Illinois Journal of Mathematics
Volume 53, Number 1, Spring 2009, Pages 339–347
S 0019-2082

DUALITY, UNIFORMITY, AND LINEAR LOCAL CONNECTIVITY

SHANSHUANG YANG

ABSTRACT. Using Väisälä's metric duality on joinability of sets, we show, among other things, that in $\mathbb{R}^3$ if the complementary domains of a surface are LLC, they are also uniform. As an application, we show that an Ahlfors regular topological sphere that admits a quasiconformal reflection is quasisymmetrically equivalent to the standard sphere.

1. Introduction and main results

In this paper, we use the metric duality theory, developed by Väisälä [9], to study the connection between uniformity and LLC properties of domains in the Euclidean space $\mathbb{R}^3$. In particular, we investigate under what conditions the following well known implications of domain properties can be reversed for domains in $\mathbb{R}^3$:

$$\text{Uniform} \Rightarrow \text{Sobolev extension} \Rightarrow \text{QED} \Rightarrow \text{LLC}. \tag{1}$$

In [12], we constructed a domain that is the image of the upper half space under a global homeomorphism of $\mathbb{R}^3$ and is a Sobolev extension domain (and hence LLC). But, it is not uniform. This example shows that the above implications cannot be reversed under the strongest possible topological conditions. Other geometric conditions are needed. As noted in [12, 3.11], the complementary domain D^* of the constructed domain D is not linearly locally connected (or LLC). A natural question to ask is if both D and D^* are LLC, are they also uniform? We give a positive answer to this question.

In order to formulate the main results, we need the following preliminaries. The standard Euclidean n-space ($n \geq 2$) will be denoted by $\mathbb{R}^n$ and its one point compactification by $\bar{\mathbb{R}}^n$. Let $B(a, r)$ denote the ball centered at a of

Received September 3, 2008; received in final form February 13, 2009.
2000 *Mathematics Subject Classification.* 30C65, 55M05.

radius r. The boundary and closure of a set A will be denoted by ∂A and $\bar{A}$, respectively.

DEFINITION 1.1. A domain D in $\bar{\mathbb{R}}^n$ $(n \geq 2)$ is said to be *uniform* if there exists a constant $c = c(D)$, $1 \leq c < \infty$, such that each pair of points $x_1, x_2 \in D \cap \mathbb{R}^n$ can be joined by a continuum (or, equivalently, a curve) β in D for which

$$\text{dia}(\beta) \leq c|x_1 - x_2| \quad \text{and} \quad \min_{j=1,2} |x_j - x| \leq c d(x, \partial D) \tag{2}$$

for each $x \in \beta$. Here $\text{dia}(\beta)$ denotes the diameter of β and $d(x, \partial D)$ the distance from x to the boundary ∂D.

DEFINITION 1.2. For $c \geq 1$, a set A in $\bar{\mathbb{R}}^n$ is said to be c-LLC$_1$ if for each finite point $a \in \mathbb{R}^n$ and $r > 0$, each pair of points in $A \cap B(a, r)$ can be joined by a continuum in $A \cap B(a, cr)$. Similarly, $A \subset \bar{\mathbb{R}}^n$ is said to be c-LLC$_2$ if for each finite point $a \in \mathbb{R}^n$ and $r > 0$, each pair of points in $A \setminus \bar{B}(a, cr)$ can be joined by a continuum in $A \setminus \bar{B}(a, r)$. Finally, $A \subset \bar{\mathbb{R}}^n$ is said to be *linearly locally connected* (or *LLC*) if it is both c-LLC$_1$ and c-LLC$_2$ for some c.

Uniform domains and LLC domains (along with several other classes of domains such as Sobolev extension domains and QED domains) have played important roles in geometric function theory and PDE theory, largely due to the fact that certain functions defined on these domains have special properties such as extension property and injectivity. These classes of domains have been extensively studied in recent years. Since we will not deal with Sobolev extension domains or QED domains directly in this paper, we omit the definitions here and refer the reader to [5], [13], [8] for further definitions and discussions of various related classes of domains.

