad + \|b_x(1-r_n)\tilde{q}\| + \|a_x(1-r_n)\tilde{q}\| \\
&< 4\varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon = 8\varepsilon.
\end{aligned}$$

We have

$$(4) \qquad \|qx - xq\| < 8\varepsilon \quad \text{for all } x \in F.$$

Since

$$\begin{aligned} qxq &= \tilde{q}(1-r_n)\tilde{q}x\tilde{q}(1-r_n)\tilde{q} \\ &\approx_{\varepsilon} \tilde{q}(1-r_n)\tilde{q}(b_x+c_x)\tilde{q}(1-r_n)\tilde{q} \\ &= \tilde{q}(1-r_n)\tilde{q}b_x\tilde{q}(1-r_n)\tilde{q} + \tilde{q}(1-r_n)\tilde{q}c_x\tilde{q}(1-r_n)\tilde{q} \\ &= \tilde{q}(1-r_n)\tilde{q}b_x\tilde{q}(1-r_n)\tilde{q} + qc_xq \\ &= \tilde{q}(1-r_n)b_x(1-r_n)\tilde{q} + qc_xq \end{aligned}$$

then

$$\|qxq - qc_xq\| < \varepsilon + \|\tilde{q}(1-r_n)\tilde{q}b_x\tilde{q}(1-r_n)\tilde{q}\| < 2\varepsilon.$$

We have

$$qxq \in_{2\varepsilon} C. \tag{5}$$

By Lemma 3.1, (3) and (4), there is a projection $r \in (1-p)I(1-p)$ such that

$$\|rx - xr\| < 24\varepsilon \quad \text{for all } x \in F$$

and

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-q-r)a(1-q-r)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(qaq)]. \tag{6}$$

Then together with (4), we have

$$\|qx - xq\| < 8\varepsilon \quad \text{and} \quad \|rx - xr\| < 24\varepsilon \quad \text{for all } x \in F. \tag{1$'$}$$

Since

$$(q+r)x(q+r) = qxq + qxr + rxq + rxr \approx_{16\varepsilon} qxq + rxr,$$

together with (5), we have

$$qxq \in_{2\varepsilon} C \quad \text{and} \quad (q+r)x(q+r) \in_{18\varepsilon} C + rIr \quad \text{for all } x \in F. \tag{2$'$}$$

(3′) is from (6), and then we complete the proof. □

Theorem 3.7. *Suppose the extension (A, I) is quasidiagonal. If* $\mathrm{TR}(I) \le 1$ *and A/I is TAI, then* $\mathrm{TR}(A) \le 1$.

Proof. For any $\varepsilon > 0$, any finite subset F of A containing a nonzero positive element a, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 0$, and any integer $n > 0$, we choose $d_i, \delta_i, i = 1, 2, 3, 4$, satisfying

$$0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < d_4 < d_3 < d_2 < d_1 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1.$$

Since A/I is TAI, there exist a projection $\bar{q}$ in A/I and a C^*-subalgebra $\bar{C}_1$ of A/I which is isomorphic to $\bigoplus_{k=1}^{l} M_{n_k} C(Y_k)$ with $1_{\bar{C}_1} = \bar{q}$, where $Y_k = \{pt\}$ or $[0, 1]$, such that

$$\|\pi(x)\bar{q} - \bar{q}\pi(x)\| < \varepsilon/64 \quad \text{for all } x \in F; \tag{1}$$

$$\bar{q}\pi(x)\bar{q} \in_{\varepsilon/64} \bar{C}_1 \quad \text{for all } x \in F; \tag{2}$$

$$n\big[f_{d_2}^{d_1}\big((1-\bar{q})\pi(a)(1-\bar{q})\big)\big] \le [f_{d_4}^{d_3}(\bar{q}\pi(a)\bar{q})]. \tag{3}$$

By Theorem 3.6, there exist a projection $q \in A$ and a C^*-subalgebra $C_1 \in \mathcal{I}^{(1)}$ of A with $1_{C_1} = q$, $\pi(q) = \bar{q}$, and a projection $r \in (1-q)I(1-q)$ such that

$$\|qx - xq\| < \varepsilon/8, \qquad \|rx - xr\| < 3/8\varepsilon \quad \text{for all } x \in F; \tag{4}$$

$$qxq \in_{1/32\varepsilon} C_1, \qquad (q+r)x(q+r) \in_{9\varepsilon/32} C_1 + rIr \quad \text{for all } x \in F; \tag{5}$$

$$n\big[f_{\delta_2}^{\delta_1}\big((1-q-r)a(1-q-r)\big)\big] \leq [f_{\delta_4}^{\delta_3}(qaq)]. \tag{6}$$

Let $G = \{rxr | x \in F\}$. Since $\mathrm{TR}(I) \leq 1$, by Lemma 2.5 we have $\mathrm{TR}(rIr) \leq 1$. Then there exist a projection $p \in rIr$ and a C^*-subalgebra $C_2 \in \mathcal{I}^{(1)}$ of rIr with $1_{C_2} = p$ such that

$$\|prxr - rxrp\| < \varepsilon/8 \quad \text{for all } x \in F; \tag{7}$$

$$pxp \in_{\varepsilon/8} C_2 \quad \text{for all } x \in F; \tag{8}$$

$$n\big[f_{\delta_2}^{\delta_1}\big((r-p)a(r-p)\big)\big] \leq [f_{\delta_4}^{\delta_3}(pap)]. \tag{9}$$

