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NONVANISHING DERIVATIVES AND THE MACLANE CLASS $\mathcal{A}$

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ABSTRACT. Let $k \geq 2$ and let f be meromorphic in the unit disc Δ, such that $f(z)f^{(k)}(z) \neq 0$ for all $z \in \Delta$ and the poles of f in Δ have bounded multiplicities. Then f has asymptotic values on a dense subset of $\partial\Delta$.

1. Introduction

Let $\Delta = B(0,1)$ denote the unit disc in the complex plane and let $\mathbb{T} = \partial\Delta$ be the boundary circle. A meromorphic function $f : \Delta \to \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is said to have the asymptotic value $a \in \mathbb{C}^*$ at $\zeta \in \mathbb{T}$ if there exists a path $z(t) : [0,\infty) \to \Delta$ such that

$$z(t) \to \zeta \quad \text{and} \quad f(z(t)) \to a \quad \text{as } t \to +\infty.$$

The MacLane class $\mathcal{A}$ is the set of all analytic functions f on D such that f has asymptotic values at each ζ in a dense subset E_f of $\mathbb{T}$ [14], [15]. The corresponding class of meromorphic functions is denoted by $\mathcal{A}_m$ [1]. Note that it is common practice to exclude constant functions from the classes $\mathcal{A}$ and $\mathcal{A}_m$, but for the present paper it is convenient to admit them. Our starting point is the following theorem [2, Theorem 2(a)].

THEOREM 1.1 ([2]). *Let f be analytic on Δ such that ff'' has no zeros in Δ. Then f'/f, $\log f$ and f are all in $\mathcal{A}$.*

The corresponding study of meromorphic functions in the plane with non-vanishing derivatives has a long history, going back at least as far as Pólya [16]. In a landmark paper on the value distribution of meromorphic functions

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and their derivatives [9], Hayman conjectured that if f is meromorphic in the plane and f and $f^{(k)}$ have no zeros for some $k \geq 2$, then

$$f(z) = e^{az+b} \quad \text{or} \quad f(z) = (az+b)^{-n}, \tag{1}$$

where $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. For entire functions and $k = 2$, this conjecture was proved by Hayman [9]. Theorem 1.1 may be regarded as an analogue for the unit disc of Hayman's result. For $k \geq 3$ and f again entire, Hayman's conjecture was proved by Clunie [4] using what is now called the Tumura–Clunie method [10], [18]. Finally, Hayman's conjecture was established for meromorphic functions for $k \geq 3$ by Frank [5], [7], and for $k = 2$ by Langley [13].

Associated with these results in the plane is a normal family analogue for plane domains in the spirit of the Bloch hypothesis [20]. The following theorem is due to Bergweiler and Langley [3], but was proved by Schwick [17] for families of analytic functions: both results rely on the Pang–Zalcman rescaling method [19], [20].

Theorem 1.2 ([3]). *Let D be a domain in $\mathbb{C}$, let $k \geq 2$ be an integer, and let $\mathcal{F}$ be the family of all meromorphic functions f on D such that f and $f^{(k)}$ have no zeros on D. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D.*

The main result of the present paper is the following theorem.

Theorem 1.3. *Let $k \geq 2$ and let f be meromorphic in $\Delta = B(0,1)$, such that $f(z)f^{(k)}(z) \neq 0$ for all $z \in \Delta$ and the poles of f in Δ have bounded multiplicities. Then $1/f \in \mathcal{A}$ and $f \in \mathcal{A}_m$.*

The hypothesis on the multiplicities of the poles may not really be needed in Theorem 1.3, but is indispensable for the present method in that it implies a separation between distinct poles of f which is sufficient for much of the machinery of [2] to be applicable, with appropriate modifications, to f'/f.

2. Preliminary lemmas

The following lemma is straightforward but we give a proof for completeness.