The main results of this paper can be stated as follows.

THEOREM 1.1. *Let D and D^* be complementary domains in $\bar{\mathbb{R}}^3$ with trivial homology groups and such that $\partial D = \partial D^*$. Then D and D^* are uniform if and only if D and D^* are LLC.*

THEOREM 1.2. *Let D and D^* be complementary domains in $\bar{\mathbb{R}}^3$ with trivial homology groups and such that $\partial D = \partial D^*$. Let D be locally collared along ∂D. Then D and D^* are uniform if and only if ∂D is LLC.*

A domain D is said to be *locally collared* along ∂D if each point $x \in \partial D \cap \mathbb{R}^n$ has a neighborhood U such that $U \cap \bar{D}$ is homeomorphic to the intersection of a ball and the closure of the upper half space.

Before proving Theorems 1.1 and 1.2 in Section 3, we recall and reformulate in Section 2 Väisälä's metric duality results on joinability and LLC property in the forms as they are needed in this paper. As applications, in Section 4, we study uniformity of higher orders, quasiconformal reflection domains, and quasisymmetric parametrizations of topological spheres.

2. Metric duality

In this section, we recall Väisälä's metric duality theory for joinability, which plays an essential role in this paper.

2.1. Topological and algebraic terminology. First, we need to establish some topological and algebraic terminology. For a set $X \subset \bar{\mathbb{R}}^n$, we let $H_p(X)$ and $H^p(X)$ denote, respectively, the reduced singular homology groups and reduced Čech cohomology groups of X with coefficients in a fixed nontrivial Abelian group (the integer group $\mathbb{Z}$, for example). Next, we say that a sequence of Abelian groups and homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is *fast* if $\ker(\beta\alpha) = \ker(\alpha)$ or, equivalently, $\ker(\beta\alpha) \subset \ker(\alpha)$. Dually, the sequence is said to be *slow* if $\mathrm{im}(\beta\alpha) = \mathrm{im}(\beta)$ or, equivalently, $\mathrm{im}(\beta) \subset \mathrm{im}(\beta\alpha)$.

2.2. Joinability. Väisälä's metric duality theory is based on the concept of joinability, which is a p-dimensional version of the LLC property. Suppose that $A \subset \bar{\mathbb{R}}^n$, $a \in \mathbb{R}^n$, $r > 0$ and $c \geq 1$. For each integer $p \geq 0$, consider the following four sequences induced by inclusions:

$$\text{(a)} \qquad H_p\big(A \cap B(a,r)\big) \longrightarrow H_p\big(A \cap B(a,cr)\big) \longrightarrow H_p(A),$$

$$\text{(b)} \qquad H_p\big(A \setminus \bar{B}(a,cr)\big) \longrightarrow H_p\big(A \setminus \bar{B}(a,r)\big) \longrightarrow H_p(A),$$

$$\text{(c)} \qquad H^p(A) \longrightarrow H^p\big(A \cap \bar{B}(a,cr)\big) \longrightarrow H^p\big(A \cap \bar{B}(a,r)\big),$$

$$\text{(d)} \qquad H^p(A) \longrightarrow H^p\big(A \setminus B(a,r)\big) \longrightarrow H^p\big(A \setminus B(a,cr)\big).$$

If the above sequence (a) is fast for every $a \in \mathbb{R}^n$ and $r > 0$, we say that A is *homologically outer* (p,c)*-joinable*. If (b) is fast for all a, r, then A is *homologically inner* (p,c)*-joinable*. Similarly, if sequence (c) is slow for all a, r, then A is *cohomologically outer* (p,c)*-joinable*. If (d) is slow for all a, r, then A is *cohomologically inner* (p,c)*-joinable*.

As in [9], we shall abbreviate the words 'homologically' and 'cohomologically' by hlog and cohlog, respectively. We say that A is *hlog* (p,c)*-joinable* if A is both hlog outer (p,c)-joinable and hlog inner (p,c)-joinable. If A is hlog outer (p,c)-joinable for some $c \geq 1$, we say that A is hlog outer (p)-joinable, and similarly for the other joinability properties. As a convention to further simplify notation, we shall omit the word 'hlog' if the set A is open and omit the word 'cohlog' if the set A is closed.