Now let $s = q + p$ and $C = C_1 + C_2$. It is easy to see that $C \in \mathcal{I}^{(1)}$ and $1_C = s$. Since

$$\begin{aligned} sx - xs &= (q+p)x - x(q+p) \\ &= qx - xq + px - xp \\ &\approx_{\varepsilon/8} px - xp \\ &= prrx - xrrp \\ &\approx_{6\varepsilon/8} prxr - rxrp \\ &\approx_{\varepsilon/8} 0, \end{aligned}$$

we have

$$\|sx - xs\| < \varepsilon \quad \text{for all } x \in F. \tag{1'}$$

Since

$$\begin{aligned} sxs &= (q+p)x(q+p) \\ &= qxq + pxp + pxq + qxp \\ &\approx_{\varepsilon/4} qxq + pxp, \end{aligned}$$

by (8) and (5), we have

$$sxs \in_{3\varepsilon/8} C \quad \text{for all } x \in F. \tag{2'}$$

Finally, with ε small enough, by Lemma 2.6, (4), (6), (7) and (9), we have

$$\begin{aligned} &n\big[f_{\sigma_2}^{\sigma_1}\big((1-s)a(1-s)\big)\big] \\ &\quad \leq n\big[f_{\delta_2}^{\delta_1}\big((1-q-r)a(1-q-r)\big)\big] + n\big[f_{\delta_2}^{\delta_1}\big((r-p)a(r-p)\big)\big] \\ &\quad \leq [f_{\delta_4}^{\delta_3}(qaq)] + [f_{\delta_4}^{\delta_3}(pap)] \\ &\quad \leq [f_{\sigma_4}^{\sigma_3}(sas)]. \end{aligned}$$

Then

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-s)a(1-s)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(sas)]. \tag{3'}$$

We complete the proof from (1′), (2′), (3′) above. ☐

Corollary 3.8. *Suppose the extension (A, I) is quasidiagonal and A/I is a unital simple C^*-algebra. If* $\mathrm{TR}(I) \le 1$ *and* $\mathrm{TR}(A/I) \le 1$, *then* $\mathrm{TR}(A) \le 1$.

Proof. Suppose A/I is a unital simple C^*-algebra. By Theorem 7.1(b) in [19], $\mathrm{TR}(A/I) \le 1$ if and only if A/I is TAI. Then the conclusion follows from Theorem 3.7. ☐

4. Tracially quasidiagonal extensions of C^*-algebras

Definition 4.1. Let I and A be as in ($\star$). The extension (A, I) is said to be tracially quasidiagonal if for any $\varepsilon > 0$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $F \subset A$ containing a nonzero positive element a, and any integer $n > 0$, there exist a projection $p \in A$ and a C^*-subalgebra C of A with $1_C = p$ such that

$$\|px - xp\| < \varepsilon \quad \text{for all } x \in F; \tag{1}$$

$$pxp \in_\varepsilon C \quad \text{for all } x \in F; \tag{2}$$

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\big] \le [f_{\sigma_4}^{\sigma_3}(pap)]; \tag{3}$$

$$C \cap I = pIp \text{ and the extension } (C, pIp) \text{ is quasidiagonal.} \tag{4}$$

The following definition was first given by Hu–Lin–Xue, which can be found in [14].

Definition 4.2. Let A be a unital C^*-algebra. We say that A has the property (P_k) if the following holds: for any $\varepsilon > 0$, any integer $n > 0$, any finite subset $F \subset A$ containing a nonzero positive element a, and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exist a projection $p \in A$ and a C^*-subalgebra C of A with $1_C = p$ and $\mathrm{TR}(C) \le k$ such that

$$\|px - xp\| < \varepsilon \quad \text{for all } x \in F; \tag{1}$$

$$pxp \in_\varepsilon C \quad \text{for all } x \in F; \tag{2}$$

$$n\big[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\big] \le 2[f_{\sigma_4}^{\sigma_3}(pap)]. \tag{3}$$

Lemma 4.3. *Suppose that D is a compact subset of* $[0, 1]$. *For any $\varepsilon > 0$ and $h \in C(D)$ with $h(x) = x$ if $x \in D$, then there is $h' \in C(D)$ such that $\|h - h'\| < \varepsilon$ and the spectrum of h' is the finite disjoint union of sets of a single point or closed intervals* (*possibly the finite disjoint union is one set of a single point or one closed interval*).

Proof. Fix $\varepsilon > 0$. Let $a_0 = \inf D$ and $b = \sup D$, then $a_0, b \in D$. Without loss of generality, we assume that $a_0 + \varepsilon \leq b$.

Case (1). If a_0 is the left interior point of D, that is, there exists $t > 0$ such that $[a_0, t] \subseteq D$. Let $a_1 = \sup\{t | [a_0, t] \subseteq D\}$, then $a_1 \in D$ and a_1 is not the left interior point of D. We define h'_0 on $[a_0, a_1]$ by

$$h'_0(x) = x, \quad a_0 \leq x \leq a_1.$$

Case (2). If a_0 is not the left interior point of D and $c = a_0 + \varepsilon/2$ is the left interior point of D, then we can choose sufficiently small δ with $0 < \delta < \varepsilon/2$ such that $a_0 + \delta < c$ and $a_0 + \delta \in D^c$. Since D^c is an open set, we can find an interval $[\alpha, \beta] \subset D^c$ such that $a_0 < \alpha < a_0 + \delta < \beta < c$.

Let $a_1 = \sup\{t | [c, t] \subset D\} = \max\{t | [c, t] \subset D\}$, then $a_1 \in D$ and $a_1 - a_0 > \varepsilon/2$. We define h'_1 on $[a_0, a_1]$ by

$$h'_0(t) = \begin{cases} a_0, & a_0 \leq x < \alpha, \\ \text{linear}, & \alpha \leq x < \beta, \\ c, & \beta \leq x < c, \\ x, & c \leq x \leq a_1. \end{cases}$$

Case (3). If both a_0 and $c = a_0 + \varepsilon/2$ are not the left interior points of D, then we can choose δ with $0 < \delta < \varepsilon/2$ such that $c + \delta \in D^c$. Let $a_1 = \sup\{t | (c + \delta, t) \cap D = \phi\}$, then $a_1 \in D$ and $a_1 - a_0 > \varepsilon/2$. We define h'_1 on $[a_0, a_1]$ by

$$h'_0(t) = \begin{cases} a_0, & a_0 \leq x < c + \delta, \\ \text{linear}, & c + \delta \leq x < a_1, \\ a_1, & x = a_1. \end{cases}$$

In any case above, h'_0 is a continuous function on $[a_0, a_1]$ with $h'_0(a_0) = a_0$, $h'_0(a_1) = a_1$ and $\|h|_{D\cap[a_0,a_1]} - h'_0|_{D\cap[a_0,a_1]}\| < \varepsilon$. Moreover, the spectrum of $h'_0|_{D\cap[a_0,a_1]}$ is the finite disjoint union of sets of a single point or closed intervals.

