HARDY SPACES OF OPERATOR-VALUED ANALYTIC FUNCTIONS

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ABSTRACT. We are concerned with Hardy and BMO spaces of operator-valued functions analytic in the unit disk of $\mathbb{C}$. In the case of the Hardy space, we involve the atomic decomposition since the usual argument in the scalar setting is not suitable. Several properties (the Garsia-norm equivalent theorem, Carleson measure, and so on) of BMOA spaces are extended to the operator-valued setting. Then, the operator-valued H^1-BMOA duality theorem is proved. Finally, by the H^1-BMOA duality we present the Lusin area integral and Littlewood–Paley g-function characterizations of the operator-valued analytic Hardy space.

1. Introduction

The classical Hardy and BMO spaces and H^1-BMO duality theorem play a crucial role in harmonic analysis (see, for example, [4], [6], [13]). The vector-valued analogue was studied by Bourgain [3] and Blasco [2] in the case of the unit disc. Recently, operator-valued (= quantum) harmonic analysis has developed considerably (e.g., see [5]). This is inspired by the works on matrix-valued harmonic analysis (e.g., see [8], [11] and references therein) and the recent development on the theory of noncommutative martingales (see the survey paper by Xu [15]). The theory of operator-valued Hardy and BMO spaces in $\mathbb{R}$ was well built by Mei [7]. The goal of this paper is to study the disk analogue of Mei's results, with an emphasis on the 'analytical' aspect.

The remainder of this paper is divided into five sections. In Section 2, we present some preliminaries, including the L^p-spaces of operator-valued measurable functions, the noncommutative Hölder inequality, operator-valued analytic functions and some properties. The operator-valued analytic Hardy

space is defined by the atomic decomposition in Section 3, since the usual argument in the scalar setting is not suitable, as was pointed out in [7]. In Section 4, we show the Garsia-norm equivalent theorem and Carleson measure characterization of the operator-valued BMOA space. One of main results in the paper is then proved in Section 5: the operator-valued H^1-BMOA duality theorem. Section 6 is devoted to the proof of another main result, the Lusin area integral and Littlewood–Paley g-function characterizations of the operator-valued analytic Hardy space. Our argument uses the atomic decomposition and is distinct from Mei's method originating from [9], in which the Lusin area integral is used for defining the Hardy space. The techniques involved here is slightly simpler and also suitable for use in obtaining the corresponding results found there. We would like to point out that the atomic decomposition of the predual of operator-valued BMO spaces has been studied by Ricard [12].

In what follows, C always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y by $X \asymp Y$, we mean that there exists a constant C such that $C^{-1} X \leq Y \leq C X$. Any notation and terminology not otherwise explained, are as used in [4] for complex harmonic analysis, and in [14] for theory of von Neumann algebras.

2. Preliminaries

2.1. Operator-valued measurable functions. Throughout this paper, $\mathcal{M}$ will always denote a von Neumann algebra, and $\mathcal{M}_+$ its positive part. Recall that a *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying:

  (1) $\tau(x + y) = \tau(x) + \tau(y)$ for arbitrary $x, y \in \mathcal{M}_+$;
  (2) $\tau(\lambda x) = \lambda \tau(x)$ for any $\lambda \in [0, \infty)$ and $x \in \mathcal{M}_+$; and
  (3) $\tau(u^* u) = \tau(u u^*)$ for all $u \in \mathcal{M}$.

τ is said to be *normal* if $\sup_\gamma \tau(x_\gamma) = \tau(\sup_\gamma x_\gamma)$ for each bounded increasing net (x_γ) in $\mathcal{M}_+$, *semifinite* if for each $x \in \mathcal{M}_+ \backslash \{0\}$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$, and *faithful* if for each $x \in \mathcal{M}_+ \backslash \{0\}$, $\tau(x) > 0$. A von Neumann algebra $\mathcal{M}$ is called *semifinite* if it admits a normal semifinite faithful trace τ, which we assume in the remainder of this paper.

Denote by $\mathcal{S}_+$ the set of all $x \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x$ is the support of x which is defined as the least projection p in $\mathcal{M}$ so that $px = x$ or equivalently $xp = x$. Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Then $\mathcal{S}$ is a $*$-subalgebra of $\mathcal{M}$ which is w^*-dense in $\mathcal{M}$. Moreover, for each $0 < p < \infty$, $x \in \mathcal{S}$ implies $|x|^p \in \mathcal{S}_+$ (and so $\tau(|x|^p) < \infty$), where $|x| = (x^* x)^{1/2}$ is the modulus of x. Now we define $\|x\|_p = [\tau(|x|^p)]^{1/p}$ for all $x \in \mathcal{S}$. One can show that $\| \cdot \|_p$ is a norm on $\mathcal{S}$ if $1 \leq p < \infty$, and a quasi-norm (more precisely, p-norm) if $0 < p < 1$. The completion of $(\mathcal{S}, \| \cdot \|_p)$ is denoted by $L^p(\mathcal{M}, \tau)$. This is the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$. For convenience, we set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. The trace τ can be

extended to a linear functional on $\mathcal{S}$, still denoted by τ. Since $|\tau(x)| \leq \|x\|_1$ for all $x \in \mathcal{S}$, τ extends to a continuous functional on $L^1(\mathcal{M}, \tau)$.

Let $0 < \gamma, p, q \leq \infty$ be such that $1/\gamma = 1/p + 1/q$. If $x \in L^p(\mathcal{M}, \tau), y \in L^q(\mathcal{M}, \tau)$ then $xy \in L^\gamma(\mathcal{M}, \tau)$ and

$$\|xy\|_\gamma \leq \|x\|_p \|y\|_q. \tag{1}$$

In particular, if $\gamma = 1$, $|\tau(xy)| \leq \|xy\|_1 \leq \|x\|_p \|y\|_q$ for arbitrary $x \in L^p(\mathcal{M}, \tau)$ and $y \in L^q(\mathcal{M}, \tau)$. This defines a natural duality between $L^p(\mathcal{M}, \tau)$ and $L^q(\mathcal{M}, \tau)$: $\langle x, y \rangle = \tau(xy)$. For any $1 \leq p < \infty$, we have $L^p(\mathcal{M}, \tau)^* = L^q(\mathcal{M}, \tau)$ isometrically. Thus, $L^1(\mathcal{M}, \tau)$ is the predual $\mathcal{M}_*$ of $\mathcal{M}$, and $L^p(\mathcal{M}, \tau)$ is reflexive for $1 < p < \infty$. (For the theory of noncommutative L^p-spaces, see the survey paper by Pisier and Xu [10] and references therein.)

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. A $\mathcal{S}$-valued function φ on Ω is said to be *simple* if it can be written as

$$\varphi = \sum_j a_j \chi_{F_j}, \tag{2}$$

where $a_j \in \mathcal{S}$ and $\{F_j\}$ is a finite set of measurable disjoint subsets of Ω with $\mu(F_j) < \infty$. Denote by $S(\Omega, \mathcal{S})$ the set of all simple $\mathcal{S}$-valued functions on Ω. For such a simple function φ, we define

$$\|\varphi\|_{L_c^p} = \left\| \left(\sum_j a_j^* a_j \mu(F_j) \right)^{\frac{1}{2}} \right\|_p = \left\| \left(\int_\Omega |\varphi|^2 \, d\mu \right)^{\frac{1}{2}} \right\|_p \tag{3}$$

and $\|\varphi\|_{L_r^p} = \|\varphi^*\|_{L_c^p}$, respectively. As shown in [7], each φ can be regarded as an element $T(\varphi)$ in $\mathcal{M} \otimes \mathcal{B}(L^2(\Omega))$ (where $\mathcal{B}(L^2(\Omega))$ is the space of all bounded operators on $L^2(\Omega)$ with the usual trace Tr) and

$$\|\varphi\|_{L_c^p} = \|T(\varphi)\|_{L^p(\mathcal{M} \otimes \mathcal{B}(L^2(\Omega)))} \tag{4}$$

(and $\|\varphi\|_{L_r^p} = \|\varphi^*\|_{L_c^p}$). This concludes that $\|\cdot\|_{L_c^p}$ (and $\|\cdot\|_{L_r^p}$) are norms on $S(\Omega, \mathcal{S})$ if $1 \leq p \leq \infty$ and p-norms if $0 < p < 1$. For $0 < p < \infty$ (or $p = \infty$), the completions of $S(\Omega, \mathcal{S})$ in $\|\cdot\|_{L_c^p}$ and $\|\cdot\|_{L_r^p}$ (or in weak*-operator topology) are denoted by $L^p(\mathcal{M}, L_c^2(\Omega))$ and $L^p(\mathcal{M}, L_r^2(\Omega))$, respectively.