One of the main ingredients in this paper is the following duality result due to Väisälä [9, Theorem 2.7] on joinability. It is proved by using exact Mayer–Vietoris sequences and Alexander duality on homology and cohomology groups.

Lemma 2.1. *Suppose that U is an open set in $\bar{\mathbb{R}}^n$ and that p is an interger with $0 \leq p \leq n-2$. Let $X = \bar{\mathbb{R}}^n \setminus U$ and $q = n-2-p$. Then*

(a) *U is outer (p,c)-joinable if and only if X is inner (q,c)-joinable*;
(b) *U is inner (p,c)-joinable if and only if X is outer (q,c)-joinable*;
(c) *U is (p,c)-joinable if and only if X is (q,c)-joinable.*

2.3. Joinability and LLC. The above joinability concept seems to be very abstract. But the homological joinability properties can be defined more explicitly in terms of cycles and chains. For example, an open set U is outer (p,c)-joinable if and only if, for any $a \in \mathbb{R}^n$ and $r > 0$, a p-cycle in $U \cap B(a,r)$ that bounds in U also bounds in $U \cap B(a,cr)$. In particular, since 0-dimensional simplexes and 0-dimensional chains can be represented by points, (0)-joinability of a set is closely related to the LLC property. We collect the following results as they are needed in this paper.

LEMMA 2.2. *Let U be an open set in $\bar{\mathbb{R}}^n$. Then U is hlog $(0,c)$-joinable if and only if every component of U is c-LLC.*

LEMMA 2.3. *Let A be a compact set in $\bar{\mathbb{R}}^n$. If A is cohlog $(0,c)$-joinable, then every component of A is c'-LLC for any $c' > c$. Conversely, if every component of A is c-LLC, then A is cohlog $(0,c')$-joinable for any $c' > c$.*

The first lemma follows from [9, Theorem 3.5], and the fact that, for an open set, pathwise connectedness and continuumwise connectedness are equivalent. The second lemma follows from [9, Theorem 3.10].

2.4. Joinability and uniformity. Next, we consider the relation between joinability and uniformity of order p. Suppose that U is an open set in $\bar{\mathbb{R}}^n$, $0 \le p \le n-2$ and $c \ge 1$. Following Väisälä's definition, we say that U is *weakly (p,c)-uniform* if for every p-cycle z bounding in U there is a $(p+1)$-chain g with $\partial g = z$ such that

(L) $$d(x,|z|) \le c d(x,\partial U) \quad \text{for all } x \in |g|$$

and

(T) $$\operatorname{dia}(|g|) \le c d(|z|).$$

The condition (L) is often referred to as the *lens condition* and (T) the *turning condition*. A domain U is said to be *(p,c)-uniform* if it is weakly (p,c)-uniform and $H_p(U) = 0$. In other words, a domain U is *(p,c)-uniform* if for every p-cycle z in U there is a $(p+1)$-chain g with $\partial g = z$ that satisfies the conditions (L) and (T).

We note that a domain U is $(0,c)$-uniform if and only if it is uniform in the ordinary sense. Uniformity of higher order is a relatively new concept. In [7], we studied uniform domains of order p based on homotopy in connection with quasiconformal reflections. Alestalo and Väisälä considered both homotopical and homological versions of uniformity of order p [1], [2]. It is easy to see that a domain $D \subset \mathbb{R}^n$ is homologically (p,c)-uniform in the sense of [1] if and only if it is (p,c)-uniform in the above sense. We also note that, as shown in [9, 5.5], U is weakly (p,c)-uniform if and only if $U \setminus \infty$ is weakly (p,c)-uniform. Another major ingredient we need is the following connection between joinability and uniformity due to Väisälä [9, 5.22].

