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# ON THE BJÖRLING PROBLEM IN A THREE-DIMENSIONAL LIE GROUP

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Abstract. We prove existence and uniqueness of the solution of the Björling problem for minimal surfaces in a three-dimensional Lie group.

## 1. Introduction

The Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ has been a fundamental tool for producing examples and proving general properties of such surfaces, since the surfaces can be parametrized by holomorphic data. In [8], the authors describe a general Weierstrass representation formula for minimal surfaces in an arbitrary Riemannian manifold. The partial differential equations involved are, in general, too complicated to be solved explicitly. However, for particular ambient manifolds, such as the Heisenberg group, the hyperbolic space and the product of the hyperbolic plane with $\mathbb{R}$, the equations are more workable and the formula can be used to produce examples (see [7, 8]).

In this note, we will show how this formula can be used, at least if the ambient manifold is a 3-dimensional Lie group, in order to prove existence and uniqueness of the solution of the Björling problem. We also give some examples for the case in which the ambient manifold is the Heisenberg group $\mathbb{H}_3$ or $\mathbb{H}^2 \times \mathbb{R}$, the product of the hyperbolic plane and the real line.

## 2. The Weierstrass representation formula

The arguments will be essentially local so we will consider, as ambient manifold $M$, the space $\mathbb{R}^3$ with a Riemannian metric $g = (g_{ij})$. We will

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denote by $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a *simply connected* domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$, and by:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right); \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

the complex derivatives.

In this situation, the general Weierstrass representation formula can be stated as follows.

THEOREM 2.1 (See [8] for a proof). *Let $f : \Omega \to M$ be a conformal minimal immersion and $g = (g_{ij})$ be the induced metric. The complex tangent vector*:

$$\frac{\partial f}{\partial z} := \phi := \sum_i \phi_i \frac{\partial}{\partial x_i}, \quad \phi_i : \Omega \to \mathbb{C},$$

*has the following properties*:

(1) $\sum_{i,j} g_{ij}\phi_i\overline{\phi_j} \neq 0$,
(2) $\sum_{i,j} g_{ij}\phi_i\phi_j = 0$,
(3) $\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,k} \Gamma^i_{jk}\overline{\phi_j}\phi_k = 0$,

*where $\{\Gamma^i_{jk}\}$ are the Christoffel symbols of the Riemannian connection.*

*Conversely, given functions $\phi_i : \Omega \to \mathbb{C}$ that verify the above conditions, then the map*:

$$f : \Omega \to M, \quad f_i(z) = 2Re\int_{z_0}^{z} \phi_i \, dz,$$

*is a well defined conformal minimal immersion of $\Omega$ into $M$ (here $z_0$ is an arbitrary fixed point of $\Omega$ and the integral is along any curve joining $z_0$ to $z$).*

REMARK 2.2. The first condition of Theorem 2.1 tells us that $f$ is an immersion, the second that $f$ is conformal and the last one that $f$ is minimal. The last condition is called the *holomorphicity condition* since it is the local coordinates version of the condition: $\tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\phi = 0$, where $\tilde{\nabla}$ is the induced connection on the pull-back bundle $f^*(TM \otimes \mathbb{C})$. In fact, we have that the section $\phi$ is holomorphic if and only if

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\left(\sum_i \phi_i \frac{\partial}{\partial x_i}\right) &= \sum_i \left\{\frac{\partial \phi_i}{\partial \bar{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\frac{\partial f}{\partial \bar{z}}}\frac{\partial}{\partial x_i}\right\} \\ &= \sum_i \left\{\frac{\partial \phi_i}{\partial \bar{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\sum_j \overline{\phi_j}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right\} \\ &= \sum_i \left\{\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,k} \Gamma^i_{jk}\overline{\phi_j}\phi_k\right\}\frac{\partial}{\partial x_i} = 0. \end{aligned} \tag{2.1}$$

In general, it is quite difficult to produce functions $\phi_i$ with the above properties since the holomorphicity condition is given by partial differential equations with *nonconstant coefficients*. If $M$ is a Lie group equipped with a left-invariant metric $g$ and $\{E_i\}$ are orthonormal left-invariant vector fields, we can write

$$\phi = \sum_i \phi_i \frac{\partial}{\partial x_i} = \sum_i \psi_i E_i, \quad \psi_i : \Omega \to \mathbb{C},$$

with $\phi_i = \sum_{i,j} A_{ij}\psi_j$ and $A = (A_{ij})$ being an invertible matrix, with function entries $A_{ij}$. In this case, the Weierstrass formula becomes the following.