If $a_1 + \varepsilon \leq b$, replacing a_0 by a_1 and repeating the process above, then we have $a_2 \in D$ and continuous function h'_1 on $[a_1, a_2]$ with $h'_1(a_1) = a_1$, $h'_1(a_2) = a_2$ and $\|h|_{D\cap[a_1,a_2]} - h'_1|_{D\cap[a_1,a_2]}\| < \varepsilon$. Moreover, $a_2 - a_0 > \varepsilon/2$ and the spectrum of $h'_1|_{D\cap[a_1,a_2]}$ is the finite disjoint union of sets of a single point or closed intervals. We may continue the process above, for example n times, and have $a_0, \ldots, a_n$ with $a_n \in D$ and $a_{n-1} + \varepsilon \leq b$, and continuous functions h'_i on $[a_i, a_{i+1}]$ with $h'_i(a_i) = a_i$, $h'_i(a_{i+1}) = a_{i+1}$ and $\|h|_{D\cap[a_i,a_{i+1}]} - h'_i|_{D\cap[a_i,a_{i+1}]}\| < \varepsilon$ $(0 \leq i \leq n)$ such that $a_{2i} - a_{2(i-1)} > \varepsilon/2$ $(i \geq 1)$ and the spectrum of $h'_i|_{D\cap[a_i,a_{i+1}]}$ is the finite disjoint union of sets of a single point or closed intervals for each i with $0 \leq i \leq n$.

Since $a_{2i}-a_{2(i-1)}>\varepsilon/2$ $(i\geq 1)$ and $|[a_0,b]|=b-a_0<1$, then we can find a positive integer n such that $a_n\leq b$ and $a_n+\varepsilon>b$. Then we define $\tilde{h}$ on $[a_0,b]$ by

$$\tilde{h}(t)=\begin{cases}h'_{i-1}, & a_{i-1}\leq x<a_i, 1\leq i\leq n,\\ a_n, & a_n<x\leq b.\end{cases}$$

Let $h'=\tilde{h}|_D$, then $h'\in C(D)$ and $\|h-h'\|<\varepsilon$. Moreover, the spectrum of h' is the finite disjoint union of sets of a single point or closed intervals, and this completes the proof. □

NOTE. The result of Lemma 4.3 is also obtained from Lemma 2.4 of [23]. But the proof here is different from that one.

THEOREM 4.4. *Suppose the extension* (A,I) *is tracially quasidiagonal. If* $\mathrm{TR}(I)\leq 1$ *and* A/I *is TAI, then* A *has the property* (P_1).

Proof. Let ε be a positive number, F be a finite subset of A containing a nonzero positive element a, σ_i $(i=1,2,3,4)$ be positive numbers with $0<\sigma_4<\sigma_3<\sigma_2<\sigma_1<1$, and n be a positive integer. We choose $\alpha_1,\dots,\alpha_{18},d_1,d_2$ satisfying $\sigma_3<d_2<d_1<\alpha_{18}<\cdots<\alpha_1<\sigma_2$. Without loss of generality, we assume that F is contained in the unit ball of A. Let γ be a positive number which will be decided later.

Since A/I is TAI, there exist a projection $\bar{p}$ in A/I and a C^*-subalgebra $\bar{C}$ of A/I with $1_{\bar{C}}=\bar{p}$ and $\bar{C}\cong\bigoplus_{k=1}^{l}(C(X_k)\otimes M_{n_k})$, where X_k is $\{pt\}$ or $[0,1]$, such that

$$\|\pi(x)\bar{p}-\bar{p}\pi(x)\|<\gamma \quad \text{for all } x\in F; \tag{1}$$

$$\bar{p}\pi(x)\bar{p}\in_{\gamma}\bar{C} \quad \text{for all } x\in F; \tag{2}$$

$$\left[f_{\alpha_{12}}^{\alpha_{11}}\left((1-\bar{p})\pi(a)(1-\bar{p})\right)\right]\leq[f_{\alpha_{14}}^{\alpha_{13}}(\bar{p}\pi(a)\bar{p})]. \tag{3}$$

Let φ be the isomorphism from $\bigoplus_{k=1}^{l}(C(X_k)\bigotimes M_{n_k})$ onto $\bar{C}$. Without loss of generality, we may assume that there exists an integer $l_0>0$ such that

$$X_k=[0,1]\quad\text{if } 1\leq k\leq l_0 \quad\text{and}\quad X_k=\{pt\}\quad\text{if } l_0<k\leq l.$$

For any k with $1\leq k\leq l$, let $\{d_{ij}^k\}_{1\leq i,j\leq n_k}$ be the matrix units of M_{n_k}. Let $\bar{D}_k=\varphi(1_{C(X_k)}\otimes M_{n_k})$, then $\varphi(\bigoplus_{k=1}^{l}(1_{C(X_k)}\otimes M_{n_k}))=\bigoplus_{k=1}^{l}\bar{D}_k$. Let $\bar{e}_{ij}^k=\varphi(1_{C(X_k)}\otimes d_{ij}^k)$, then $\bar{p}=\bigoplus_{k=1}^{l}(\bigoplus_{i=1}^{n_k}\bar{e}_{ii}^k)$. Let a_{ij}^k be the element in A such that $\pi(a_{ij}^k)=\bar{e}_{ij}^k$. Moreover, for $1\leq k\leq l_0$, we let $\bar{h}_k=\varphi(h_k\otimes(\sum_{i=1}^{n_k}d_{ii}^k))$, where $h_k\in C[0,1]$ is the identity function. Then

$$\bar{C}=\left(\bigoplus_{k=1}^{l_0}C^*(\bar{h}_k,\bar{e}_{ij}^k:1\leq i,j\leq n_k)\right)\oplus\left(\bigoplus_{k=l_0+1}^{l}C^*(\bar{e}_{ij}^k:1\leq i,j\leq n_k)\right).$$