LEMMA 2.4. *Let $0 \le p \le n-2$ and U be an open set in $\bar{\mathbb{R}}^n$. If U is (k,c)-joinable for $p \le k \le n-2$ and $H_k(U)=0$ for $p+1 \le k \le n-1$, then U is weakly (p,c')-uniform with $c'=c'(c,n,p)$.*

3. Uniformity and LLC

In this section, we study the relation between uniformity and the LLC property of domains and their complements in $\bar{\mathbb{R}}^n$. As a consequence, we derive the main results stated in Section 1. Another ingredient needed is the following 'push to the boundary' result on the LLC property, which we believe has its own interest. A slightly different form of this result was established in [11]. But, we give a proof here for completeness.

THEOREM 3.1. *Let an open set U in $\bar{\mathbb{R}}^n$ be c-LLC. Then its closure $\bar{U}$ is c'-LLC for any $c'>c$.*

Proof. To show that $\bar{U}$ is LLC_1, we let $a \in \mathbb{R}^n$, $r>0$ and fix $x,y \in \bar{U} \cap B(a,r)$. For any $\varepsilon>0$, choose $x' \in U \cap B(x,\varepsilon r)$ and $y' \in U \cap B(y,\varepsilon r)$. Then $x',y' \in U \cap B(a,(1+\varepsilon)r)$. Since U is c-LLC, there is a continuum E with

$$x',y' \in E \subset U \cap B\big(a,c(1+\varepsilon)r\big).$$

Next, choose a sequence $x_n \in U$ with $x_1=x'$ and $|x_n-x|<\frac{\varepsilon r}{n}$ for $n=1,2,\dots$. By the c-LLC property of U again, there is a sequence of continua F_n with

$$x_n,x_{n+1} \in F_n \subset U \cap B(x,c\varepsilon r).$$

Then the set $F=\bigcup \bar{F}_n$ is a continuum with

$$x',x \in F \subset \bar{U} \cap \bar{B}(x,c\varepsilon r) \subset \bar{U} \cap B\big(a,c(1+\varepsilon)r\big).$$

Similarly, one can obtain a continuum $F' \subset \bar{U} \cap B(a,c(1+\varepsilon)r)$ connecting y' and y. Thus, the set $F \cup E \cup F'$ is a continuum connecting x and y in $\bar{U} \cap B(a,c(1+\varepsilon)r)$. This shows that $\bar{U}$ is c'-LLC_1 for any $c'>c$.

To show that $\bar{U}$ is LLC_2, we let $a \in \mathbb{R}^n$, $r>0$ and fix $x,y \in \bar{U} \setminus \bar{B}(a,r)$. Similar to the above argument, one can choose sequences $x_n,y_n \in U \setminus \bar{B}(a,r)$ with $x_n \to x$ and $y_n \to y$. Thus, by invoking the c-LLC_2 property of U, one can construct as above continua $E,F,F' \subset \bar{U} \setminus B(a,r/c)$ with $x_1,y_1 \in E$, $x,x_1 \in F$ and $y_1,y \in F'$. Thus, the set $F \cup E \cup F'$ is a desired continuum connecting x and y in $\bar{U} \setminus B(a,r/c)$. This shows that $\bar{U}$ is c'-LLC_2 for any $c'>c$ and completes the proof of Theorem 3.1. □

Theorem 3.1 can be thought of as a 'push to the boundary' result on LLC. Conversely, we also have the following 'push to the inside' result on LLC, which will also be needed.

THEOREM 3.2. *Let D be a domain in $\bar{\mathbb{R}}^n$ that is locally collared along the boundary* (*as defined in Section 1*). *If $\bar{D}$ is c-LLC, then D is also c-LLC.*

Proof. Let $r>0$ and $x_0 \in \mathbb{R}^n$. Fix $x_1, x_2 \in D \cap B(x_0, r)$. Since $\bar{D}$ is c-LLC, there is a continuum $E \subset \bar{D} \cap B(x_0, cr)$ with $x_1, x_2 \in E$. We shall construct a continuum joining x_1 and x_2 in $D \cap B(x_0, cr)$ by pushing E into D using local collars.