LEMMA 1 (Proposition 1.1 in [7]). *Let $0 < \gamma, p, q \leq \infty$ be such that $1/\gamma = 1/p + 1/q$. If $f \in L^p(\mathcal{M}, L_c^2(\Omega)), g \in L^q(\mathcal{M}, L_c^2(\Omega))$, then*

$$\langle f, g \rangle = \int_\Omega f^* g \, d\mu \tag{5}$$

exists as an element in $L^\gamma(\mathcal{M}, \tau)$ and

$$\|\langle f, g \rangle\|_\gamma \leq \|f\|_{L_c^p} \|g\|_{L_c^q}. \tag{6}$$

A similar statement also holds for $L^p(\mathcal{M}, L_r^2(\Omega))$.

For $1 \le p < \infty$ with $q = p/(p-1)$, by Lemma 1 we have

$$L^p(\mathcal{M}, L_c^2(\Omega))^* = L^q(\mathcal{M}, L_c^2(\Omega))$$

isometrically via the anti-duality

(7)
$$(f, g) = \tau(\langle f, g \rangle) = \tau\left(\int_\Omega f^* g \, d\mu\right)$$

for $f \in L^p(\mathcal{M}, L_c^2(\Omega))$ and $g \in L^q(\mathcal{M}, L_c^2(\Omega))$. Similarly,

$$L^p(\mathcal{M}, L_r^2(\Omega))^* = L^q(\mathcal{M}, L_r^2(\Omega)).$$

(For details, see [7].)

Also, by the convexity of the operator-valued function $x \to |x|^2$, we have

(8)
$$\left|\int_\Omega fg \, d\mu\right|^2 \le \int_\Omega |f|^2 \, d\mu \int_\Omega |g|^2 \, d\mu$$

for every $f \in L^p(\mathcal{M}, L_c^2(\Omega))$ and $g \in L^2(\Omega, \mu)$ (e.g., see (1.13) in [7]).

LEMMA 2. *If $f \in L^1(\mathcal{M}, L_c^2(\Omega))$, then*

(9)
$$\left(\int_\Omega |\tau(f)|^2 \, d\mu\right)^{1/2} \le \|f\|_{L_c^1}.$$

Consequently, $L^1(\mathcal{M}, L_c^2(\Omega)) \subset L^2(\Omega, L^1(\mathcal{M}))$.

Proof. Since

$$\|g\|_{L_c^\infty} = \left\|\int_\Omega |g|^2 \, d\mu\right\|_\infty^{1/2} \le \left(\int_\Omega \|g\|_\infty^2 \, d\mu\right)^{1/2} = \|g\|_{L^2(\Omega, L^\infty(\mathcal{M}))},$$

we conclude from (7) that

$$\left(\int_\Omega |\tau(f)|^2 \, d\mu\right)^{1/2} \le \|f\|_{L^2(\Omega, L^1(\mathcal{M}))}$$

$$= \sup_{g \in L^2(\Omega, L^\infty(\mathcal{M})), \|g\|_2 \le 1} \left|\tau\left(\int_\Omega fg^* \, d\mu\right)\right|$$

$$\le \sup_{g \in L^\infty(\mathcal{M}, L_c^2(\Omega)), \|g\|_{L_c^\infty} \le 1} \left|\tau\left(\int_\Omega fg^* \, d\mu\right)\right|$$

$$= \|f\|_{L_c^1}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For $0 < p \le 2$, we set

(10)
$$L^p(\mathcal{M}, L_{cr}^2(\Omega)) = L^p(\mathcal{M}, L_c^2(\Omega)) + L^p(\mathcal{M}, L_r^2(\Omega)),$$

equipped with the sum norm

$$\|f\|_{L_{cr}^p} = \inf\{\|g\|_{L_c^p} + \|h\|_{L_r^p}\},$$

where the infimum is taken over all $g \in L^p(\mathcal{M}, L_c^2(\Omega))$ and $h \in L^p(\mathcal{M}, L_r^2(\Omega))$ such that $f = g + h$. For $2 < p \leq \infty$,

$$L^p(\mathcal{M}, L_{cr}^2(\Omega)) = L^p(\mathcal{M}, L_c^2(\Omega)) \cap L^p(\mathcal{M}, L_r^2(\Omega)), \tag{11}$$

equipped with the maximum norm

$$\|f\|_{L_{cr}^p} = \max\{\|f\|_{L_c^p}, \|f\|_{L_r^p}\}.$$

Then, for $1 \leq p < \infty$ with $q = p/(p-1)$ we have

$$L^p(\mathcal{M}, L_{cr}^2(\Omega))^* = L^q(\mathcal{M}, L_{cr}^2(\Omega))$$

isometrically via the anti-duality (7).

2.2. Operator-valued analytic functions. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. $dm = d\theta/2\pi$ will denote the normalized Lebesgue measure on $\mathbb{T}$. The kernel

$$P_z(w) = \frac{1 - |z|^2}{|1 - \bar{z}w|^2} \quad (z \in \mathbb{D}, w \in \mathbb{D} \cup \mathbb{T}) \tag{12}$$

is called the Poisson kernel in $\mathbb{D}$. By (6), we can define the Poisson integral $P[f]$ of a function $f \in L^p(\mathcal{M}, L_c^2(\mathbb{T}))$ or $L^p(\mathcal{M}, L_r^2(\mathbb{T}))$ by

$$P[f](z) = \int_{\mathbb{T}} P_z(t) f(t) \, dm(t) \in L^p(\mathcal{M}) \tag{13}$$

for $z \in \mathbb{D}$. For simplicity, we still denote $P[f]$ by f. Similarly, we define the Cauchy integral $\mathfrak{C}(f)$ of a function $f \in L^p(\mathcal{M}, L_c^2(\mathbb{T}))$ or $L^p(\mathcal{M}, L_r^2(\mathbb{T}))$ by

$$\mathfrak{C}(f)(z) = \int_{\mathbb{T}} \frac{f(t)}{1 - tz} \, dm(t) \in L^p(\mathcal{M}) \tag{14}$$

for $z \in \mathbb{D}$.

We let $\mathrm{Aut}(\mathbb{D})$ be the Möbius group of all automorphisms of $\mathbb{D}$. Every Möbius transformation can be written as

$$\psi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z} \quad (z \in \mathbb{D})$$

with θ real and $|\bar{z}_0| < 1$.

Let $1 \leq p \leq \infty$. Recall that $f : \mathbb{D} \to L^p(\mathcal{M})$ is said to be analytic if f is the sum of a power series in the $L^p(\mathcal{M})$-norm, that is, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for every $z \in \mathbb{D}$, where $a_n \in L^p(\mathcal{M})$. The class of all such functions is denoted by $\mathcal{H}(\mathbb{D}, L^p(\mathcal{M}))$. If $f \in \mathcal{H}(\mathbb{D}, L^p(\mathcal{M}))$, then all order partial derivatives of f exist and belong still to $\mathcal{H}(\mathbb{D}, L^p(\mathcal{M}))$. By Lemma 1, one has $\mathfrak{C}(f) \in \mathcal{H}(\mathbb{D}, L^p(\mathcal{M}))$ for any f in $L^p(\mathcal{M}, L_c^2(\mathbb{T}))$ or $L^p(\mathcal{M}, L_r^2(\mathbb{T}))$. We set

$$\mathcal{H}^p(\mathcal{M}, L_c^2(\mathbb{T})) = \{f \in L^p(\mathcal{M}, L_c^2(\mathbb{S})) : P[f] \in \mathcal{H}(\mathbb{D}, L^p(\mathcal{M}))\}.$$

Similarly, we have $\mathcal{H}^p(\mathcal{M}, L_r^2(\mathbb{T}))$ and $\mathcal{H}^p(\mathcal{M}, L_{cr}^2(\mathbb{T}))$.

For any f in $L^p(\mathcal{M}, L_c^2(\mathbb{T}))$, the gradient $\nabla f(z)$ is the $L^p(\mathcal{M})$-valued vector $(\partial f/\partial x, \partial f/\partial y)$ and

$$|\nabla f(z)|^2 = \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \in L^{p/2}(\mathcal{M})$$

with $z = x + iy$. In this notation, we have

(15) $$|\nabla f(z)|^2 = 2|f'(z)|^2$$

if f is analytic.