By (3), there exist the elements $\bar{x}_s$ $(1\leq s\leq n+1)$ in A/I such that $\bar{x}_s^*\bar{x}_s=f_{\alpha_{12}}^{\alpha_{11}}((1-\bar{p})\pi(a)(1-\bar{p}))$, $\bar{x}_s\bar{x}_s^*\in\mathrm{Her}(f_{\alpha_{14}}^{\alpha_{13}}(\bar{p}a\bar{p}))$, and $\bar{x}_s\bar{x}_s^*$ $(1\leq s\leq n+1)$

are mutually orthogonal. Let x_s $(1 \le s \le n+1)$ be the elements in A with $\|x_s\| \le 1$ such that $\pi(x_s) = \bar{x}_s$. Let p' be the element in A with $0 \le p' \le 1_A$ such that $\pi(p') = \bar{p}$, and b_k $(1 \le k \le l_0)$ be the positive elements in A such that $\pi(b_k) = \bar{h}_k$. Let

$$\begin{aligned} F' = F &\cup \{a_{ij}^k | 1 \le k \le l, 1 \le i,j \le n_k\} \cup \{x_s, x_s^* | 1 \le s \le n+1\} \\ &\cup \{b_k, b_k^* | 1 \le k \le l_0\} \cup \{p'\}, \end{aligned}$$

and let γ_1 be a positive number with $0 < \gamma_1 < 1/54$ which will be decided later.

Since the extension (A, I) is tracially quasidiagonal, there exist a projection q in A and a C^*-subalgebra B of A with $1_B = q$ such that $B \cap I = qIq$, the extension (B, qIq) is quasidiagonal, and

$$\|qx - xq\| < \gamma_1 \quad \text{for all } x \in F'; \tag{4}$$

$$qxq \in_{\gamma_1} B \quad \text{for all } x \in F'; \tag{5}$$

$$n\big[f_{\alpha_2}^{\alpha_1}\big((1-q)a(1-q)\big)\big] \le [f_{\alpha_4}^{\alpha_3}(qaq)]. \tag{6}$$

For any k with $1 \le k \le l$, we can choose $\bar{c}_{ij}^k$ $(1 \le i,j \le n_k)$ of $\pi(B)$ such that $\|\bar{c}_{ij}^k\| \le 1$, $\bar{c}_{ii}^k$ is a positive element and $\pi(q)\bar{e}_{ij}^k\pi(q) \approx_{\gamma_1} \bar{c}_{ij}^k$. Since

$$\begin{aligned} (\bar{c}_{ii}^k)^2 - \bar{c}_{ii}^k &\approx_{2\gamma_1} \bar{c}_{ii}^k \pi(q)\bar{e}_{ii}^k\pi(q) - \pi(q)\bar{e}_{ii}^k\pi(q) \\ &\approx_{\gamma_1} \pi(q)\bar{e}_{ii}^k\pi(q)\bar{e}_{ii}^k\pi(q) - \pi(q)\bar{e}_{ii}^k\pi(q) \\ &\approx_{\gamma_1} \pi(q)\bar{e}_{ii}^k\pi(q) - \pi(q)\bar{e}_{ii}^k\pi(q) \\ &= 0, \end{aligned}$$

then we have

$$\|(\bar{c}_{ii}^k)^2 - \bar{c}_{ii}^k\| < 4\gamma_1.$$

Since $\gamma_1 < 1/108$, then we have

$$\|(\bar{c}_{ii}^k)^2 - \bar{c}_{ii}^k\| < 4\gamma_1 < 1/4.$$

By Lemma 2.5.5 in [16], there exists a projection $\bar{p}_{ii}^k$ in $\pi(B)$ such that $\|\bar{c}_{ii}^k - \bar{p}_{ii}^k\| < 8\gamma_1$. Then we have

$$\|\pi(q)\bar{e}_{ii}^k\pi(q) - \bar{p}_{ii}^k\| < 9\gamma_1$$

and

$$\big\|\bar{e}_{ii}^k - \bar{p}_{ii}^k - \big(1-\pi(q)\big)\bar{e}_{ii}^k\big(1-\pi(q)\big)\big\| < 10\gamma_1.$$

Let s and t be integers with $1 \le t \le l$ and $1 \le s \le n_t$, then we have a projection $\bar{p}_{ss}^t$ in $\pi(B)$ such that $\|\pi(q)\bar{e}_{ss}^t\pi(q) - \bar{p}_{ss}^t\| < 9\gamma_1$ and $\|\bar{e}_{ss}^t - \bar{p}_{ss}^t - (1-\pi(q))\bar{e}_{ss}^t(1-\pi(q))\| < 10\gamma_1$. If $k \ne t$, or $k = t$ but $i \ne s$, then we have

$$\begin{aligned} \bar{p}_{ii}^k\bar{p}_{ss}^t &\approx_{18\gamma_1} \pi(q)\bar{e}_{ii}^k\pi(q)\pi(q)\bar{e}_{ss}^t\pi(q) \\ &\approx_{\gamma_1} \pi(q)\bar{e}_{ii}^k\bar{e}_{ss}^t\pi(q) \\ &= 0. \end{aligned}$$

That is, $\|\bar{p}_{ii}^k \bar{p}_{ss}^t\| < 19\gamma_1$. By Lemma 2.5.6 in [16], with γ_1 sufficiently small, we may assume that $\bar{p}_{ii}^k$ and $\bar{p}_{ss}^t$ are mutually orthogonal if $k \neq t$, or $k = t$ but $i \neq s$.