Lemma 2.1. *Let $\mathcal{F}$ be a normal family of meromorphic functions on the unit disc Δ. Let d, c_1, c_2 be real numbers with $0 < d < 1$ and $0 \leq c_1 < c_2$. Then there exist positive real numbers b_j such that the following properties hold for all $u \in \mathcal{F}$.*

(i) *If $z_1 \in B(0,d)$ and $|u(z_1)| \leq c_1$, we have $|u(z)| \leq c_2$ for all $z \in B(z_1, b_1)$.*

(ii) *For any zero z_1 of u in $B(0,d)$, there are no zeros z of u which satisfy $0 < |z - z_1| < b_2 s$, where $s = \min\{1, |u'(z_1)|\}$.*

Proof. Part (i) follows simply from the equicontinuity of $\mathcal{F}$. For part (ii), let $z_1 \in B(0,d)$ be a zero of u and apply (i) with $c_1 = 0, c_2 = 1$. This gives a positive constant B_1, independent of u, such that

$$|u(z)| \le 1 \quad \text{for } |z - z_1| \le 2B_1.$$

Assume now that z_2 is a zero of u with $0 < |z_2 - z_1| \le B_1$. Then

$$|h(z)| \le \frac{1}{(2B_1)(B_1)} \quad \text{on } \partial B, \text{ where } h(z) = \frac{u(z)}{(z - z_1)(z - z_2)}$$

is analytic on the disc $B = B(z_1, 2B_1)$. It follows that

$$|u'(z_1)| = |(z_2 - z_1)h(z_1)| \le \frac{|z_2 - z_1|}{2B_1^2},$$

which gives a lower bound for $|z_2 - z_1|$ and completes the proof. □

The next lemma is an analogue for the unit disc of a standard result in the plane setting [12, Lemma 7.7].

LEMMA 2.2. *Let k and m be positive integers, let $A_0, \dots, A_{k-1}$ be meromorphic functions on the unit disc Δ, and assume that the equation*

$$w^{(k)} + A_{k-1}w^{(k-1)} + \cdots + A_0 w = 0 \tag{2}$$

has a fundamental set $f_1, \dots, f_k$ of solutions meromorphic in Δ and satisfying

$$T(r, f_j) = O(1-r)^{-m}$$

as $r \to 1$ for each j. Then

$$m(r, A_p) = O\left(\log \frac{1}{1-r}\right) \tag{3}$$

as $r \to 1$ for each p.

Proof. This uses induction on k and the familiar reduction of order procedure. If $k = 1$, then the result follows immediately from [10, Lemma 2.3], applied to f_1. Assume now that $k \ge 2$ and that the result has been proved for $k - 1$, and write $w = vf_1$ and $u = v'$. Then the functions

$$g_j = \left(\frac{f_j}{f_1}\right)', \quad j = 2, \dots, k,$$

are linearly independent solutions of the equation

$$u^{(k-1)} + B_{k-2}u^{(k-2)} + \cdots + B_0 u = 0,$$

where

$$B_{k-2} = k\frac{f_1'}{f_1} + A_{k-1}, \qquad \dots, \qquad B_0 = k\frac{f_1^{(k-1)}}{f_1} + \cdots + A_1. \tag{4}$$

The induction hypothesis gives (3) for $p=0,\dots,k-2$, but with A_p replaced by B_p, and (4) then leads to (3) for $p=k-1,\dots,1$. Finally, (3) for $p=0$ follows from dividing (2) by w. □

3. Estimates for logarithmic derivatives

Throughout this section, let f be meromorphic on the unit disc Δ such that f and $f^{(k)}$ have no zeros there, for some $k\geq 2$. Let

$$\psi(z)=\frac{f'(z)}{f(z)}. \tag{5}$$

LEMMA 3.1. *There exists $c_1>0$ such that*

$$\rho(\psi(z))=\frac{|\psi'(z)|}{1+|\psi(z)|^2}\leq\frac{c_1}{(1-|z|)^2}\quad \textit{on } \Delta. \tag{6}$$