LEMMA 3. If $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$, then

(16) $$\int_{\mathbb{T}} |f - f(0)|^2 \, dm = \frac{1}{\pi} \int_{\mathbb{D}} |\nabla f(z)|^2 \log \frac{1}{|z|} \, dx \, dy.$$

Consequently, if $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$ then

(17) $$\int_{\mathbb{T}} |f - f(w)|^2 P_w \, dm \asymp \int_{\mathbb{D}} P_w(z) |\nabla f(z)|^2 (1 - |z|^2) \, dx \, dy$$

for every $w \in \mathbb{D}$.

Proof. By Lemma 3.1 in [4], we have

$$\int_{\mathbb{T}} \overline{(\phi - \phi(0))} (\varphi - \varphi(0)) \, dm = \frac{1}{\pi} \int_{\mathbb{D}} \overline{\nabla \phi(z)} \nabla \varphi(z) \log \frac{1}{|z|} \, dx \, dy$$

for all $\phi, \varphi \in L^2(\mathbb{T})$. Then, for any $\mathcal{S}_{\mathcal{M}}$-valued simple function $f = \sum_j a_j \chi_{F_j}$ on $\mathbb{T}$, one has that

$$\int_{\mathbb{T}} |f - f(0)|^2 \, dm = \int_{\mathbb{T}} \left| \sum_j a_j \left(\chi_{F_j} - \chi_{F_j}(0) \right) \right|^2 dm$$

$$= \sum_{j,k} a_j^* a_k \int_{\mathbb{T}} \overline{\left(\chi_{F_j} - \chi_{F_j}(0) \right)} \left(\chi_{F_k} - \chi_{F_k}(0) \right) dm$$

$$= \frac{1}{\pi} \sum_{j,k} a_j^* a_k \int_{\mathbb{D}} \overline{\nabla \chi_{F_j}(z)} \nabla \chi_{F_k}(z) \log \frac{1}{|z|} \, dx \, dy$$

$$= \frac{1}{\pi} \int_{\mathbb{D}} |\nabla f(z)|^2 \log \frac{1}{|z|} \, dx \, dy.$$

Since the set of $\mathcal{S}_{\mathcal{M}}$-valued simple functions is dense in $L^1(\mathcal{M}, L_c^2(\mathbb{T}))$ and τ is faithful, it is concluded that (16) holds for all $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$.

To prove (17), we first prove the case of $w = 0$, that is,

(18) $$\int_{\mathbb{T}} |f - f(0)|^2 \, dm \asymp \int_{\mathbb{D}} |\nabla f(z)|^2 (1 - |z|^2) \, dx \, dy.$$

To this end, we note that $1 - |z|^2 \leq 2\log(1/|z|), |z| < 1$. Hence, by (16) one has that

$$\int_{\mathbb{D}} |\nabla f(z)|^2 (1 - |z|^2) \, dx \, dy \leq 2\pi \int_{\mathbb{T}} |f - f(0)|^2 \, dm.$$

To prove the reverse inequality, suppose that the integral on the left hand is finite and denoted by A. For $|z| > 1/4$, we have the reverse inequality $\log(1/|z|) \leq C(1 - |z|^2)$. This yields that

$$\int_{1/4 < |z| < 1} |\nabla f(z)|^2 \log \frac{1}{|z|} \, dx \, dy \leq CA.$$

Also, let $\mathbb{H}$ be the Hilbert space on which $\mathcal{M}$ acts. For $|z| < 1/4$, the subharmonicity of $z \to \|\nabla f(h)(z)\|^2$ for each $h \in \mathbb{H}$ gives that

$$
\begin{aligned}
\langle |\nabla f(z)|^2(h), h \rangle = \|\nabla f(h)(z)\|^2 \\
\leq \frac{16}{\pi} \int_{|w-z|<1/4} \|\nabla f(h)(w)\|^2 \, dx \, dy \\
\leq \frac{32}{\pi} \int_{|w|<1/2} \|\nabla f(h)(w)\|^2 (1 - |w|^2) \, dx \, dy \\
\leq \frac{32}{\pi} \langle A(h), h \rangle.
\end{aligned}
$$

Hence,

$$\int_{|z| \leq 1/4} |\nabla f(z)|^2 \log \frac{1}{|z|} \, dx \, dy \leq \frac{32}{\pi} A \int_{|z| \leq 1/4} \log \frac{1}{|z|} \, dx \, dy \leq CA.$$

Using (16), we conclude the proof of (18).

Now, fix $w \in \mathbb{D}$. The identity

$$(19) \qquad \int_{\mathbb{T}} |f - f(w)|^2 P_w \, dm = \frac{1}{\pi} \int_{\mathbb{D}} |\nabla f(z)|^2 \log \left| \frac{1 - \bar{w}z}{z - \bar{w}} \right| \, dx \, dy$$

has the same proof as (16). Using the identity

$$1 - \left| \frac{z - w}{1 - \bar{w}z} \right|^2 = (1 - |z|^2) P_w(z),$$

we similarly obtain (17). This completes the proof. □

For every $\alpha > 1$ and $t \in \mathbb{T}$, let

$$\Gamma_\alpha(t) = \{z \in \mathbb{D} : |t - z| < \alpha(1 - |z|)\}.$$

In what follows, we fix $\alpha > 1$.

LEMMA 4. *If* $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$, *then*

$$(20) \qquad \int_{\mathbb{T}} \left(\int_{\Gamma_\alpha(t)} |\nabla f(z)|^2 \, dx \, dy \right) dm \asymp \int_{\mathbb{T}} |f - f(0)|^2 \, dm.$$

Proof. Let $\mathbb{H}$ be the Hilbert space on which $\mathcal{M}$ acts. Note that as $|z| \to 1$, $m\{t \in \mathbb{T} : z \in \Gamma_\alpha(t)\} \asymp 1 - |z|^2$. By Lemma 3, one has that

$$
\left\langle \int_{\mathbb{T}} \left(\int_{\Gamma_\alpha(t)} |\nabla f(z)|^2 \, dx \, dy \right) dm(h), h \right\rangle
$$

$$
= \int_{\mathbb{D}} \langle |\nabla f(z)|^2(h), h \rangle m\{t \in \mathbb{D} : z \in \Gamma_\alpha(t)\} \, dx \, dy
$$

$$
\asymp \int_{\mathbb{D}} \langle |\nabla f(z)|^2(h), h \rangle (1 - |z|^2) \, dx \, dy
$$

$$
\asymp \left\langle \int_{\mathbb{T}} |f - f(0)|^2 \, dm(h), h \right\rangle
$$

for all $h \in \mathbb{H}$. $\square$

3. Operator-valued Hardy space

Throughout, we always denote by I a subarc of $\mathbb{T}$ and $|I| = m(I)$.

DEFINITION 5. $a \in L^1(\mathcal{M}, L^2_c(\mathbb{T}))$ is said to be an $\mathcal{M}_c$-atom, if

  (i) a is supported in a subarc I of $\mathbb{T}$,
 (ii) $\int_I a \, d\sigma = 0$, and
(iii) $\|a\|_{L^1_c} \le |I|^{-1/2}$.

By Lemma 2, one concludes that an $\mathcal{M}_c$-atom is also an $L^1(\mathcal{M})$-valued 2-atom on $\mathbb{T}$. Hence, for each $\mathcal{M}_c$-atom a one has

$$(21) \qquad \|a\|_{L^1(\mathbb{T}, L^1(\mathcal{M}))} \le |I|^{-1/2} \|a\|_{L^2(\mathbb{T}, L^1(\mathcal{M}))} \le 1.$$

Then, we define $\mathrm{H}^1_c(\mathbb{T}, \mathcal{M})$ as the space of all $f \in L^1(\mathbb{T}, L^1(\mathcal{M}))$ which admit a decomposition

$$
f = \sum_k \lambda_k a_k,
$$

where for each k, a_k is an $\mathcal{M}_c$-atom or an element in $L^1(\mathcal{M})$ of norm less than 1, and $\lambda_k \in \mathbb{C}$ so that $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$
\|f\|_{\mathrm{H}^1_c} = \inf \left\{ \sum_k |\lambda_k| : f = \sum_k \lambda_k a_k \right\},
$$

where the infimum is taken over all decompositions of f described above.

PROPOSITION 6. *Let* $\mathrm{H}^1(\mathbb{T}, L^1(\mathcal{M}))$ *be the Hardy space of* $L^1(\mathcal{M})$-*valued functions on* $\mathbb{T}$. *If* $f \in \mathrm{H}^1_c(\mathbb{T}, \mathcal{M})$, *then*

$$(22) \qquad\qquad\qquad \|f\|_{\mathrm{H}^1} \le \|f\|_{\mathrm{H}^1_c}.$$

Consequently, $\mathrm{H}^1_c(\mathbb{T}, \mathcal{M}) \subset \mathrm{H}^1(\mathbb{T}, L^1(\mathcal{M}))$ *and is a Banach space under the norm* $\| \cdot \|_{\mathrm{H}^1_c}$.