Since D is locally collared at the boundary, for each $y \in E \cap \partial D$, there is a neighborhood $U_y \subset B(x_0, cr)$ such that $U_y \cap \bar{D}$ is homeomorphic to $B(0,1) \cap \bar{\mathbb{H}}$, the intersection of the unit ball and the closure of the upper half space. We may further assume that $x_1, x_2 \notin U_y$ for each y. Next, we fix a finite open cover $\{U_{y_k}\}_{k=1}^m$ for the compact set $E \cap \partial D$. For each y_k, let

$$f_k : U_{y_k} \cap \bar{D} \to B(0,1) \cap \bar{\mathbb{H}}$$

be a homeomorphism. Then we can fix an embedding $g : B(0,1) \cap \bar{\mathbb{H}} \to B(0,1) \cap \mathbb{H}$ such that g is identity on the upper half unit sphere and that $B(0,1) \cap \bar{\mathbb{H}}$ is mapped into $B(0,1) \cap \mathbb{H}$. Thus, we obtain an embedding

$$F_k = f_k^{-1} \circ g \circ f_k : U_{y_k} \cap \bar{D} \to U_{y_k} \cap D$$

which is identity when x approaches $\partial U_{y_k} \cap D$.

Finally, define maps $G_k : E \to \bar{D}$ as follows.

$$G_1(x) = \begin{cases} x, & x \notin U_{y_1}, \\ F_1(x), & x \in U_{y_1}, \end{cases}$$

$$G_k(x) = \begin{cases} G_{k-1}(x), & x \notin U_{y_k}, \\ F_k(x), & x \in U_{y_k}. \end{cases}$$

Then one can see that $E_1 = G_1(E)$ is a continuum in $\bar{D} \cap B(x_0, cr)$ which does not intersect the boundary ∂D in U_{y_1}. By induction, the set $E_m = G_m(E)$ is a continuum joining x_1 and x_2 in $D \cap B(x_0, cr)$. This shows that D is c-LLC$_1$. Similarly, one can also show that D is c-LLC$_2$. This completes the proof of Theorem 3.2. □

Note that the result proved above is stronger than a similar result for n-manifolds in $\mathbb{R}^n$ established by Väisälä [10, Lemma 5.8], in the sense that Theorem 3.2 requires that ∂D is only locally collared in D. With these preliminary results, we can proceed to prove the main results stated in the Introduction.

Proof of Theorem 1.1. It suffices to show that if D and D^* are LLC, then they are also uniform. Assume that D and D^* are c-LLC. By Lemma 2.2, they are (0)-joinable. Furthermore, it follows from Theorem 3.1 that $\bar{D}$ and $\bar{D}^*$ are LLC. Thus, Lemma 2.3 yields that $\bar{D}$ and $\bar{D}^*$ are also (0)-joinable.

Invoking Lemma 2.1, the metric duality result with $p=1$ and $q=0$, we conclude that D and D^* are both (1)-joinable. Therefore, Lemma 2.4 implies that D and D^* are weakly (p)-uniform for $p=0,1$, and hence uniform as desired. □

Proof of Theorem 1.2. Assume that D and D^* are uniform. Then they are LLC and, by Theorem 3.1 and Lemma 2.3, $\bar{D}$ and $\bar{D}^*$ are also (0)-joinable. Thus, it follows from Lemma 2.1 that D^*, D, and hence $D \cup D^*$ are (1)-joinable. By the metric duality again, ∂D is (0)-joinable, and therefore LLC.

For the converse, assume that ∂D is LLC. Then it is (0)-joinable. Applying the metric duality (Lemma 2.1) repeatedly, we deduce that D, D^* are (1)-joinable and that $\bar{D}$ and $\bar{D}^*$ are (0)-joinable. Since D is locally collared at ∂D, Theorem 3.2 implies that D, D^* are also (0)-joinable. Finally, Lemma 2.4 yields that D, D^* are (p)-uniform for $p=0,1$. In particular, they are uniform in the ordinary sense. □

4. Examples and applications

4.1. Uniformity of higher order. Recall that a domain D is (p,c)-*uniform* (in homology sense) if for every singular p-cycle z in D there is a singular $(p+1)$-chain g with $\partial g = z$ that satisfies the lens condition (L) and turning condition (T) as defined in Section 2. Similarly, according to [7] and [1], a domain D is *homotopically* (p,c)-*uniform* if each pair of maps $f_1, f_2 : \mathbf{S}^p \to D$ are homotopical to each other by a homotopy F that satisfies a lens condition and a turning condition, or, equivalently, if each map $f : \mathbf{S}^p \to D$ has an extension $g : \bar{\mathbf{B}}^{p+1} \to D$ satisfying a lens condition and a turning condition.