Furthermore, there exists $\delta\in(0,1/2)$ such that, for all $z_0\in\Delta$,

$$(1-|z_0|)|\psi(z_0)|\geq 2 \quad\Rightarrow\quad (1-|z_0|)|\psi(z)|\geq 1 \quad \textit{for } z\in B\big(z_0,2\delta(1-|z_0|)\big). \tag{7}$$

Finally, suppose in addition that the poles of f have bounded multiplicities. Then δ may be chosen so that for each $z_0\in\Delta$ the function f has at most one pole, possibly multiple, in $B(z_0,2\delta(1-|z_0|))$.

Proof. Let $z_0\in\Delta$ and set

$$g(z)=f\big(z_0+(1-|z_0|)z\big),\quad G(z)=\frac{g'(z)}{g(z)}=(1-|z_0|)\psi\big(z_0+(1-|z_0|)z\big).$$

Then g belongs to the family $\mathcal{H}$ of functions h which are meromorphic on Δ with $hh^{(k)}\neq 0$ there, and G belongs to the family $\{h'/h : h\in\mathcal{H}\}$, which is normal by Theorem 1.2. Thus, $\rho(G(0))\leq c_1$ for some c_1 independent of f and z_0, which implies (6). Now the existence of δ satisfying (7) follows from Lemma 2.1(i) applied to $H=1/G$ with $z_1=0$. Finally, if the poles of f have bounded multiplicities, then there exists $c_2>0$ such that $H(z)=0$ implies that $|H'(z)|\geq c_2$. If u_1,u_2 are distinct poles of f in $B(z_0,(1-|z_0|)/2)$, define v_1,v_2 by $u_j=z_0+(1-|z_0|)v_j$. Then v_1,v_2 are distinct zeros of H in $B(0,1/2)$, and it follows from Lemma 2(ii) that $|v_1-v_2|\geq c_3>0$, where c_3 is independent of z_0. This proves Lemma 3.1. □

Observe next that (6) gives, in the terminology of [10, p. 12],

$$A(r,\psi)=O\left(\frac{1}{1-r}\right)^3,\quad T(r,\psi)=O\left(\frac{1}{1-r}\right)^2 \tag{8}$$

as $r\to 1$. It then follows using [10, p. 36] that

$$m(r,\psi'/\psi)=O\left(\log\frac{1}{1-r}\right),\qquad T\big(r,\psi^{(j)}\big)=O\left(\frac{1}{1-r}\right)^2 \tag{9}$$

as $r \to 1$, for each $j \in \mathbb{N}$.

PROPOSITION 3.1. *If $k \geq 3$, then*

$$T(r,\psi) = O\left(\log \frac{1}{1-r}\right) \tag{10}$$

as $r \to 1$. The same conclusion holds for $k = 2$ if, in addition,

$$\overline{N}(r,f) = O\left(\log \frac{1}{1-r}\right) \tag{11}$$

as $r \to 1$.

We make several remarks concerning Proposition 3.1. First, it will be shown in Section 5 that (11) automatically holds if the poles of f have bounded multiplicities. On the other hand, it seems likely that Proposition 3.1 holds for $k = 2$ without the additional hypothesis (11), although the present method does not suffice for this.

Next, the case $k \geq 3$ is essentially not new, and may be derived directly from the methods of [5], [7]: however, it is much simpler to do this once the estimates (8) and (9) are available, and we will outline the proof in the next section.

4. Proof of Proposition 3.1

Let f satisfy the hypotheses of Proposition 3.1 for some $k \geq 2$, and define ψ by (5). We first dispose of the case $k = 2$. If f is given by (1), then the estimate (10) is obvious, while in the contrary case (10) follows at once from (9), (11), and [9, Theorem 4] (see also [10, p. 60]).