THEOREM 2.3 (See [8] for a proof). *Given functions $\psi_i : \Omega \to \mathbb{C}$ such that*:

(1) $\sum_i |\psi_i|^2 \neq 0$,
(2) $\sum_i \psi_i^2 = 0$,
(3) $\frac{\partial \psi_i}{\partial \bar{z}} + \sum_{j,k} L^i_{jk} \overline{\psi_j} \psi_k = 0$,

*where $L^i_{jk} := g(\nabla_{E_j} E_k, E_i)$, then the map*:

$$f : \Omega \to M, \quad f_i(z) = 2Re\left( \int_{z_0}^{z} \sum_j A_{ij}\psi_j \, dz \right),$$

*defines a conformal minimal immersion.*

The advantage of having partial differential equations with constant coefficients is not really a great gain, in principle, since we still have to compute the integrand $A_{ij}\psi_j$ along the solutions. However, in certain cases, as for example the hyperbolic space, the Heisenberg group and $\mathbb{H}^2 \times \mathbb{R}$, this problem may be overcome by ad hoc arguments, as shown (for example) in [8].

## 3. The Björling problem for three-dimensional Lie groups

In this section, we will suppose that $M$ is a three-dimensional Lie group endowed with a left-invariant Riemannian metric $g$. Let $\beta : I \subseteq \mathbb{R} \to M$ be a regular analytic curve in $M$ and $V : I \to TM$ a unitary real analytic vector field along $\beta$, such that $g(\dot{\beta}, V) \equiv 0$. The Björling problem is the following:

*Determine a minimal surface $f : I \times (-\varepsilon, \varepsilon) = \Omega \subseteq \mathbb{C} \to M$, such that:*

- $f(u, 0) = \beta(u)$,
- $N(u, 0) = V(u)$,

*for all $u \in I$, where $N : \Omega \to TM$ is the Gauss map of the surface.*

We observe that if $\beta$ is parameterized by arc-length and $\ddot{\beta} := \nabla_{\dot{\beta}} \dot{\beta}$, we have that $V = \|\ddot{\beta}\|^{-1} \ddot{\beta}$ is a unit vector field along the curve such that $g(\dot{\beta}, V) \equiv 0$. Then the Björling problem is a generalization of the problem of finding a minimal surface which contains a given curve as a geodesic.

THEOREM 3.1. *The Björling problem has a unique solution.*[1]

[1] Unique up to fixing the domain.

*Proof.* In order to prove the theorem, we must analyze Theorem 2.3 carefully. In this theorem, we have essentially four conditions on the three functions $\psi_i$ (the first condition is "generically satisfied"). We will start showing that these conditions are dependent.

LEMMA 3.2. *Let $\psi_i : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, $i = 1, 2$, be two differentiable functions and $\psi_3^2 = -\psi_1^2 - \psi_2^2$. We suppose that $\psi_i$, $i = 1, 2$, satisfy the two first equations of the third item of Theorem 2.3. Then $\psi_3$ satisfies the third equation.*

*Proof.* Deriving with respect to $\bar{z}$ the equation

$$-\psi_3^2 = (\psi_1^2 + \psi_2^2),$$

and using the fact that the two first equations of the third item of Theorem 2.3 are satisfied, we have:

$$\begin{aligned} -\psi_3 \frac{\partial \psi_3}{\partial \bar{z}} &= \psi_1 \frac{\partial \psi_1}{\partial \bar{z}} + \psi_2 \frac{\partial \psi_2}{\partial \bar{z}} \\ &= -\sum_{j,k=1}^{3} (L_{jk}^1 \psi_1 + L_{jk}^2 \psi_2) \bar{\psi}_j \psi_k. \end{aligned}$$

Therefore, to prove the lemma it suffices to show that

$$\sum_{j,k=1}^{3} (L_{jk}^1 \psi_1 + L_{jk}^2 \psi_2 + L_{jk}^3 \psi_3) \bar{\psi}_j \psi_k = 0.$$

Writing the above sum as:

$$\sum_{j,k=1}^{3} L_{jk}^k \bar{\psi}_j \psi_k^2 + \sum_{\substack{j,k,l=1 \\ k<l}}^{3} (L_{jk}^l + L_{jl}^k) \bar{\psi}_j \psi_k \psi_l,$$

and using the relation $L_{jk}^l + L_{jl}^k = 0$, where $j, k, l \in \{1, 2, 3\}$, we conclude the proof. □

We go back now to the proof of Theorem 3.1. Consider the system:

$$\begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} + \sum_{j,k=1}^{3} L_{jk}^1 \overline{\psi_j} \psi_k = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} + \sum_{j,k=1}^{3} L_{jk}^2 \overline{\psi_j} \psi_k = 0, \end{cases} \tag{3.1}$$

where $\psi_i : \Omega \to \mathbb{C}$ and $\psi_3^2 = -\psi_1^2 - \psi_2^2$.