For any k with $1 \leq k \leq l$, since

$$\begin{aligned} \bar{e}_{1j}^k &\approx_{\gamma_1} \pi(q)\bar{e}_{1j}^k\pi(q) + \big(1-\pi(q)\big)\bar{e}_{1j}^k\big(1-\pi(q)\big) \\ &\approx_{\gamma_1} \bar{c}_{1j}^k + \big(1-\pi(q)\big)\bar{e}_{1j}^k\big(1-\pi(q)\big), \end{aligned}$$

then

$$\bar{e}_{1j}^k \approx_{2\gamma_1} \bar{c}_{1j}^k + \big(1-\pi(q)\big)\bar{e}_{1j}^k\big(1-\pi(q)\big)$$

and

$$\bar{e}_{j1}^k \approx_{2\gamma_1} (\bar{c}_{1j}^k)^* + \big(1-\pi(q)\big)\bar{e}_{j1}^k\big(1-\pi(q)\big).$$

So we have

$$\begin{aligned} \bar{e}_{1j}^k &= \bar{e}_{11}^k\bar{e}_{1j}^k\bar{e}_{jj}^k \\ &\approx_{10\gamma_1} \big(\bar{p}_{11}^k + \big(1-\pi(q)\big)\bar{e}_{11}^k\big(1-\pi(q)\big)\big)\bar{e}_{1j}^k\bar{e}_{jj}^k \\ &\approx_{10\gamma_1} \big(\bar{p}_{11}^k + \big(1-\pi(q)\big)\bar{e}_{11}^k\big(1-\pi(q)\big)\big)\bar{e}_{1j}^k\big(\bar{p}_{jj}^k + \big(1-\pi(q)\big)\bar{e}_{jj}^k\big(1-\pi(q)\big)\big) \\ &\approx_{2\gamma_1} \big(\bar{p}_{11}^k + \big(1-\pi(q)\big)\bar{e}_{11}^k\big(1-\pi(q)\big)\big)\big(\bar{c}_{1j}^k + \big(1-\pi(q)\big)\bar{e}_{1j}^k\big(1-\pi(q)\big)\big) \\ &\qquad \times \big(\bar{p}_{jj}^k + \big(1-\pi(q)\big)\bar{e}_{jj}^k\big(1-\pi(q)\big)\big) \\ &= \bar{p}_{11}^k\bar{c}_{1j}^k\bar{p}_{jj}^k + \big(1-\pi(q)\big)\bar{e}_{11}^k\big(1-\pi(q)\big)\bar{e}_{1j}^k\big(1-\pi(q)\big)\bar{e}_{jj}^k\big(1-\pi(q)\big). \end{aligned}$$

Set $z_{1j}^k = \bar{p}_{11}^k\bar{c}_{1j}^k\bar{p}_{jj}^k$ and $f_{1j}^k = (1-\pi(q))\bar{e}_{11}^k(1-\pi(q))\bar{e}_{1j}^k(1-\pi(q))\bar{e}_{jj}^k(1-\pi(q))$, then we have

$$\bar{e}_{1j}^k \approx_{22\gamma_1} z_{1j}^k + f_{1j}^k, \qquad \bar{e}_{j1}^k \approx_{22\gamma_1} (z_{1j}^k)^* + (f_{1j}^k)^*. \tag{7}$$

Therefore, we have

$$\bar{e}_{11}^k = \bar{e}_{1j}^k\bar{e}_{j1}^k \approx_{44\gamma_1} z_{1j}^k(z_{1j}^k)^* + f_{1j}^k(f_{1j}^k)^*$$

and

$$\bar{e}_{jj}^k = \bar{e}_{j1}^k\bar{e}_{1j}^k \approx_{44\gamma_1} (z_{1j}^k)^*z_{1j}^k + (f_{1j}^k)^*f_{1j}^k.$$

Since

$$\bar{p}_{11}^k + \big(1-\pi(q)\big)\bar{e}_{11}^k\big(1-\pi(q)\big) \approx_{10\gamma_1} \bar{e}_{11}^k \approx_{44\gamma_1} z_{1j}^k(z_{1j}^k)^* + f_{1j}^k(f_{1j}^k)^*$$

and

$$\bar{p}_{jj}^k + \big(1-\pi(q)\big)\bar{e}_{jj}^k\big(1-\pi(q)\big) \approx_{10\gamma_1} \bar{e}_{jj}^k \approx_{44\gamma_1} (z_{1j}^k)^*z_{1j}^k + (f_{1j}^k)^*f_{1j}^k,$$

then we have

$$\|\bar{p}_{11}^k - z_{1j}^k(z_{1j}^k)^*\| < 54\gamma_1 \quad \text{and} \quad \|\bar{p}_{jj}^k - (z_{1j}^k)^*z_{1j}^k\| < 54\gamma_1.$$

Since $\gamma_1 < 1/54$, then we have

$$\|\bar{p}_{11}^k - z_{1j}^k(z_{1j}^k)^*\| < 54\gamma_1 < 1 \quad \text{and} \quad \|\bar{p}_{jj}^k - (z_{1j}^k)^*z_{1j}^k\| < 54\gamma_1 < 1.$$

By Lemma 2.5.3 of [16], we have $\bar{p}^k_{11}$ and $\bar{p}^k_{jj}$ are Murray–von Neumann equivalent. Since

$$0 \le \bar{p}^k_{jj} - |z^k_{1j}| \le \bar{p}^k_{jj} - (z^k_{1j})^* z^k_{1j} < 54\gamma_1,$$

then $|z^k_{1j}|$ is invertible in $\bar{p}^k_{jj}\pi(B)\bar{p}^k_{jj}$. Set $\tilde{v}^k_{1j} = z^k_{1j}|z^k_{1j}|^{-1}$, then

$$\tilde{v}^k_{1j}(\tilde{v}^k_{1j})^* = \bar{p}^k_{11}, \qquad (\tilde{v}^k_{1j})^*\tilde{v}^k_{1j} = \bar{p}^k_{jj} \quad \text{and} \quad \|\tilde{v}^k_{1j} - z^k_{1j}\| < \frac{54\gamma_1}{1-54\gamma_1}.$$