Assume henceforth that $k \geq 3$. The notation $S(r)$ will be used to denote any function $S : [0,1) \to [0,\infty)$ which satisfies

$$S(r) = O\left(\log \frac{1}{1-r}\right)$$

as $r \to 1$. Then (9) gives

$$m\big(r, \psi^{(j)}/\psi\big) = S(r)$$

for each $j \in \mathbb{N}$. Denote by Λ the collection of meromorphic functions λ on Δ such that

$$T(r,\lambda) = S(r).$$

Then Λ is a field closed under differentiation.

Frank's method [5], [7] depends on properties of the Wronskian determinant [12, Section 1.4]. Define analytic functions f_j, g, h and w_j on Δ by

$$f_j(z) = z^{j-1}, \qquad g^k = \frac{f}{f^{(k)}}, \qquad h = -\left(\frac{f'}{f}\right)g = -\psi g, \tag{12}$$

$$w_j = f_j' g + f_j h.$$

Then we have, with c_k a nonzero constant,

$$W(f_1,\dots,f_k,f)=c_k f^{(k)}=c_k f g^{-k}$$

and so

$$\frac{c_k}{(fg)^k}=W(f_1/f,\dots,f_k/f,1)=(-1)^k W\big((f_1/f)',\dots,(f_k/f)'\big).$$

Multiplying through by $(fg)^k$ then gives

$$(-1)^k c_k = W\big((f_1/f)'(fg),\dots,(f_k/f)'fg\big)=W(w_1,\dots,w_k).$$

It follows that $w_1,\dots,w_k$ are linearly independent solutions of an equation (2), in which the coefficients A_p are analytic in Δ and $A_{k-1}\equiv 0$. Moreover, we have $A_p\in\Lambda$, by (9), (12), and Lemma 2.2. The key to Frank's method is then to observe that there is a system of equations

$$T_\mu(G)=S_\mu(H)=\sum_{j=0}^{k-\mu} c_{j,\mu}H^{(j)},\quad \mu=0,\dots,k-1, \tag{13}$$

with the following properties [3, Lemma 2.4] (see also [6, Lemma 6] and [8, Lemma C]).

(i) The system (13) is solved by $G=g, H=h$.

(ii) The T_μ and S_μ are homogeneous linear differential operators, and their coefficients are rational functions in the A_p and their derivatives and so are in Λ.

(iii) If G,H are any solutions of (13), then the functions

$$f_1'G+f_1H,\dots,f_k'G+f_kH$$

are solutions of the equation (2) and so linear combinations of the w_j.

(iv) Taking $\mu=k-1$ gives

$$S_{k-1}(H)=H'=T_{k-1}(G)=U(G)=-(k-1)G''/2-A_{k-2}G/k. \tag{14}$$

There are then two cases to consider (for the details see [3, pp. 358–361]). In the first case, suppose that we have $c_{0,\nu}\not\equiv 0$ for at least one $\nu\in\{0,\dots,k-1\}$. Then (12), (13), and (14) give

$$h=-\psi g=(c_{0,\nu})^{-1}\left(T_\nu(g)-\sum_{j=1}^{k-\nu} c_{j,\nu}\frac{d^{j-1}}{dz^{j-1}}(U(g))\right)=V(g), \tag{15}$$

and g solves a system of equations

$$U(g)=\frac{d}{dz}(V(g)),\qquad S_\mu(V(g))=T_\mu(g),\quad \mu=0,\dots,k-2, \tag{16}$$

with coefficients in Λ. If the dimension of the solution space of (16) is 1, then a standard reduction procedure [11, p. 126] shows that g solves a first order homogeneous linear differential equation with coefficients in Λ, in which case g'/g is in Λ and therefore so is ψ, by (15). On the other hand, if the system (16) has a solution G with G/g nonconstant, then G and $H=V(G)$ solve (13). Hence, the functions $f_j'G+f_jH$ are solutions of (2) and so linear combinations of the w_p, and so there are polynomials g_j with

$$f_j'G+f_jH=g_j'g+g_jh$$

for $j=1,\dots,k$. The standard argument due to Frank [3, p. 360] (see also [6, p. 424]) then shows that this system of linear equations has rank 3, and $\psi=-h/g$ is a rational function and so obviously satisfies (10).