It was shown by Alestalo [1, Theorem 1.7] that if a domain D is homotopically (1)-uniform, then it is also homologically (1)-uniform. But the converse is not true. As shown in [1, Section 5], the complement D of a wild bounded turning arc in $\mathbb{R}^n$, $n \geq 4$, is homologically (1)-uniform. However, such a domain D is not homotopically (1)-uniform due to the fact that its fundamental group $\pi_1(D)$ is nontrivial. We will illustrate an example which shows that the converse is not true even for domains with trivial fundamental group π_1. First, we state two corollaries which follow directly from the proofs of Theorems 1.1 and 1.2. We say that a triple $(D, D^*, \partial D)$ is a *homologically trivial partition* of $\bar{\mathbb{R}}^3$ if D and D^* are complementary domains in $\bar{\mathbb{R}}^3$ with trivial homology groups such that $\partial D = \partial D^*$.

Corollary 4.1. *Let $(D, D^*, \partial D)$ be a homologically trivial partition of $\mathbb{R}^3$. If D and D^* are LLC, then they are both uniform and homologically* (1)*-uniform.*

Corollary 4.2. *Let $(D, D^*, \partial D)$ be a homologically trivial partition of $\mathbb{R}^3$ with D being locally collared at ∂D. If ∂D is LLC, then D and D^* are both uniform and homologically* (1)*-uniform.*

The following example shows that the domains D and D^* in the above corollaries may fail to be *homotopically* (1)-uniform even if the partition

$(D, D^*, \partial D)$ is homotopically trivial (i.e., D and D^* have trivial homotopy groups). According to [7, Example 6.6], there is a domain D in $\mathbb{R}^3$ which has the following properties:

(a) $(D, D^*, \partial D)$ is a homotopically trivial partition of $\mathbb{R}^3$;
(b) both D and D^* are homeomorphic to an open half space;
(c) D^* is bi-Lipschitz equivalent to an open half space;
(d) D is uniform, but not homotopically (1)-uniform.

By Corollary 4.1, both D and D^* are homologically (1)-uniform. This seems to be the first example where homological uniformity does not imply homotopical uniformity even with trivial fundamental groups.

4.2. Quasiconformal reflection domains. A domain D in $\bar{\mathbb{R}}^n$ is called a *quasiconformal (QC) reflection domain* if there is an orientation reversing quasiconformal mapping $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ such that $f(D) = D^*(= \bar{\mathbb{R}}^n \setminus \bar{D})$ and that f is identity on ∂D. In this case, we also say that ∂D admits a QC reflection. In the plane, these domains are precisely quasi-disks [4]. In higher dimensions, they were studied in connection with uniformity [7], [14]. Here, we deduce another result in this direction by using the duality.

THEOREM 4.1. *Let Σ be a topological sphere in $\mathbb{R}^3$ that admits a QC reflection. Then the complementary domains D and D^* are homologically (p, c)-uniform for $p = 0, 1$.*

Proof. Since Σ is a topological sphere, it follows from the Alexander duality that $(D, D^*, \partial D)$ is a homologically trivial partition of $\bar{\mathbb{R}}^3$. Furthermore, by [14, Theorem 3.1], D and D^* are uniform domains, and hence LLC. Hence, it follows from Corollary 4.1 that they are also homologically (1)-uniform. □