For $1 \le i,j \le n_k$, set $\tilde{e}^k_{ij} = (\tilde{v}^k_{1i})^*\tilde{v}^k_{1j}$, then $\{\tilde{e}^k_{ij}\}_{1\le i,j\le n_k}$ $(1 \le k \le l)$ are mutually orthogonal matrix units in $\pi(B)$. Since

$$\begin{aligned}
\tilde{e}^k_{ij} &= (\tilde{v}^k_{1i})^*\tilde{v}^k_{1j} \\
&\approx_{\frac{108\gamma_1}{1-54\gamma_1}} (z^k_{1i})^* z^k_{1j} \\
&\approx_{44\gamma_1} \pi(q)(\bar{e}^k_{1i})^*\pi(q)\pi(q)\bar{e}^k_{1j}\pi(q) \quad \text{(by (7))} \\
&\approx_{\gamma_1} \pi(q)(\bar{e}^k_{1i})^*\bar{e}^k_{1j}\pi(q) \\
&= \pi(q)\bar{e}^k_{ij}\pi(q),
\end{aligned}$$

then we have

$$\|\pi(q)\bar{e}^k_{ij}\pi(q) - \tilde{e}^k_{ij}\| < 45\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1} \quad \text{for } 1 \le k \le l,\ 1 \le i,j \le n_k. \tag{8}$$

For any k with $1 \le k \le l_0$, by (5), we can find an element g_k in B with $0 \le g_k \le q$ such that $\|qb_kq - g_k\| < \gamma_1$. Let

$$\tilde{g}_k = \pi(g_k) \quad \text{and} \quad \tilde{h}_k = \bigoplus_{i=1}^{n_k} \tilde{e}^k_{i1}\tilde{g}_k\tilde{e}^k_{i1} \quad \text{for any } k \text{ with } 1 \le k \le l_0,$$

then $\tilde{g}_k, \tilde{h}_k \in \pi(B)$ and $\tilde{h}_k \le \bigoplus_{i=1}^{n_k} \tilde{e}^k_{ii} = \tilde{e}_k \le \pi(q)$, where $\tilde{e}_k$ is the unit of the finite dimensional C^*-subalgebra spanned by $\{\tilde{e}^k_{ij} | 1 \le i,j \le n_k\}$. It is easy to check that $\tilde{h}_k$ commutes with $\tilde{e}^k_{ij}$ $(1 \le i,j \le n_k)$. Since

$$\begin{aligned}
\tilde{h}_k &= \bigoplus_{i=1}^{n_k} \tilde{e}^k_{i1}\tilde{g}_k\tilde{e}^k_{1i} \approx_{\gamma_1} \bigoplus_{i=1}^{n_k} \tilde{e}^k_{i1}\pi(q)\bar{h}_k\pi(q)\tilde{e}^k_{1i} \\
&\approx_{2n_k(45\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1})} \sum_{i=1}^{n_k} \pi(q)\bar{e}^k_{i1}\pi(q)\bar{h}_k\pi(q)\bar{e}^k_{1i}\pi(q) \quad \text{(by (8))} \\
&\approx_{2n_k\gamma_1} \sum_{i=1}^{n_k} \pi(q)\bar{e}^k_{i1}\bar{h}_k\bar{e}^k_{1i}\pi(q) = \pi(q)\bar{h}_k\pi(q) \quad \text{(by (4))},
\end{aligned}$$

then for any k with $1 \le k \le l_0$ we have

$$\|\tilde{h}_k - \pi(q)\bar{h}_k\pi(q)\| < n_k\left(93\gamma_1 + \frac{216\gamma_1}{1-54\gamma_1}\right). \tag{9}$$

For any k with $1 \le k \le l_0$, let $A_k = C^*(\tilde{h}_k, \tilde{e}^k_{ij} : 1 \le i,j \le n_k)$ and $A_{k,e} = C^*(\tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11}, \tilde{e}^k_{11})$. Since $\{d^k_{ij}\}_{1\le i,j\le n_k}$ is a matrix unit for M_{n_k} and $\{\tilde{e}^k_{ij}\}_{1\le i,j\le n_k}$ is a matrix unit in A_k, then we define a map α_k from $A_{k,e}\otimes M_{n_k}$ to A_k by

$$\alpha_k(a\otimes d^k_{ij}) = \tilde{e}^k_{i1}a\tilde{e}^k_{1j}.$$

By Theorem 6.1.2 and Remark 6.1.3 in [27], α_k is an isomorphism from $A_{k,e}\otimes M_{n_k}$ onto A_k. By the Gelfand theorem for commutative C^*-algebras, there is an isomorphism β_k from $A_{k,e}$ onto $C(\sigma(\tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11}))$ such that ${\beta_k}^{-1}(f_k) = \tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11}$, where f_k the identity function on $\sigma(\tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11})$.

Let $\eta > 0$ which will be decided later, by Lemma 4.3, there exists $f'_k \in C(\sigma(\tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11}))$ such that $\|f_k - f'_k\| < \eta$ and the spectrum of f'_k is the finite disjoint union of sets of a single point or closed intervals. Then we have

$$ {\beta_k}^{-1}(f'_k) \in A_{k,e}, \qquad \|\tilde{e}^k_{11}\tilde{h}_k\tilde{e}^k_{11} - \beta_k^{-1}(f'_k)\| < \eta,$$

and

$$C^*({\beta_k}^{-1}(f'_k), \tilde{e}^k_{11}) \cong C(\sigma(\beta_k^{-1}(f'_k))).$$