In the second case, we have $c_{0,\mu}\equiv 0$ for each μ in the system (13), which is then solved by taking $G=0, H=1$. Hence, the functions $f_j'G+f_jH=f_j$ are solutions of (2), and so the w_j are rational functions, from which it follows that so is ψ.

5. Proof of Theorem 1.3

Let f satisfy the hypotheses of Theorem 1.3 and define ψ by (5). We follow the construction of [2], but with modifications to take account of the poles of ψ. Denote positive constants by c_j, d_j. Choose a small positive δ as in Lemma 3.1, and define t, r_n and q_n by setting, for $n=1,2,\dots,$

$$t=1-\frac{\delta}{8},\qquad r_n=1-t^n,\qquad q_n=\left[\frac{16\pi r_n}{\delta t^n}\right]+1,\qquad \theta_n=\frac{2\pi}{q_n}, \tag{17}$$

where $[x]$ denotes the greatest integer not exceeding x. The logarithmic rectangles $B_{n,q}$ are then defined, for $n=1,2,\dots$ and $q=0,\dots,q_n-1$, by

$$B_{n,q}=\{re^{i\theta}: r_n\le r\le r_{n+1}, q\theta_n\le\theta\le(q+1)\theta_n\}. \tag{18}$$

Following [2] we obtain, from (18),

$$\operatorname{diam} B_{n,q}\le r_{n+1}-r_n+r_n\theta_n<\frac{\delta t^n}{4}<\frac{\delta(1-r_{n+1})}{2}. \tag{19}$$

Thus, (19) implies that

$$z_0\in B_{n,q}\quad\Rightarrow\quad B_{n,q}\subseteq B\left(z_0,\frac{\delta(1-|z_0|)}{2}\right). \tag{20}$$

It now follows from Lemma 3.1 and (20) that

(21) f has at most one pole, possibly multiple, in each $B_{n,q}$.

By (21), the number of distinct poles z of f satisfying $r_n\le|z|\le r_{n+1}$ is at most $q_n=O(t^{-n})$. For $r_n\le r\le r_{n+1}$ we deduce using (17) that

$$\overline{n}(r,f)\le c_1(1+t^{-1}+\cdots+t^{-n})\le c_2t^{-n}\le\frac{c_3}{1-r_n}\le\frac{c_3}{1-r}. \tag{22}$$

This leads at once to (11), and proves the first assertion made following Proposition 3.1.

5.1. An exceptional set. Let $w_1, w_2, \ldots$ be the distinct poles of f in the set $\{z \in \mathbb{C} : 1/4 \le |z| < 1\}$, arranged in order of nondecreasing modulus. Let σ_1 be small and positive and set

$$\Omega_j = \left\{ z \in \mathbb{C} : \left| \arg \frac{z}{w_j} \right| \le \sigma_1 (1 - |w_j|)^2, \left| \log \left| \frac{z}{w_j} \right| \right| \le \sigma_1 (1 - |w_j|)^2 \right\},$$

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j. \tag{23}$$

Then there exist small positive constants σ_2, σ_3 such that

$$\sigma_2 \le \frac{|z - w_j|}{|w_j|(1 - |w_j|)^2} \le \sigma_3 \quad \text{for all } z \in \partial\Omega_j. \tag{24}$$

By choosing σ_1 small enough, we may therefore assume in view of Lemma 3.1 that the Ω_j are pairwise disjoint.