4.3. Quasisymmetric parameterizations. Finally, we close this paper with an application in quasisymmetric parameterization. The problem of characterizing topological (or metric) spheres that admit a quasisymmetric (QS) parameterization by the standard sphere has drawn considerable attention in recent years due to its role in analysis, geometry and topology (see [6], [3]). In the positive direction, Bonk and Kleiner showed that an Ahlfors 2-regular metric sphere Σ admits a QS parameterization by the standard 2-sphere $\mathbf{S}^2$ if and only if it is LLC. A metric measure space (X, μ) is called (Ahlfors) *n-regular* if $C^{-1}R^n \le \mu(B_R) \le CR^n$ for some constant $C \ge 1$ and for all closed balls B_R of radius $0 < R < \operatorname{diam} X$. We shall derive that certain topological spheres in $\mathbb{R}^3$ that admit quasiconformal (QC) reflections are QS equivalent to $\mathbf{S}^2$. A topological 2-sphere in $\mathbb{R}^3$ is said to be 2-*regular* if it is 2-regular with respect to the 2-dimensional Hausdorff measure in the above sense.

THEOREM 4.2. *Let Σ be a 2-regular topological sphere in $\mathbb{R}^3$ that admits a QC reflection. Then Σ admits a QS parametrization by the standard sphere* $\mathbf{S}^2$.

Proof. Let D_1 and D_2 denote the two complementary domains to Σ. By [14, Theorem 3.1], D_1 and D_2 are uniform, and hence LLC. Thus, by invoking Theorem 3.1 and Lemma 2.1, we deduce that D_1 and D_2 are (1)-joinable, and hence $D_1 \cup D_2$ is also (1)-joinable. By duality (Lemma 2.1) again, we conclude that Σ, as the complement of $D_1 \cup D_2$, is (0)-joinable (or LLC). Finally, [3, Theorem 1.1] implies that Σ is QS equivalent to $\mathbf{S}^2$ as desired. □

REFERENCES

[1] P. Alestalo, *Uniform domains of higher order*, Ann. Acad. Sci. Fenn. Ser. A I Math. (Dissertation) **94** (1994), 1–48. MR 1283431

[2] P. Alestalo and J. Väisälä, *Uniform domains of higher order II*, Ann. Acad. Sci. Fenn. Ser. A I Math. **21** (1996), 411–437. MR 1404095

[3] M. Bonk and B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. **150** (2002), 127–183. MR 1930885

[4] F. W. Gehring, *Characterizations of quasidisks*, Quasiconformal geometry and dynamics, Banach Center Publ., vol. 48, Polish Acad. Sci., Warsaw, 1999, pp. 11–41. MR 1709972

[5] F. W. Gehring and O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings*, J. Analyse Math. **45** (1985), 181–206. MR 0833411

[6] J. Heinonen and S. Semmes, *Thirty-three yes or no questions about mappings*, Conform. Geom. Dyn. **1** (1997), 1–12. MR 1452413

[7] J. Heinonen and S. Yang, *Strongly uniform domains and periodic quasiconformal maps*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), 123–148. MR 1304111

[8] D. Herron and P. Koskela, *Uniform and Sobolev extension domains*, Proc. Amer. Math. Soc. **114** (1992), 483–489. MR 1075947

[9] J. Väisälä, *Metric duality in Euclidean spaces*, Math. Scand. **80** (1997), 249–288. MR 1481106

[10] J. Väisälä, *Quasimöbius maps*, J. Analyse Math. **44** (1984/85), 218–234. MR 0801295

[11] M. F. Walker, *Linearly locally connected sets and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **11** (1986), 77–86. MR 0826350

[12] S. Yang, *A Sobolev extension domain that is not uniform*, Manuscripta Math. **120** (2006), 241–251. MR 2234252

[13] S. Yang, *QED domains and NED sets in* $\bar{\mathbb{R}}^n$, Trans. Amer. Math. Soc. **334** (1992), 97–120. MR 1065605

[14] S. Yang, *Quasiconformal reflection, uniform and quasiconformal extension domains*, Complex Variables **17** (1992), 277–286. MR 1147057

SHANSHUANG YANG, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GA 30322, USA

E-mail address: syang@mathcs.emory.edu