Since $\sigma(\beta_k^{-1}(f'_k))$ is the finite disjoint union of sets of a single point or closed intervals, we may assume there exist an integer $m_k > 0$ and disjoint sets Y^k_i $(1 \le i \le m_k)$ which are single points or closed intervals such that $\sigma(\beta_k^{-1}(f'_k)) = \bigcup_{i=1}^{m_k} Y^k_i$, then $C(\sigma(\beta_k^{-1}(f'_k)))$ is isomorphic to $\bigoplus_{i=1}^{m_k} C(Y^k_i)$. Since a closed interval is homeomorphic to $[0,1]$, we have

$$C^*({\beta_k}^{-1}(f'_k), \tilde{e}^k_{11}) \otimes M_{n_k} \cong \left(\bigoplus_{i=1}^{m_k} C(X^k_i)\right) \otimes M_{n_k},$$

where each X^k_i is a single point or $[0,1]$. For any k with $1 \le k \le l_0$, let

$$\widetilde{C}_k = \alpha_k\big(C^*({\beta_k}^{-1}(f'_k), \tilde{e}^k_{11}) \otimes M_{n_k}\big) \quad \text{and} \quad \tilde{h}'_k = \sum_{i=1}^{n_k} \tilde{e}^k_{i1}\beta_k^{-1}(f'_k)\tilde{e}^k_{1i}.$$

Since $\|\tilde{h}_k - \tilde{h}'_k\| = \|\sum_{i=1}^{n_k} \tilde{e}^k_{i1}(\beta_k^{-1}(f_k) - \beta_k^{-1}(f'_k))\tilde{e}^k_{1i}\| < \eta$, together with (9), we have

$$\|\pi(q)\bar{h}_k\pi(q) - \tilde{h}'_k\| < n_k\left(93\gamma_1 + \frac{216\gamma_1}{1-54\gamma_1}\right) + \eta \quad \text{for } 1 \le k \le l_0. \tag{10}$$

Since

$$\alpha_k(\tilde{e}^k_{11} \otimes d^k_{ij}) = \tilde{e}^k_{i1}\tilde{e}^k_{11}\tilde{e}^k_{1i} = \tilde{e}^k_{ij}$$

and

$$\begin{aligned}\alpha_k\left(\beta_k^{-1}(f'_k) \otimes \left(\sum_{i=1}^{n_k} d^k_{ii}\right)\right) &= \sum_{i=1}^{n_k}\big(\alpha_k\big(\beta_k^{-1}(f'_k)\otimes d^k_{ii}\big)\big)\\ &= \sum_{i=1}^{n_k} \tilde{e}^k_{i1}\beta_k^{-1}(f'_k)\tilde{e}^k_{1i} = \tilde{h}'_k,\end{aligned}$$

we have

$$\widetilde{C}_k = C^*(\tilde{h}'_k, \tilde{e}^k_{ij} : 1 \le i, j \le n_k), \quad 1 \le k \le l_0.$$

Let

$$\widetilde{C} = \left(\bigoplus_{k=1}^{l_0} \widetilde{C}_k\right) \oplus \left(\bigoplus_{k=l_0+1}^{l} C^*(\tilde{e}^k_{ij} : 1 \le i, j \le n_k)\right),$$

then $\widetilde{C} \subseteq \pi(C)$. Let $\tilde{p} = 1_{\widetilde{C}}$, then $\tilde{p} = \bigoplus_{k=1}^{l}(\bigoplus_{i=1}^{n_k} \tilde{e}^k_{ii}) \le \pi(q) = 1_{\pi(C)}$. By (8) and $\bar{p} = \bigoplus_{k=1}^{l}(\bigoplus_{i=1}^{n_k} \bar{e}^k_{ii})$, we have

$$\|\tilde{p}\bar{e}^k_{ij}\tilde{p} - \tilde{e}^k_{ij}\| < 45\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1} \tag{11}$$

and

$$\|\pi(q)\bar{p}\pi(q) - \tilde{p}\| < M\left(45\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1}\right), \tag{12}$$

where $M = n_1 + n_2 + \cdots + n_l$. Moreover, by (4) and (12), we have

$$\|\pi(q)\bar{p} - \tilde{p}\| < M\left(46\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1}\right). \tag{13}$$

For any $x \in F$, by (13), we have

$$\begin{aligned}
&\tilde{p}\pi(q)\pi(x)\pi(q) - \pi(q)\pi(x)\pi(q)\tilde{p}\\
&\quad\approx_{2M(46\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1})} \pi(q)\bar{p}\pi(x)\pi(q) - \pi(q)\pi(x)\bar{p}\pi(q)\\
&\quad\approx_{\gamma} 0.
\end{aligned}$$

Therefore, with γ_1 and γ sufficiently small, we have

$$\|\tilde{p}y - y\tilde{p}\| < \varepsilon \quad \text{for all } y \in \pi(qFq). \tag{14}$$

For any $x \in F$, by (2) there exists $\bar{x} \in \bar{C}$ such that $\|\bar{p}\pi(x)\bar{p} - \bar{x}\| < \gamma$. Then by (13), we have

$$\begin{aligned}
\tilde{p}\pi(q)\pi(x)\pi(q)\tilde{p} &\approx_{2M(46\gamma_1 + \frac{108\gamma_1}{1-54\gamma_1})} \pi(q)\bar{p}\pi(x)\bar{p}\pi(q)\\
&\approx_{\gamma} \pi(q)\bar{x}\pi(q).
\end{aligned}$$

By (8), (10), and the definitions of $\bar{C}$ and $\tilde{C}$, for any $\lambda > 0$, with γ_1 and η sufficiently small, we can find $\tilde{x} \in \widetilde{C}$ such that $\|\pi(q)\bar{x}\pi(q) - \tilde{x}\| < \lambda$. Therefore, with γ_1, γ, η and λ sufficiently small, we have

$$\tilde{p}y\tilde{p} \in_{\varepsilon} \widetilde{C} \quad \text{for all } y \in \pi(qFq). \tag{15}$$