Lemma 5.1. *We have*

$$\log |\psi(z)| \le O\left(\frac{1}{1-r} \log \frac{1}{1-r} \right) \quad \textit{for } |z| = r \ge \frac{1}{2}, \quad z \notin \operatorname{int} \Omega. \tag{25}$$

Proof. Let z be as in (25) and apply the Poisson–Jensen formula to ψ in $B(0, R)$, where $1 - R = (1 - r)/2$. Ignoring the contribution from the zeros of ψ, which in any case is nonpositive, and observing that each pole of f is a simple pole of ψ, we obtain

$$\log |\psi(z)| \le \left(\frac{R + r}{R - r} \right) \big(T(R, \psi) + T(R, 1/\psi) \big) + \sum_{|w_j| < R} \log \frac{4}{|z - w_j|} + O(1).$$

But $|z - w_j| \ge c_4 (1 - r)^2$ for all $j \in \mathbb{N}$, by (24), and so (25) follows using (10) and (22). ☐

5.2. A growth estimate for $1/f$.

Lemma 5.2. *We have, for* $|z| = r \ge \frac{1}{2}$,

$$\log^+ \log^+ \frac{1}{|f(z)|} = O\left(\frac{1}{1-r} \log \frac{1}{1-r} \right). \tag{26}$$

Proof. Let $z_0 = r_0 e^{i\theta_0}$ with $3/4 \le r_0 < 1$ and $\theta_0 \in [0, 2\pi)$ and define the closed set S_0 as follows. First, take the line segment L_0 from $(3/4)e^{i\theta_0}$ to z_0, and let M_0 be the component of $L_0 \cup \Omega$ which contains z_0. Finally, define S_0 by $S_0 = M_0 \setminus \operatorname{int} \Omega$. Then S_0 is a connected subset of $B(0, R)$, where $1 - R = (1 - r_0)/2$, using the fact that the Ω_j are pairwise disjoint in (23). By construction the total arc length of S_0 is at most c_5, and since (25) holds on S_0, integration of $-\psi$ gives (26) on S_0, with r replaced by r_0. But z_0 either lies on S_0 or in the interior of some Ω_j which meets L_0, in which case $\partial\Omega_j \subseteq S_0$. Since $1/f$ is analytic on Δ, the lemma follows. ☐

5.3. Application of Harnack's inequality. Fix a small positive constant ε and a large positive integer N. We modify the classification of [2] as follows. A box $B_{n,q}$ will be called *bad* if $n \geq N$ and there exists

$$z_0 \in B_{n,q} \setminus \Omega \quad \text{with } \log|\psi(z_0)| > \frac{12}{(1-|z_0|)^{1-\varepsilon}}. \tag{27}$$

Lemma 5.3. *Let $B_{n,q}$ be a bad box. Then*

$$\log|\psi(z)| \geq \frac{1}{(1-|z|)^{1-\varepsilon}} \quad \textit{for all } z \in B_{n,q}. \tag{28}$$

Proof. Take z_0 satisfying (27). By Lemma 3.1, (27), and the fact that N is large, we have

$$|\psi(z)| \geq \frac{1}{1-|z_0|} \quad \text{for all } z \in B\big(z_0, 2\delta(1-|z_0|)\big), \tag{29}$$

and there is at most one pole w^* of ψ in $B(z_0, 2\delta(1-|z_0|))$. If there is no such pole w^*, or if $|w^* - z_0| \geq \delta(1-|z_0|)$, set

$$h(z) = \log|\psi(z)|, \qquad U = B\big(z_0, \delta(1-|z_0|)\big).$$

On the other hand, if $|w^* - z_0| < \delta(1-|z_0|)$ set

$$h(z) = \log\left|\frac{\psi(z)(z-w^*)}{\delta}\right|, \qquad U = B\big(z_0, 2\delta(1-|z_0|)\big).$$