Proof. As noted above, $\mathcal{M}_c$-atoms are all $L^1(\mathcal{M})$-valued 2-atoms. This implies the required result. $\square$

DEFINITION 7. We define

$$\mathcal{H}_c^1(\mathbb{T}, \mathcal{M}) = \{f \in \mathrm{H}_c^1(\mathbb{T}, \mathcal{M}) : P(f) \in \mathcal{H}(\mathbb{D}, L^1(\mathcal{M}))\},$$

equipped with the norm $\|f\|_{\mathrm{H}_c^1}$.

By (22) one concludes that $\mathcal{H}_c^1(\mathbb{T}, \mathcal{M}) \subset \mathcal{H}^1(\mathbb{T}, L^1(\mathcal{M}))$, the Hardy space of $L^1(\mathcal{M})$-valued functions on $\mathbb{T}$, the Poisson integral of which are analytic in $\mathbb{D}$. Hence, $\mathcal{H}_c^1(\mathbb{T}, \mathcal{M})$ is a Banach space.

Similarly, we can define $\mathrm{H}_r^1(\mathbb{T}, \mathcal{M})$ and $\mathcal{H}_r^1(\mathbb{T}, \mathcal{M})$.

DEFINITION 8. The Hardy space of operator-valued analytic functions in the unit disc $\mathbb{D}$ is defined by

$$\mathcal{H}_{cr}^1(\mathbb{T}, \mathcal{M}) = \mathcal{H}_c^1(\mathbb{T}, \mathcal{M}) + \mathcal{H}_r^1(\mathbb{T}, \mathcal{M}),$$

equipped with the sum norm

$$\|f\|_{\mathcal{H}_{cr}^1} = \inf\{\|g\|_{\mathrm{H}_c^1} + \|h\|_{\mathrm{H}_r^1}\},$$

where the infimum is taken over all $g \in \mathcal{H}_c^1(\mathbb{T}, \mathcal{M})$ and $h \in \mathcal{H}_r^1(\mathbb{T}, \mathcal{M})$ such that $f = g + h$.

Clearly, $\mathcal{H}_{cr}^1(\mathbb{T}, \mathcal{M})$ is a Banach space. Let $\mathcal{H}^1(\mathbb{D})$ denote the analytic Hardy space in the unit disc $\mathbb{D}$.

PROPOSITION 9. *If $f \in \mathcal{H}_{cr}^1(\mathbb{T}, \mathcal{M})$ and $m \in L^\infty(\mathcal{M})$, then $\tau(mf) \in \mathcal{H}^1(\mathbb{D})$ and*

(23)
$$\|\tau(mf)\|_{\mathrm{H}^1} \leq \|m\|_\infty \|f\|_{\mathrm{H}_{cr}^1}.$$

Proof. It is sufficient to prove that

$$\|\tau(ma)\|_{\mathrm{H}^1} \leq \|m\|_\infty$$

for each $\mathcal{M}_c$-atom a. Let a be an $\mathcal{M}_c$-atom supported in I. By Lemma 2, we have

$$\left(\int_I |\tau(ma)|^2\, dm\right)^{1/2} \leq \|m\|_\infty \tau\left(\int_I |a|^2\, dm\right)^{1/2} \leq \|m\|_\infty |I|^{-1/2}.$$

This shows that $\tau(ma)/\|m\|_\infty$ is an 2-atom supported in I. The proof is complete. $\square$

REMARK 1. The $\mathcal{M}_c$-atom is a noncommutative analogue of the classical 2-atom for H^1. However, the noncommutative analogues of classical p-atoms seem to be of no meaningfulness.

4. Operator-valued BMOA space

Let $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$. Then, by (6) one concludes that $\langle f, g \rangle \in \mathcal{M}$ for each $g \in L^\infty(\mathbb{T})$. Thus, for a subarc I of $\mathbb{T}$ we can define the mean value f_I of f over I by $f_I = \int_I f \, dm / |I| \in \mathcal{M}$. Set

$$\|f\|_{*,c} = \sup_I \left\| \left(\frac{1}{|I|} \int_I |f - f_I|^2 \, dm \right)^{1/2} \right\|_{\mathcal{M}},$$

where the supremum is taken over all subarcs I of $\mathbb{T}$. We define

$$\mathrm{BMO}_c(\mathbb{T}, \mathcal{M}) = \{ f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T})) : \|f\|_{*,c} < \infty \}$$

equipped with the norm

$$\|f\|_{\mathrm{BMO}_c} = \left\| \int_{\mathbb{T}} f \, d\sigma \right\|_{\mathcal{M}} + \|f\|_{*,c}.$$

PROPOSITION 10. *If* $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$, *then*

$$(24) \qquad \|f\|_{*,c} \leq \sup_I \left(\frac{1}{|I|} \int_I \|f - f_I\|_{\mathcal{M}}^2 \, dm \right)^{1/2}$$

and

$$(25) \qquad \left\| \int_{\mathbb{T}} f \, dm \right\|_{\mathcal{M}} \leq \|f\|_{L_c^\infty} \leq \left\| \int_{\mathbb{T}} f \, dm \right\|_{\mathcal{M}} + \|f\|_{*,c}.$$

Consequently, $\mathrm{BMO}(\mathbb{T}, \mathcal{M}) \subset \mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$ *and* $\mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$ *is a Banach space. Here,* $\mathrm{BMO}(\mathbb{T}, \mathcal{M})$ *is the* BMO *space of* $\mathcal{M}$-*valued functions on* $\mathbb{T}$.

Proof. Let $\mathbb{H}$ be the Hilbert space on which $\mathcal{M}$ acts. For every $h \in \mathbb{H}$, one has

$$\int_I \|f(h) - f(h)_I\|^2 \, dm = \int_I \langle |f - f_I|^2(h), h \rangle \, dm = \left\langle \left(\int_I |f - f_I|^2 \, dm \right)(h), h \right\rangle,$$

which yields that

$$(26) \qquad \|f\|_{*,c} = \sup_{h \in \mathbb{H}, \|h\| \leq 1} \|f(h)\|_{\mathrm{BMO}(\mathbb{T},\mathbb{H})},$$

where $\mathrm{BMO}(\mathbb{T}, \mathbb{H})$ is the $\mathbb{H}$-valued BMO space on $\mathbb{T}$. This concludes (24) and so, $\mathrm{BMO}(\mathbb{T}, \mathcal{M}) \subset \mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$.

Since

$$\|f\|_{L_c^\infty}^2 = \left\| \int_{\mathbb{T}} |f|^2 \, dm \right\|_{\mathcal{M}} = \sup_{\|h\| \leq 1} \left\langle \int_{\mathbb{T}} |f|^2 \, dm(h), h \right\rangle = \sup_{\|h\| \leq 1} \int_{\mathbb{T}} \|f(h)\|^2 \, dm,$$

it is concluded that

$$\|f\|_{L_c^\infty} = \sup_{\|h\| \leq 1} \left(\int_{\mathbb{T}} \|f(h)\|^2 \, dm \right)^{1/2} \geq \sup_{\|h\| \leq 1} \int_{\mathbb{T}} \|f(h)\| \, dm \geq \left\| \int_{\mathbb{T}} f \, dm \right\|_{\mathcal{M}}$$

and

$$\|f\|_{L_c^\infty} = \sup_{\|h\|\leq 1} \left(\int_{\mathbb{T}} \|f(h)\|^2 \, dm \right)^{1/2}$$

$$\leq \sup_{\|h\|\leq 1} \left(\int_{\mathbb{T}} \left\| f(h) - \int_{\mathbb{T}} f(h) \, dm \right\|^2 dm \right)^{1/2} + \sup_{\|h\|\leq 1} \left\| \int_{\mathbb{T}} f(h) \, dm \right\|$$

$$\leq \left\| \int_{\mathbb{T}} f \, dm \right\|_{\mathcal{M}} + \|f\|_{*,c}.$$

This completes the proof of (25). □

DEFINITION 11. We define

$$\mathrm{BMOA}_c(\mathbb{T}, \mathcal{M}) = \{ f \in \mathrm{BMO}_c(\mathbb{T}, \mathcal{M}) : P[f] \in \mathcal{H}(\mathbb{D}, \mathcal{M}) \},$$

equipped with the norm $\|\cdot\|_{\mathrm{BMO}_c}$.

By (24) and (26), we conclude that $\mathrm{BMOA}(\mathbb{T}, \mathcal{M}) \subset \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$ and $\mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$ is Banach space.