Since this system is of Cauchy–Kovalevskaya type (see [10] for a proof of the Cauchy–Kovalevskaya theorem), fixing the initial data $\psi_i(u, 0)$, $i = 1, 2$, it has, locally, a unique solution. This solution gives, via Theorem 2.3 and Lemma 3.2, a minimal surface. Thus, we must find initial conditions so that this surface has the required properties. Observe that, if $f$ is a solution of the Björling problem, we have:

$$\phi(u,0) := \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right)(u,0) = \frac{1}{2}\big(\dot\beta(u) + i\dot\beta(u)\wedge V(u)\big). \tag{3.2}$$

Therefore, the initial data for the system is:

$$\psi(u,0) = A^{-1}(\beta(u))\phi(u,0). \tag{3.3}$$

Note that the initial condition implies $\frac{\partial f}{\partial u}(u,0) = \dot\beta(u)$. Hence, up to a constant determined by the constant of integration in Theorem 2.3, we have $f(u,0) = \beta(u)$. Also, the initial condition forces the choice of one of the determinations of $\psi_3^2 = -\psi_1^2 - \psi_2^2$.

Up to now, we have proved the existence of a local solution to the problem. Using compactness of $I$ and local uniqueness, we have existence and uniqueness of the solution when $\beta(I)$ is contained in a coordinate neighborhood, for $\varepsilon$ sufficiently small. Covering $I$ with a finite number of inverse images, via $\beta$, of coordinate neighborhoods and using (again) the uniqueness of the local problem, the result is proved for the general case. □

## 4. Examples in the space $\mathbb{H}^2 \times \mathbb{R}$

Let $\mathbb{H}^2$ be the hyperbolic plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$ endowed with the metric, of constant Gauss curvature $-1$, given by $g_{\mathbb{H}} = (dx^2 + dy^2)/y^2$. The hyperbolic plane $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left-invariant. Then the product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the product structure given by

$$(x,y,z) * (x',y',z') = (x'y + x, yy', z + z')$$

and the product metric $g = g_{\mathbb{H}} + dz^2$ is left-invariant. The Lie algebra of the infinitesimal isometries of $(\mathbb{H}^2 \times \mathbb{R}, g)$ admits the following bases of Killing vector fields

$$\begin{aligned}
X_1 &= \frac{(x^2 - y^2)}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\\
X_4 &= \frac{\partial}{\partial z}.
\end{aligned}$$

With respect to the metric $g$, an orthonormal basis of left-invariant vector fields is:

$$E_1 = y\frac{\partial}{\partial x}, \qquad E_2 = y\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

Also, the matrix $A$ is given by:

$$A = \begin{pmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the nonzero $L^k_{ij}$ are $L^1_{12} = -1$ and $L^2_{11} = 1$. Consequently, system (3.1) becomes:

$$\begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} - \overline{\psi_1}\psi_2 = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} + |\psi_1|^2 = 0. \end{cases} \tag{4.1}$$

Example 4.1 (The horizontal plane $z = c$). First of all, we consider the curve

$$\beta(u) = (\cos u, \sin u, c), \qquad u \in (0, \pi), c \in \mathbb{R},$$

and the unit vector field $V(u) = -E_3$. As $\dot{\beta}(u) = -\sin u E_1 + \cos u E_2$, it results that $g(\dot{\beta}, V) \equiv 0$ and, also, using the equations (3.2) and (3.3), we have that the initial data for the system (4.1) is:

$$\psi(u, 0) = \left( -\frac{(\sin u + i \cos u)}{2 \sin u}, \frac{(\cos u - i \sin u)}{2 \sin u}, 0 \right).$$