In either case, we have $h(z) > 0$ on ∂U, using (29), and the function h is positive and harmonic on U. Furthermore, the fact that $z_0 \notin \Omega$ gives

$$h(z_0) \geq \frac{12}{(1-|z_0|)^{1-\varepsilon}} - c_6 \log\frac{1}{1-|z_0|} - c_6 \geq \frac{6}{(1-|z_0|)^{1-\varepsilon}},$$

again since N is large. Applying Harnack's inequality now yields

$$h(z) \geq \frac{2}{(1-|z_0|)^{1-\varepsilon}} \quad \text{for } |z-z_0| < \frac{\delta(1-|z_0|)}{2},$$

from which (28) follows using (20). □

For $\theta \in [0, 2\pi]$, let

$$R_\theta = \{re^{i\theta} : 0 \leq r < 1\}.$$

For $n = N, N+1, \dots$, let E_n be the union of the bad boxes $B_{n,q}$ and let

$$F_n = \{\theta \in [0,2\pi] : r_n e^{i\theta} \in E_n\} = \{\theta \in [0,2\pi] : R_\theta \cap E_n \neq \emptyset\},$$

using (18). Then (10) and (28) give

$$c_7 \log\frac{1}{1-r_n} \geq m(r_n, \psi) \geq \frac{1}{2\pi}\int_{F_n} \log^+|\psi(r_n e^{i\theta})|\, d\theta \geq \frac{|F_n|}{2\pi(1-r_n)^{1-\varepsilon}},$$

using $|X|$ for the Lebesgue measure of $X \subseteq \mathbb{R}$, and so we obtain, recalling (17),

$$|F_n| \le c_8(1-r_n)^{1-\varepsilon} \log \frac{1}{1-r_n} = c_9 n t^{n(1-\varepsilon)}.$$

Next, for $n \ge N$ let E_n^* be the union of all those Ω_j which meet the half-open annulus given by $r_n \le |z| < r_{n+1}$, and let

$$F_n^* = \{\theta \in [0,2\pi] : R_\theta \cap E_n^* \ne \emptyset\}.$$

It follows from (21) and (23) that the number of Ω_j which make up E_n^* is not greater than $q_{n-1} + q_n + q_{n+1} = O(t^{-n})$, and that

$$|F_n^*| \le d_1(1-r_n)^2 t^{-n} \le d_2 t^n.$$

Now set

$$\widetilde{E}_n = E_n \cup E_n^*, \qquad \widetilde{F}_n = F_n \cup F_n^*,$$

for $n \ge N$, so that

$$|\widetilde{F}_n| \le d_3 n t^{n(1-\varepsilon)}, \qquad \sum_{n=N}^{\infty} |\widetilde{F}_n| < \infty. \tag{30}$$

Then

$$\widetilde{F} = \{\theta \in [0,2\pi] : R_\theta \text{ meets infinitely many } \widetilde{E}_n\} = \bigcap_{m=N}^{\infty} \bigcup_{n=m}^{\infty} \widetilde{F}_n$$

has Lebesgue measure $|\widetilde{F}| = 0$. Set $\widetilde{E}_{N-1} = \Delta$, $\widetilde{F}_{N-1} = [0,2\pi]$ and

$$G_n = \{\theta \in [0,2\pi] : R_\theta \cap \widetilde{E}_{n-1} \ne \emptyset,\ R_\theta \cap \widetilde{E}_m = \emptyset \text{ for all } m \ge n\}$$

for $n \ge N$. Then the G_n are pairwise disjoint with union $[0,2\pi] \setminus \widetilde{F}$, and for $n > N$ we have

$$G_n \subseteq \widetilde{F}_{n-1} \quad \text{and} \quad |G_n| \le |\widetilde{F}_{n-1}| \le d_3 n t^{n(1-\varepsilon)} \tag{31}$$

by (30).