For $f \in \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$, we introduce the set of functions

$$M_f = \{ g \in \mathcal{H}(\mathbb{D}, \mathcal{M}) : g = f \circ \psi - f \circ \psi(0), \psi \in \mathrm{Aut}(\mathbb{D}) \},$$

and for $0 < \gamma < 1$, we denote by f_γ the dilated function defined for $|z| < 1/\gamma$ by $f_\gamma(z) = f(\gamma z)$. Also, for $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$ set

$$\|f\|_{**,c} = \sup_{z \in \mathbb{D}} \left\| \left(\int_{\mathbb{T}} |f - f(z)|^2 \, dm_z \right)^{1/2} \right\|_{\mathcal{M}},$$

where $dm_z(t) = P_z(t) \, dm(t)$ for $z \in \mathbb{D}$.

PROPOSITION 12. If $f \in \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$, then

(27) $$\|f\|_{*,c} \asymp \sup_{g \in M_f} \sup_{0 < \gamma < 1} \|g_\gamma\|_{L_c^\infty} \asymp \|f\|_{**,c}.$$

Proof. Let $\mathbb{H}$ be the Hilbert space on which $\mathcal{M}$ acts. By merely reproducing the proof in the case of scalars (for details, see [1]), we can obtain (27) for $\mathbb{H}$-valued functions. Then, by (26) we have

$$\|f\|_{*,c} = \sup_{h \in \mathbb{H}, \|h\| \leq 1} \|f(h)\|_{\mathrm{BMO}(\mathbb{T}, \mathbb{H})}$$

$$\asymp \sup_{\|h\| \leq 1} \sup_{g \in M_f} \sup_{0 < \gamma < 1} \left(\int_{\mathbb{T}} \|g_\gamma(h)\|^2 \, dm \right)^{1/2}$$

$$= \sup_{g \in M_f} \sup_{0 < \gamma < 1} \|g_\gamma\|_{L_c^\infty}.$$

This concludes the first equivalence.

Similarly, for $f \in \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$ we have

$$
\begin{aligned}
\|f\|_{*,c} &= \sup_{h \in \mathbb{H}, \|h\| \leq 1} \|f(h)\|_{\mathrm{BMO}(\mathbb{T}, \mathbb{H})} \\
&\asymp \sup_{\|h\| \leq 1} \sup_{z \in \mathbb{D}} \left(\int_{\mathbb{T}} \|f(h) - f(h)(z)\|^2 \, dm_z \right)^{1/2} \\
&= \sup_{z \in \mathbb{D}} \left\| \left(\int_{\mathbb{T}} |f - f(z)|^2 \, dm_z \right)^{1/2} \right\|_{\mathcal{M}}.
\end{aligned}
$$

This proves that $\|f\|_{*,c} \asymp \|f\|_{**,c}$ and completes the proof. $\qquad\square$

Similarly, we can define $\mathrm{BMO}_r(\mathbb{T}, \mathcal{M})$ and $\mathrm{BMOA}_r(\mathbb{T}, \mathcal{M})$, by letting

$$
\|f\|_{*,r} = \|f^*\|_{*,c} \quad \text{and} \quad \|f\|_{\mathrm{BMO}_r} = \|f^*\|_{\mathrm{BMO}_c},
$$

respectively. Evidently, we have

PROPOSITION 13. *If* $f \in \mathrm{BMOA}_r(\mathbb{T}, \mathcal{M})$, *then*

(28)
$$
\|f\|_{*,r} \asymp \sup_{g \in M_f} \sup_{0 < \gamma < 1} \|g_\gamma\|_{L_r^\infty} \asymp \|f\|_{**,r}.
$$

Here, $\|f\|_{**,r} = \|f^*\|_{**,c}$.

We define

$$
\mathrm{BMO}_{cr}(\mathbb{T}, \mathcal{M}) = \mathrm{BMO}_c(\mathbb{T}, \mathcal{M}) \cap \mathrm{BMO}_r(\mathbb{T}, \mathcal{M}),
$$

quipped with the norm $\|f\|_{\mathrm{BMO}_{cr}} = \max\{\|f\|_{\mathrm{BMO}_c}, \|f\|_{\mathrm{BMO}_r}\}$. As shown above, $\mathrm{BMO}(\mathbb{T}, \mathcal{M}) \subset \mathrm{BMO}_{cr}(\mathbb{T}, \mathcal{M})$.

DEFINITION 14. The operator-valued BMOA space on $\mathbb{T}$ is defined as follows:

$$
\mathrm{BMOA}_{cr}(\mathbb{T}, \mathcal{M}) = \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M}) \cap \mathrm{BMOA}_r(\mathbb{T}, \mathcal{M}),
$$

quipped with the norm $\|f\|_{\mathrm{BMO}_{cr}}$.

$\mathrm{BMOA}_{cr}(\mathbb{T}, \mathcal{M})$ is evidently a Banach space under the norm $\| \cdot \|_{\mathrm{BMO}_{cr}}$. Moreover, $\mathrm{BMOA}(\mathbb{T}, \mathcal{M}) \subset \mathrm{BMOA}_{cr}(\mathbb{T}, \mathcal{M})$.

It is well-known that BMO spaces are related to the so-called Carleson measures (e.g., see [4], [13]). In the sequel, we do this in the operator-valued setting on the circle, with an emphasis on the 'analytical' aspect.

For $t \in \mathbb{T}$ and $\delta > 0$, we introduce the set

$$
\hat{I}(t_0, \delta) = \{rt \in \mathbb{D} : 1 - \delta \leq r < 1, |t - t_0| < \delta\},
$$

whose closure intersects $\mathbb{T}$ at the subarc $I(t_0, \delta) = \{t \in \mathbb{T} : |t - t_0| < \delta\}$. $\hat{I}(t, \delta)$ is said to be a Carleson tube at t.

DEFINITION 15. An $\mathcal{M}_+$-valued Borel measure ν in $\mathbb{D}$ is called a Carleson measure if there exists a constant $C > 0$ such that

$$\|\nu(\hat{I}(t,\delta))\|_{\mathcal{M}} \le C\delta$$

for all $t \in \mathbb{T}$ and $\delta > 0$. Set

$$\|\nu\|_c = \sup_I \frac{\|\nu(\hat{I}(t,\delta))\|_{\mathcal{M}}}{\delta},$$

which is called the Carleson norm of ν.

Since $|I(t,\delta)| \asymp \delta$ $(0 < \delta < 1)$ and $\hat{I}(t,\delta) = \mathbb{D}$ provided $\delta > 1$, it is concluded that an $\mathcal{M}_+$-valued Borel measure ν in $\mathbb{D}$ is a Carleson measure if and only if

$$(29) \qquad \sup\left\{ \frac{\|\nu(\hat{I}(t,\delta))\|_{\mathcal{M}}}{|I(t,\delta)|} : t \in \mathbb{T}, \delta > 0 \right\} < \infty.$$

PROPOSITION 16. *An $\mathcal{M}_+$-valued Borel measure ν in $\mathbb{D}$ is a Carleson measure if and only if*

$$\mathcal{N}(\nu) = \sup_{z \in \mathbb{D}} \left\| \int_{\mathbb{D}} P_z(w)\, d\nu(w) \right\|_{\mathcal{M}} < \infty.$$

The constant $\mathcal{N}(\nu)$ satisfies

$$C_1 \mathcal{N}(\nu) \le \|\nu\|_c \le C_2 \mathcal{N}(\nu)$$

for absolute constants C_1 and C_2.

The proof is the same as in the scalar case (e.g., see Lemma 3.3 in [4]) and omitted. Proposition 16 shows the conformally invariant character of Carleson measures.

The following theorem is the operator-valued version of one of the most fundamental results in the theory of BMO spaces, which characterizes BMO spaces in terms of Carleson measures.

THEOREM 17. *Let $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$. Then, the following assertions are equivalent:*

(1) *f is in $\mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$.*
(2) *$d\nu_f(z) = |\nabla f(z)|^2 (1 - |z|^2)\, dx\, dy$ is a Carleson measure in $\mathbb{D}$.*
(3) *$d\lambda_f(z) = |\nabla f(z)|^2 \log\frac{1}{|z|}\, dx\, dy$ is a Carleson measure in $\mathbb{D}$.*

In this case,

$$\|\nu_f\|_c \asymp \|f\|_{*,c}^2 \asymp \|\lambda_f\|_c.$$

Consequently, if $f \in \mathcal{H}^\infty(\mathcal{M}, L_{cr}^2(\mathbb{T}))$ then $f \in \mathrm{BMOA}_{cr}(\mathbb{T}, \mathcal{M})$ if and only if both $(1 - |z|^2)|f'(z)|^2\, dx\, dy$ and $(1 - |z|^2)|f'(z)^*|^2\, dx\, dy$ are $\mathcal{M}_+$-valued Carleson measures in $\mathbb{D}$ if and only if

$$|f'(z)|^2 \log\frac{1}{|z|}\, dx\, dy \quad and \quad |f'(z)^*|^2 \log\frac{1}{|z|}\, dx\, dy$$

are both $\mathcal{M}_+$-valued Carleson measures in $\mathbb{D}$.