By (13), we may choose sufficiently small γ_1 such that

$$\big\|\big(\pi(q) - \tilde{p}\big)\pi(a)\big(\pi(q) - \tilde{p}\big) - \pi(q)(1-\bar{p})\pi(a)(1-\bar{p})\pi(q)\big\| < \delta(\alpha_9, \alpha_{10}); \tag{16}$$

and

$$\|\pi(q)\bar{p}\pi(a)\bar{p}\pi(q)-\tilde{p}\pi(a)\tilde{p}\|<\delta(\alpha_{17},\alpha_{18}), \tag{17}$$

where $\delta(\alpha_9,\alpha_{10})$ and $\delta(\alpha_{17},\alpha_{18})$ are given by Lemma 2.4(5). Then by (16) and Lemma 2.4(5), we have

$$\big[f_{\alpha_8}^{\alpha_7}\big(\big(\pi(q)-\tilde{p}\big)\pi(a)\big(\pi(q)-\tilde{p}\big)\big)\big] \le \big[f_{\alpha_{10}}^{\alpha_9}\big(\pi(q)(1-\bar{p})\pi(a)(1-\bar{p})\pi(q)\big)\big]. \tag{18}$$

By (4), we may choose sufficiently small γ_1 such that for any $y\in\{\bar{x}_s|1\le s\le n+1\}\cup\{\bar{p}\pi(a)\bar{p},(1-\bar{p})\pi(a)(1-\bar{p})\}$

$$\|\pi(q)y-y\pi(q)\|<\delta(n+1,\alpha_9,\alpha_{10},\dots,\alpha_{15},\alpha_{16}), \tag{19}$$

where $\delta(n+1,\alpha_9,\alpha_{10},\dots,\alpha_{15},\alpha_{16})$ is given by Lemma 2.7. Since

$$(n+1)\big[f_{\alpha_{12}}^{\alpha_{11}}\big((1-\bar{p})\pi(a)(1-\bar{p})\big)\big] \le [f_{\alpha_{14}}^{\alpha_{13}}(\bar{p}\pi(a)\bar{p})],$$

we have

$$\begin{aligned}
&(n+1)\big[f_{\alpha_{10}}^{\alpha_9}\big(\pi(q)(1-\bar{p})\pi(a)(1-\bar{p})\pi(q)\big)\big]\\
&\quad\le [f_{\alpha_{16}}^{\alpha_{15}}(\pi(q)\bar{p}\pi(a)\bar{p}\pi(q))] \quad \text{(by (19) and Lemma 2.7)}\\
&\quad\le [f_{\alpha_{18}}^{\alpha_{17}}(\tilde{p}\pi(a)\tilde{p})] \quad \text{(by (17) and Lemma 2.4(5))}.
\end{aligned}$$

So we have

$$(n+1)\big[f_{\alpha_8}^{\alpha_7}\big(\big(\pi(q)-\tilde{p}\big)\pi(a)\big(\pi(q)-\tilde{p}\big)\big)\big] \le [f_{\alpha_{18}}^{\alpha_{17}}(\tilde{p}\pi(a)\tilde{p})]. \tag{20}$$

By (5), without loss of generality, we may assume $qFq\subseteq B$. Since the extension (B,qIq) is quasidiagonal, by (14), (15), (20) and Theorem 3.6 there exist a projection p and a C^*-subalgebra $C\in\mathcal{I}^{(1)}$ of B with $1_C=p$, $\pi(p)=\tilde{p}$ and $\pi(C)=\widetilde{C}$, and a projection $r\in(q-p)I(q-p)$ such that

$$\|px-xp\|<8\varepsilon \quad\text{and}\quad \|rx-xr\|<24\varepsilon \quad \text{for all } x\in qFq; \tag{21}$$

$$pxp\in_{2\varepsilon} C \quad\text{and}\quad (p+r)x(p+r)\in_{18\varepsilon} C+rIr \quad \text{for all } x\in qFq; \tag{22}$$

$$(n+1)\big[f_{\alpha_6}^{\alpha_5}\big((q-p-r)a(q-p-r)\big)\big] \le [f_{d_2}^{d_1}(pap)]. \tag{23}$$

Let $C_0=C+rIr$. Since $\mathrm{TR}(I)\le 1$ and $r\in I$, we have $\mathrm{TR}(rIr)\le 1$. Set $p_0=p+r$, we have $\mathrm{TR}(C_0)\le 1$ and $1_{C_0}=p_0$. Since

$$\begin{aligned}
&n\big[f_{\sigma_2}^{\sigma_1}\big((1-p_0)a(1-p_0)\big)\big]\\
&\quad\le n\big[f_{\alpha_2}^{\alpha_1}\big((1-q)a(1-q)\big)\big]+n\big[f_{\alpha_2}^{\alpha_1}\big((q-p_0)a(q-p_0)\big)\big]\\
&\quad\le [f_{\alpha_4}^{\alpha_3}(qaq)]+n\big[f_{\alpha_2}^{\alpha_1}\big((q-p_0)a(q-p_0)\big)\big]\\
&\quad\le [f_{\alpha_6}^{\alpha_5}(p_0ap_0)]+(n+1)\big[f_{\alpha_6}^{\alpha_5}\big((q-p_0)a(q-p_0)\big)\big]\\
&\quad\le [f_{\sigma_4}^{\sigma_3}(p_0ap_0)]+[f_{d_2}^{d_1}(pap)] \quad \text{(by Lemma 2.4(2) and 2.4(6))}\\
&\quad\le 2[f_{\sigma_4}^{\sigma_3}(p_0ap_0)],
\end{aligned}$$

then it follows that A has the property (P_1) and it completes the proof. $\square$

NOTE. If the Cuntz semi-group of A is weakly unperforated, then $n = 2m$ and $n[a] \le 2[b]$ imply that $m[a] \le [b]$. Therefore, (3) of Definition 4.2 will be the same as (3) in the definition of $\mathrm{TR}(A) \le 1$. That is, Theorem 4.4 implies $\mathrm{TR}(A) \le 1$.

COROLLARY 4.5. *Suppose that (A, I) is tracially quasidiagonal, A is unital and A/I is a unital simple C^*-algebra. If $\mathrm{TR}(I) \le 1$ and $\mathrm{TR}(A/I) \le 1$, then A has the property (P_1).*

Proof. Suppose A/I is a unital simple C^*-algebra. By Theorem 7.1(b) in [19], it was shown that $\mathrm{TR}(A/I) \le 1$ if and only if A/I is TAI. Then the conclusion follows from the Theorem 4.4. □

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