Thus, it follows that $\psi(u, v) = \psi(u, 0)$ and, integrating, we obtain the conformal immersion of the totally geodesic plane $z = c$ given by

$$f(u, v) = (e^v \cos u, e^v \sin u, c).$$

Example 4.2 (The helicoid). Consider the curve $\beta(u) = (0, 1, 2u)$ and the unit vector field $V(u) = \cos(2u)E_1 + \sin(2u)E_2$. As $\dot{\beta}(u) = 2E_3$, it results that $g(\dot{\beta}, V) \equiv 0$. Also, using the equations (3.2) and (3.3), we have that the initial data for the system (4.1) is:

$$\psi(u, 0) = (-i \sin(2u), i \cos(2u), 1)$$

and moreover $\psi_3 = (-\psi_1^2 - \psi_2^2)^{1/2}$. Consequently, the solution is given by

$$\begin{aligned} \psi_1(u, v) &= \frac{2i \sin(2u) - 2(\cos(2u) + \sin(2v)) \tan(2v)}{-2 + \sin(2u - 2v) - \sin(2u + 2v)}, \\ \psi_2(u, v) &= \frac{\sec(2v)[2i(\cos(2u) + \sin(2v)) + \sin(2u)\sin(4v)]}{2 - \sin(2u - 2v) + \sin(2u + 2v)}, \\ \psi_3(u, v) &= 1. \end{aligned} \tag{4.2}$$

After integration, we have the immersion of the minimal helicoid in $\mathbb{H}^2 \times \mathbb{R}$ described in [8], given by:

$$\begin{aligned} f_1(u, v) &= \frac{2 \sin(2u) \sin(2v)}{2 - \sin(2u - 2v) + \sin(2u + 2v)}, \\ f_2(u, v) &= \frac{2 \cos(2v)}{2 - \sin(2u - 2v) + \sin(2u + 2v)}, \qquad f_3(u, v) = 2u. \end{aligned}$$

## 5. Examples in the Heisenberg group $\mathbb{H}_3$

We now consider the Heisenberg group

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\},$$

equipped with the left-invariant metric given by

$$g = dx^2 + dy^2 + \left( \frac{1}{2} y\, dx - \frac{1}{2} x\, dy + dz \right)^2.$$

An orthonormal basis of left-invariant vector fields is given by:

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad E_3 = \frac{\partial}{\partial z}.$$

The matrix $A$ takes the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 1 \end{pmatrix}$$

and the nonzero $L_{ij}^k$ are:

$$\begin{aligned} L_{12}^3 &= \frac{1}{2}, & L_{21}^3 &= -\frac{1}{2}, \\ L_{13}^2 &= -\frac{1}{2}, & L_{31}^2 &= -\frac{1}{2}, \\ L_{23}^1 &= \frac{1}{2}, & L_{32}^1 &= \frac{1}{2}. \end{aligned}$$

Thus, the system (3.1) becomes:

$$\begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} + Re(\psi_2 \overline{\psi_3}) = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} - Re(\psi_1 \overline{\psi_3}) = 0. \end{cases} \tag{5.1}$$

Example 5.1 (Helicoids). We consider

$$\beta(u) = (\rho(u), 0, b) \quad \text{and} \quad V(u) = \frac{(\rho^2 - 2c)}{2\rho'} E_2 + \frac{\rho}{\rho'} E_3,$$

where $b, c \in \mathbb{R}$ and the real-valued function $\rho = \rho(u)$ satisfies:

$$\sqrt{(\rho')^2 - \rho^2} = \rho^2/2 - c.$$

Since $\dot{\beta}(u) = \rho'(u) E_1$, we obtain that $g(\dot{\beta}, V) \equiv 0$. Using (3.2) and (3.3), we have that the initial data for the system (5.1) is

$$\psi(u, 0) = \frac{1}{2}\big(\rho'(u), -i\rho(u), i\big(\rho(u)^2 - 2c\big)/2\big).$$

Consequently, $\psi_3 = (-\psi_1^2 - \psi_2^2)^{1/2}$ and

$$\psi(u,v) = \frac{1}{2}\big(\rho'(u)\cos v + i\rho(u)\sin v, \rho'(u)\sin v - i\rho(u)\cos v, i\big(\rho(u)^2 - 2c\big)/2\big).$$

Using the fact that:

$$\phi_1 = \psi_1, \qquad \phi_2 = \psi_2, \qquad \phi_3 = -\frac{y}{2}\psi_1 + \frac{x}{2}\psi_2 + \psi_3,$$

and:

$$\begin{cases} f_1(z) = 2Re\int_{z_0}^{z} \psi_1\, dz, \\ f_2(z) = 2Re\int_{z_0}^{z} \psi_2\, dz, \\ f_3(z) = 2Re\int_{z_0}^{z} \big(\psi_3 - \frac{f_2}{2}\psi_1 + \frac{f_1}{2}\psi_2\big)\, dz, \end{cases} \tag{5.2}$$

we obtain the minimal immersion:

$$f(u,v) = \big(\rho(u)\cos v, \rho(u)\sin v, cv + b\big).$$

Therefore, if $c \neq 0$ we have the parametrization of a helicoid, while, if $c = 0$ we obtain the horizontal plane $z = b$.