Proof. By Lemma 3, Propositions 12 and 16 we get the equivalence of (1) and (2) and $\|\nu_f\|_c \asymp \|f\|^2_{*,c}$. What remains to prove is the equivalence of (2) and (3). Half of this task is trivial because the inequality $1 - |z|^2 \le 2\log(1/|z|)$ shows that $\|\nu_f\|_c \le 2\|\lambda_f\|_c$. For $|z| > 1/4$, we have the reverse inequality $\log(1/|z|) \le C(1 - |z|^2)$, which shows that $\|\lambda_f(\hat{I})\| \le C\|\nu_f(\hat{I})\|$ for subarcs $I = I(t, \delta)$ provided $\delta \le 3/4$, because $|z| > 1 - \delta \ge 1/4$ for all $z \in \hat{I}$. Then, to prove that the equivalence of (2) and (3) it suffices to prove $\|\lambda_f(|z| \le 1/4)\| \le C\|\nu_f(|z| \le 1/2)\|$. However, as shown in the proof of Lemma 3 we have

$$|\nabla f(z)|^2 \le C \int_{|z|<1/2} |\nabla f(w)|^2 (1 - |w|^2)\, dx\, dy = C\nu_f(|z| \le 1/2)$$

for all $|z| < 1/4$. Hence,

$$\lambda_f(|z| \le 1/4) = \int_{|z|\le 1/4} |\nabla f(z)|^2 \log \frac{1}{|z|}\, dx\, dy \le C\nu_f(|z| \le 1/2).$$

This gives that $\|\lambda_f(|z| \le 1/4)\| \le C\|\nu_f(|z| \le 1/2)\|$ and therefore $\|\lambda_f\|_c \le C\|\nu_f\|_c$. $\qquad\square$

5. Operator-valued H^1-BMOA duality

In this section, we show the operator-valued H^1-BMOA duality.

THEOREM 18. *We have*

$$\mathrm{H}^1_c(\mathbb{T}, \mathcal{M})^* = \mathrm{BMO}_c(\mathbb{T}, \mathcal{M}), \qquad \mathcal{H}^1_c(\mathbb{T}, \mathcal{M})^* = \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M}).$$

Similarly, $\mathrm{H}^1_r(\mathbb{T}, \mathcal{M})^* = \mathrm{BMO}_r(\mathbb{T}, \mathcal{M})$ *and* $\mathcal{H}^1_r(\mathbb{T}, \mathcal{M})^* = \mathrm{BMOA}_r(\mathbb{T}, \mathcal{M})$. *Consequently,* $\mathcal{H}^1_{cr}(\mathbb{T}, \mathcal{M})^* = \mathrm{BMOA}_{cr}(\mathbb{T}, \mathcal{M})$.

Proof. Let $g \in \mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$. For any $\mathcal{M}_c$-atom a supported in I, we have by Lemma 1

$$\left|\tau\left(\int_{\mathbb{T}} g^* a\, dm\right)\right| = \left|\tau\left(\int_I (g - g_I)^* a\, dm\right)\right|$$
$$\le \|a\|_{L^1(\mathcal{M}, L^2_c(I))} \|g - g_I\|_{L^\infty(\mathcal{M}, L^2_c(I))}$$
$$\le \|g\|_{*,c}.$$

On the other hand, for any $a \in L^1(\mathcal{M})$ with $\|a\| \le 1$ we have

$$\left|\tau\left(\int_{\mathbb{T}} g^* a\, dm\right)\right| \le \left\|\int_{\mathbb{T}} g^*\, dm\right\| \tau(|a|) \le \left\|\int_{\mathbb{T}} g\, dm\right\|.$$

Thus, we deduce that

$$(30) \qquad \left|\tau\left(\int_{\mathbb{T}} g^* f\, dm\right)\right| \le \|g\|_{\mathrm{BMO}_c} \|f\|_{\mathrm{H}^1_c}$$

for all $f \in \mathrm{H}^1_c(\mathbb{T}, \mathcal{M})$. Hence, $\mathrm{BMO}_c(\mathbb{T}, \mathcal{M}) \subset \mathrm{H}^1_c(\mathbb{T}, \mathcal{M})^*$.

Conversely, let $l \in \mathrm{H}_c^1(\mathbb{T}, \mathcal{M})^*$. Since $L^1(\mathcal{M}, L_c^2(\mathbb{T})) \subset \mathrm{H}_c^1(\mathbb{T}, \mathcal{M})$, l induces a continuous functional on $L^1(\mathcal{M}, L_c^2(\mathbb{T}))$ with the norm smaller than $\|l\|_{(\mathrm{H}_c^1)^*}$, that is, $|l(f)| \leq \|l\|_{(\mathrm{H}_c^1)^*} \|f\|_{L_c^1}$ for all $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$. Hence, by the Hahn–Banach extension theorem there exists a unique $g \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$ with $\|g\|_{L_c^\infty} \leq \|l\|_{(\mathcal{H}_c^1)^*}$ such that

$$(31) \qquad l(f) = \tau\left(\int_{\mathbb{T}} g^* f \, dm\right)$$

for all $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$. We need to show that $g \in \mathrm{BMO}_c(\mathbb{T}, \mathcal{M})$.

Let $\mathbb{H}$ be the Hilbert space on which $\mathcal{M}$ acts. Recall that the predual space $L^1(\mathcal{M})$ of $\mathcal{M}$ is the quotient space $L^1(\mathcal{B}(\mathbb{H}))/\mathcal{M}_\perp$, where $\mathcal{M}_\perp = \{a \in L^1(\mathcal{B}(\mathbb{H})) : \mathrm{Tr}(ab) = 0, \forall b \in \mathcal{M}\}$ is the pre-annihilator of $\mathcal{M}$. The quotient map is dented by $\pi : L^1(\mathcal{B}(\mathbb{H})) \to L^1(\mathcal{B}(\mathbb{H}))/\mathcal{M}_\perp$. For $u, v \in \mathbb{H}$ we define $|u\rangle\langle v|$ by $|u\rangle\langle v|(h) = \langle h, v\rangle u$ for all $h \in \mathbb{H}$. Then, for every $a \in \mathcal{M}$ one has

$$\tau(a\pi[|u\rangle\langle v|]) = \tau(\pi[a|u\rangle\langle v|]) = \mathrm{Tr}(a|u\rangle\langle v|) = \langle a(u), v\rangle.$$

Consequently, by (26) and the Hilbert space-valued H^1-BMOA duality theorem we have

$$\|g\|_{*,c} = \sup_{h \in \mathbb{H}, \|h\|=1} \|g(h)\|_{\mathrm{BMO}(\mathbb{T}, \mathbb{H})}$$

$$= \sup_{\|h\|=1} \sup_{f \in \mathrm{H}^1(\mathbb{T}, \mathbb{H}), \|f\|_{\mathrm{H}^1} \leq 1} \left| \int_{\mathbb{T}} \langle g(h), f\rangle \, dm \right|$$

$$= \sup_{\|h\|=1} \sup_{\|f\|_{\mathrm{H}^1} \leq 1} \left| \int_{\mathbb{T}} \mathrm{Tr}[g|h\rangle\langle f|] \, dm \right|$$

$$= \sup_{\|h\|=1} \sup_{\|f\|_{\mathrm{H}^1} \leq 1} \left| \tau\left(\int_{\mathbb{T}} g\pi[|h\rangle\langle f|] \, dm\right) \right|$$

$$\leq \sup_{\|f\|_{\mathrm{H}_c^1} \leq 1} \left| \tau\left(\int_{\mathbb{T}} g f^* \, dm\right) \right|$$

$$= \|l\|_{(\mathrm{H}_c^1)^*}.$$

Also,

$$\left\| \int_{\mathbb{T}} g \, dm \right\| = \sup_{y \in L^1(\mathcal{M}), \|y\|_1 \leq 1} \left| \tau\left(\int_{\mathbb{T}} g^* y \, dm\right) \right|$$

$$\leq \sup_{f \in L_c^1, \|f\|_{L_c^1} \leq 1} \left| \tau\left(\int_{\mathbb{T}} g^* f \, dm\right) \right| \leq \|l\|_{(\mathrm{H}_c^1)^*}.$$

This shows that $\|g\|_{\mathrm{BMO}_c} \leq 2\|l\|_{(\mathrm{H}_c^1)^*}$ and concludes the first assertion.