Let $n \ge N$ and $\theta \in G_n$. Then we estimate $1/f(z)$ on R_θ as follows. For $z \in R_\theta$ with $|z| \ge r_n$, we have $z \notin \widetilde{E}_m = E_m \cup E_m^*$ for all $m \ge n$, so that $z \notin \Omega$ and

$$\log|\psi(z)| \le \frac{12}{(1-|z|)^{1-\varepsilon}},$$

because otherwise z would lie in a bad box. In view of (26), this gives

$$\log \frac{1}{|f(z)|} \le \exp\left(\frac{d_4}{1-r_n} \log \frac{1}{1-r_n}\right) + \exp\left(\frac{12}{(1-r)^{1-\varepsilon}}\right)$$

for $z \in R_\theta, |z| = r > r_n$. Using (26) again, the fact that N is large, and the inequalities

$$x + y \le xy \quad (x,y \ge 2), \qquad (a+b)^{1+\varepsilon} \le (2a)^{1+\varepsilon} + (2b)^{1+\varepsilon} \quad (a,b>0),$$

we obtain

$$\begin{aligned}
I_\theta &= \int_0^1 \left(\log^+ \log^+ \frac{1}{|f(z)|}\right)^{1+\varepsilon} dr \\
&\le d_5 + \int_{\frac{1}{2}}^{r_n} \left(\frac{d_4}{1-r} \log \frac{1}{1-r}\right)^{1+\varepsilon} dr \\
&\quad + \int_{r_n}^1 \left(\frac{d_4}{1-r_n} \log \frac{1}{1-r_n} + \frac{12}{(1-r)^{1-\varepsilon}}\right)^{1+\varepsilon} dr \\
&\le \frac{d_6}{(1-r_n)^\varepsilon} \left(\log \frac{1}{1-r_n}\right)^{1+\varepsilon} + \int_{r_n}^1 \left(\frac{2d_4}{1-r_n} \log \frac{1}{1-r_n}\right)^{1+\varepsilon} dr \\
&\quad + \int_{r_n}^1 \left(\frac{24}{(1-r)^{1-\varepsilon}}\right)^{1+\varepsilon} dr \\
&\le \frac{d_7}{(1-r_n)^\varepsilon} \left(\log \frac{1}{1-r_n}\right)^{1+\varepsilon} + \int_{r_n}^1 \left(\frac{d_8}{(1-r)^{1-\varepsilon^2}}\right) dr \\
&\le \frac{d_7}{(1-r_n)^\varepsilon} \left(\log \frac{1}{1-r_n}\right)^{1+\varepsilon} + d_9(1-r_n)^{\varepsilon^2} \\
&\le d_{10} n^{1+\varepsilon} t^{-n\varepsilon}.
\end{aligned}$$

Recalling (31) and the fact that $\widetilde{F}$ has measure 0, we arrive finally at

$$\begin{aligned}
I &= \int\int_\Delta \left(\log^+ \log^+ \frac{1}{|f(z)|}\right)^{1+\varepsilon} dx\, dy \\
&\le \sum_{n=N}^\infty \int_{G_n} I_\theta \, d\theta \\
&\le d_{11} + \sum_{n=N+1}^\infty d_3 n t^{n(1-\varepsilon)} \cdot d_{10} n^{1+\varepsilon} t^{-n\varepsilon} \\
&= d_{11} + d_3 d_{10} \sum_{n=N+1}^\infty n^{2+\varepsilon} t^{n(1-2\varepsilon)} < \infty,
\end{aligned}$$

from which it follows that $1/f \in \mathcal{A}$ [2, Lemma 4]. This proves Theorem 1.3.

We conclude the paper by observing that Proposition 3.1 and Theorem 1.3 together answer a question from [2]. Suppose that f is analytic in Δ and f and $f^{(k)}$ have no zeros for some $k \ge 3$. Then $\psi = f'/f$ satisfies (10) and $\psi \in \mathcal{A}$ by [2, Lemma 2(a)]. Also, Theorem 1.3 gives $1/f \in \mathcal{A}$ and so f and $\log f$ are in $\mathcal{A}$. If, in addition, f' has no zeros, then f satisfies the hypotheses of [2, Lemma 3(b)].

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