EXAMPLE 5.2 (Catenoid-type surface). We consider the curve in $\mathbb{H}_3$ given by:

$$\beta(u) = (g\cos l, g\sin l, \tilde{h}),$$

where $g = g(u)$, $l = l(u)$ and $\tilde{h} = \tilde{h}(u)$ are real-valued functions such that

$$g'^2 = \frac{g^2(g^4 - 16) - 4}{g^2 - 4}$$

and

$$\tilde{h}' = h = \sqrt{\frac{g^2+4}{g^2-4}}, \qquad l' = \frac{2h}{g^2+4},$$

with $g^2 > 4$. Let

$$V(u) = -\frac{(gg'\sin l + 2h\cos l)}{g(g^2+4)}E_1 + \frac{(gg'\cos l - 2h\sin l)}{g(g^2+4)}E_2 + \frac{2g'}{g(g^2+4)}E_3$$

be a unitary vector field. As

$$\dot{\beta}(u) = (g'\cos l - gl'\sin l)E_1 + (g'\sin l + gl'\cos l)E_2 + 2l'E_3,$$

it is easy to check that $g(\dot{\beta}, V) \equiv 0$ and

$$\dot{\beta} \wedge V = 2g\sin l E_1 - 2g\cos l E_2 + g^2 E_3.$$

Therefore, using (3.2) and (3.3), it follows that the initial data for system (5.1) is

$$\psi(u,0) = \frac{1}{2}(g'\cos l - gl'\sin l + 2ig\sin l, g'\sin l + gl'\cos l - 2ig\cos l, 2l' + ig^2).$$

Hence,

$$\begin{aligned}\psi_1(u,v) &= \frac{1}{2}\big(g'\cos(l+2v) - gl'\sin(l+2v) + 2ig\sin(l+2v)\big),\\ \psi_2(u,v) &= \frac{1}{2}\big(g'\sin(l+2v) + gl'\cos(l+2v) - 2ig\cos(l+2v)\big),\\ \psi_3(u,v) &= \frac{1}{2}(2l' + ig^2).\end{aligned}$$

After integration, we have the catenoid-type minimal surface given by

$$f(u,v) = \big(g\cos(l+2v), g\sin(l+2v), \widetilde{h}\big).$$

## 6. Final comments

If the ambient space is $\mathbb{R}^3$ with the flat metric, the solution of the Björling problem can be given by an explicit formula. This fact has been used to prove a *reflection principle* and a nice application of this is the characterization of the helicoid as the unique ruled minimal surface, besides the plane (see [3, 11]).

In our case, the partial differential equations involved are more complicated than the Cauchy–Riemann ones, and we were not able to find an "explicit" formula for the solution of the Björling problem. It is not clear that a reflection principle holds for a generic three-dimensional Lie group. For example, in the case of the Heisenberg group $\mathbb{H}_3$, there are many minimal surfaces ruled by translated of 1-parameter subgroups. Such surfaces were classified in [2] and [5] (see also [6] and [9] for the constant mean curvature case). If we consider the developable ones, i.e., the ones with Gauss map of rank 1, then they are (up to isometries of the ambient space) the graphs of the following functions:

$$f(x,y) = \begin{cases} \frac{xy}{2} + k\big[\ln\big(y + \sqrt{1+y^2}\big) + y\sqrt{1+y^2}\big] \\ \text{or} \\ 2ky - \frac{xy}{2}, \quad k \in \mathbb{R}. \end{cases}$$

Since these are complete graphs, the Bernstein Theorem does not hold. A classification of complete minimal graphs has been recently given in [4], in terms of the generalized Hopf differentials introduced in [1].

We could also consider minimal surfaces of a Lie group, in particular of $\mathbb{H}_3$, ruled by geodesics. Very little is known for such surfaces and the classification problem seems to be more difficult. However, our feeling is that this is the right context for a reflection principle.

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