By (30), each $g \in \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$ induces evidently a bounded linear functional on $\mathcal{H}_c^1(\mathbb{T}, \mathcal{M})$ via (31). Conversely, if $l \in \mathcal{H}_c^1(\mathbb{T}, \mathcal{M})^*$, then by the Hahn–Banach extension theorem, l can be extended to a bounded linear functional

on $H_c^1(\mathbb{T}, \mathcal{M})$ and hence there exists a $g \in L^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$ such that (31) holds true for all (analytic) polynomials $f = \sum_{k=1}^n b_k z^k$, where $b_k \in L^1(\mathcal{M})$. This concludes that

$$(32) \qquad l(P) = \tau\left(\int_{\mathbb{T}} \mathfrak{C}[g]^* P \, dm\right)$$

for all polynomials $P(z) = \sum_{k=1}^n b_k z^k$, where $b_k \in L^1(\mathcal{M})$. By merely reproducing the above proof, we can prove that $\mathfrak{C}[g] \in \mathrm{BMOA}_c(\mathbb{T}, \mathcal{M})$. $\square$

6. Area integral characterizations

We define for each $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$:

(i) The *column* area function

$$[A_c(f, \alpha)](t) = \left(\int_{\Gamma_\alpha(t)} |\nabla f(z)|^2 \, dx \, dy\right)^{1/2} \qquad (t \in \mathbb{T}),$$

(ii) The *column* Littlewood–Paley g-function

$$[g_c(f)](t) = \left(\int_0^1 |\nabla f(rt)|^2 (1 - r^2) \, dr\right)^{1/2} \qquad (t \in \mathbb{T}).$$

Similarly, we define the *row* area function $A_r(f, \alpha) = A_c(f^*, \alpha)$ and *row* Littlewood–Paley g-function $g_r(f) = g_c(f^*)$.

Also, we need two technical variants of $A_c(f, \alpha)$ and $g_c(f)$ as following:

$$[A_c(f, \alpha)](t, \delta) = \left(\int_{\Gamma_\alpha(t, \delta)} |\nabla f(z)|^2 \, dx \, dy\right)^{1/2},$$

where $\Gamma_\alpha(t, \delta) = \{z \in \mathbb{D} : |t - z| < \alpha(1 - |z|), |z| < \delta\}$, and

$$[g_c(f)](t, \delta) = \left(\int_0^\delta |\nabla f(rt)|^2 (1 - r^2) \, dr\right)^{1/2}$$

for $0 < \delta \le 1, t \in \mathbb{T}$.

For simplicity, we usually denote by $A_c(f) = A_c(f, \alpha)$ in the sequel.

LEMMA 19. *There is a constant $C > 0$ such that*

$$[g_c(f)](t, \delta) \le C[A_c(f)]\big(t, (1 + \delta)/2\big) \quad (0 < \delta \le 1, \ t \in \mathbb{T})$$

for all $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$. In particular, $g_c(f) \le C A_c(f)$.

Proof. It suffices to prove the associated inequality for the case of $t = 1$. Since $\Gamma_\alpha(1, (1 + \delta)/2)$ contains a small disc centered at 0, there exists a constant $0 < c_\alpha < 1$ such that for each $0 < r < \delta$,

$$D_r \triangleq \{z \in \mathbb{D} : |z - r| < c_\alpha[(1 + \delta)/2 - r]\} \subset \Gamma_\alpha\big(1, (1 + \delta)/2\big).$$

The harmonicity of ∇f gives that

$$\nabla f(r) = \frac{1}{c_\alpha^2 \pi [(1+\delta)/2 - r]^2} \int_{D_r} \nabla f(z) \, dx \, dy.$$

This follows from (8) that

$$|\nabla f(r)|^2 \leq \frac{C}{(1-r)^2} \int_{D_r} |\nabla f(z)|^2 \, dx \, dy.$$

Hence,

$$[g_c(f)](1,\delta) = \int_0^\delta |\nabla f(r)|^2 (1-r^2) \, dr \leq C \int_0^\delta \frac{dr}{1-r} \int_{D_r} |\nabla f(z)|^2 \, dx \, dy.$$

Since

$$(1-|z|)/(1+c_\alpha) < 1 - r < (1-|z|)/(1-c_\alpha)$$

for $z \in D_r$, by Fubini's theorem we have

$$[g_c(f)](1,\delta) \leq C \int_{\Gamma_\alpha(1, \frac{1+\delta}{2})} |\nabla f(z)|^2 \int_{\frac{1-|z|}{1+c_\alpha}}^{\frac{1-|z|}{1-c_\alpha}} \frac{dr}{r} \, dx \, dy \leq C[A_c(f)]\left(1, \frac{1+\delta}{2}\right).$$

This completes the proof. $\qquad\square$

THEOREM 20. *Let* $f \in L^1(\mathcal{M}, L_c^2(\mathbb{T}))$. *The following assertions are equivalent:*

(1) $f \in \mathrm{H}_c^1(\mathbb{T}, \mathcal{M})$.
(2) $A_c(f) \in L^1(L^\infty(\mathbb{T}) \otimes \mathcal{M})$.
(3) $g_c(f) \in L^1(L^\infty(\mathbb{T}) \otimes \mathcal{M})$.

In this case,

$$(33) \qquad \|f\|_{\mathrm{H}_c^1} \asymp \|f(0)\|_{L^1(\mathcal{M})} + \|A_c(f)\|_{L^1} \asymp \|f(0)\|_{L^1(\mathcal{M})} + \|g_c(f)\|_{L^1}$$

for all $f \in \mathrm{H}_c^1(\mathbb{T}, \mathcal{M})$.

Consequently, if $f \in \mathcal{H}^1(\mathcal{M}, L_c^2(\mathbb{T}))$, *then* $f \in \mathcal{H}_c^1(\mathbb{T}, \mathcal{M})$ *if and only if* $A_c(f) \in L^1(\mathbb{T}, L^1(\mathcal{M}))$ *if and only if* $g_c(f) \in L^1(\mathbb{T}, L^1(\mathcal{M}))$.

The same statements hold also true for $\mathrm{H}_r^1(\mathbb{T}, \mathcal{M})$ *and* $\mathcal{H}_r^1(\mathbb{T}, \mathcal{M})$.

Proof. We will show that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and finally (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). We will show that there exists $C > 0$ such that

$$(34) \qquad \|f(0)\|_{L^1(\mathcal{M})} + \|A_c(f)\|_{L^1} \leq C\|f\|_{\mathrm{H}_c^1}$$

for all $f \in \mathrm{H}_c^1(\mathbb{T}, \mathcal{M})$. Since

$$\|f(0)\|_{L^1(\mathcal{M})} \leq C\|f\|_{\mathrm{H}^1} \leq C\|f\|_{\mathrm{H}_c^1}$$

by Proposition 6, it suffices to show that there exists $C > 0$ such that

$$\|A_c(a)\|_{L^1} \leq C$$

for all $\mathcal{M}_c$-atom a.

Given an $\mathcal{M}_c$-atom a supported in $I = I(t_0, \delta)$. By (6) and Lemma 4, we have

$$\tau\left(\int_{2I} A_c(a)\, dm\right) \leq |2I|^{1/2}\tau\left(\int_{2I} |A_c(a)|^2\, dm\right)^{1/2} \leq C|I|^{1/2}\|a\|_{L_c^1} \leq C,$$

where $2I = I(t_0, 2\delta)$, and by (8)

$$\tau\left(\int_{\mathbb{T}\backslash 2I} A_c(a)\, dm\right)$$

$$= \int_{\mathbb{T}\backslash 2I} \tau\left[\left(\int_{\Gamma_\alpha(t)} |\nabla a(z)|^2\, dx\, dy\right)^{\frac{1}{2}}\right] dm(t)$$

$$= \int_{\mathbb{T}\backslash 2I} \tau\left[\left(\int_{\Gamma_\alpha(t)} \left|\int_I (\nabla[P_z(s) - P_z(t_0)])a(s)\, dm(s)\right|^2 dx\, dy\right)^{\frac{1}{2}}\right] dm(t)$$

$$\leq \|a\|_{L_c^1}\int_{\mathbb{T}\backslash 2I}\left(\int_{\Gamma_\alpha(t)}\int_I |\nabla[P_z(s) - P_z(t_0)]|^2\, dm(s)\, dx\, dy\right)^{\frac{1}{2}} dm(t)$$

$$\leq \int_{\mathbb{T}\backslash 2I}\left(\int_{\Gamma_\alpha(t)}\sup_{s\in I}|\nabla[P_z(s) - P_z(t_0)]|^2\, dx\, dy\right)^{\frac{1}{2}} dm(t).$$

An immediate computation yields that

$$\frac{\partial P_z(t)}{\partial x} = -\frac{2x}{|1 - \bar{t}z|^2} + \frac{2(1 - |z|^2)\operatorname{Re}(t - z)}{|1 - \bar{t}z|^4},$$

$$\frac{\partial P_z(t)}{\partial y} = -\frac{2y}{|1 - \bar{t}z|^2} + \frac{2(1 - |z|^2)\operatorname{Im}(t - z)}{|1 - \bar{t}z|^4},$$

and hence,

$$\left|\frac{\partial}{\partial\gamma}\left[\frac{\partial P_z(t_0 + \gamma t)}{\partial x}\right]\right|, \left|\frac{\partial}{\partial\gamma}\left[\frac{\partial P_z(t_0 + \gamma t)}{\partial y}\right]\right| \leq \frac{C|t|}{|1 - (t_0 + \gamma t)\bar{z}|^3}$$

for $0 < \gamma < 1$. Then,

$$\left|\nabla[P_z(s) - P_z(t_0)]\right|^2 = \left|\frac{\partial P_z(s)}{\partial x} - \frac{\partial P_z(t_0)}{\partial x}\right|^2 + \left|\frac{\partial P_z(s)}{\partial y} - \frac{\partial P_z(t_0)}{\partial y}\right|^2$$

$$\leq \frac{C|s - t_0|^4}{|1 - [t_0 + \gamma(s - t_0)]\bar{z}|^6}$$

for some $0 < \gamma < 1$. Since

$$|s' - t_0| = \gamma|s - t_0| \leq \delta,$$

where $s' = t_0 + \gamma(s - t_0)$, it is concluded that

$$|t - t_0| \leq 2|t - s'| \leq C(|1 - t\bar{z}| + |1 - s'\bar{z}|) \leq C(1 - |z| + |1 - s'\bar{z}|) \leq C|1 - s'\bar{z}|$$

for $t \in \mathbb{T} \setminus 2I$ and $z \in \Gamma_\alpha(t)$, the first inequality is a consequence of the triangle inequality and the hypotheses. Hence,

$$|\nabla[P_z(s) - P_z(t_0)]|^2 \leq \frac{C\delta^4}{|1 - t_0\bar{t}|^6}$$

for $s \in I(t_0, \delta), t \in \mathbb{T} \setminus 2I$ and $z \in \Gamma_\alpha(t)$. This concludes that

$$\tau\left(\int_{\mathbb{T} \setminus 2I} A_c(a)\, dm\right) \leq \int_{\mathbb{T} \setminus 2I} \frac{C\delta^2}{|1 - t_0\bar{t}|^3}\, dm(t) \leq C,$$

which completes the proof of that $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$. This follows from Lemma 19.

$(3) \Rightarrow (1)$. Since $\|f(0)\|_{\mathrm{H}_c^1} \leq \|f(0)\|_{L^1(\mathcal{M})}$, by Theorem 18 it suffices to show that

$$\left|\tau\left(\int_{\mathbb{T}} fg^*\, dm\right)\right| \leq C\|g_c(f)\|_{L^1(\mathbb{T}, L^1(\mathcal{M}))}\|g\|_{\mathrm{BMO}_c}$$

for all $f \in L_0^1(\mathcal{M}, L_c^2(\mathbb{T}))$ and $g \in L_0^\infty(\mathcal{M}, L_c^2(\mathbb{T}))$. By Lemma 3 and the Cauchy–Schwarz inequality, we have

$$\left|\tau\left(\int_{\mathbb{T}} fg^*\, dm\right)\right|$$

$$= \frac{1}{\pi}\left|\tau\left(\int_{\mathbb{D}} \nabla f(z)(\nabla g)^*(z) \log \frac{1}{|z|}\, dx\, dy\right)\right|$$

$$= \frac{2}{\pi}\left|\tau\left(\int_0^1 r\log \frac{1}{r}\, dr \int_{\mathbb{T}} \nabla f(rt)(\nabla g)^*(rt)\, dm(t)\right)\right|$$

$$\leq \frac{2}{\pi}\left[\tau\left(\int_0^1 r\log \frac{1}{r}\, dr \int_{\mathbb{T}} [g_c(f)(r,t)]^{-\frac{1}{2}}|\nabla f(z)|^2[g_c(f)(r,t)]^{-\frac{1}{2}}\, dm(t)\right)\right]^{\frac{1}{2}}$$

$$\times \left[\tau\left(\int_0^1 r\log \frac{1}{r}\, dr \int_{\mathbb{T}} [g_c(f)(r,t)]^{\frac{1}{2}}|\nabla g(z)|^2[g_c(f)(r,t)]^{\frac{1}{2}}\, dm(t)\right)\right]^{\frac{1}{2}}$$

$$= \frac{2}{\pi}\left[\tau\left(\int_0^1 r\log \frac{1}{r}\, dr \int_{\mathbb{T}} [g_c(f)(r,t)]^{-1}|\nabla f(z)|^2\, dm(t)\right)\right]^{\frac{1}{2}}$$

$$\times \left[\tau\left(\int_0^1 r\log \frac{1}{r}\, dr \int_{\mathbb{T}} g_c(f)(r,t)|\nabla g(z)|^2\, dm(t)\right)\right]^{\frac{1}{2}}$$

$$\triangleq \frac{2}{\pi} A \cdot B.$$

For A, since $r\log(1/r) \leq C(1 - r^2)$ for $0 < r < 1$, we have

$$A^2 \leq C\tau\left(\int_{\mathbb{T}} \int_0^1 [g_c(f)(r,t)]^{-1}|\nabla f(rt)|^2(1 - r^2)\, dr\, dm(t)\right)$$

$$= C\tau\left(\int_{\mathbb{T}} \int_0^1 [g_c(f)(t,r)]^{-1}\frac{dg_c^2(f)(t,r)}{dr}\, dr\, dm(t)\right)$$

$$= C\tau\left(\int_{\mathbb{T}}\int_0^1 \frac{dg_c(f)(t,r)}{dr}\, dr\, dm(t)\right)$$

$$= C\|g_c(f)\|_{L^1(L^\infty(\mathbb{T})\otimes\mathcal{M})}.$$

To estimate B, we define

$$D(j,k) = \{(e^{2\pi i\theta}, r) : (j-1)2^{-k} \le \theta < j2^{-k},\ 2^{-k-1} < 1-r \le 2^{-k}\},$$

where $j = 1,\ldots,2^k$ and $k = 0,1,2,\ldots$. Let $c_{jk} = (e^{2\pi i\theta_{jk}}, 1-2^{-k})$ with $\theta_{jk} = (j-1/2)2^{-k}$. Set

$$[\tilde{A}_c(f,\alpha)](e^{2\pi i\theta}, r) \triangleq [A_c(f,3\pi\alpha)](c_{jk}) \quad \forall(e^{2\pi i\theta}, r) \in D(j,k)$$

and

$$d_k(t) \triangleq [\tilde{A}_c(f,\alpha)](t, 1-2^{-k-1}) - [\tilde{A}_c(f,\alpha)](t, 1-2^{-k}) \ge 0,$$

respectively. Since

$$|z - e^{2\pi i\theta_{jk}}| \le |z - e^{2\pi i\theta}| + |e^{2\pi i\theta} - e^{2\pi i\theta_{jk}}| \le \alpha(1-|z|) + 2\pi(1-|z|)$$
$$\le 3\pi\alpha(1-|z|)$$

for $z \in \Gamma_\alpha(e^{2\pi i\theta}, 1-2^{-k})$ with $(j-1)2^{-k} \le \theta < j2^{-k}$, it is concluded that

$$[A_c(f,\alpha)](t, 1-2^{-k}) \le [\tilde{A}_c(f,\alpha)](t, 1-2^{-k}) \le [A_c(f,5\pi\alpha)](t, 1-2^{-k})$$

for $t \in \mathbb{T}$. Then, we have

$$B^2 = \tau\left(\int_0^1 \int_{\mathbb{T}} g_c(f)(t,r)|\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\, dm(t)\right)$$

$$\le C\tau\left(\int_0^1 \int_{\mathbb{T}} A_c(f)\big(t, (1+r)/2\big)|\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\, dm(t)\right)$$

$$= C\tau\left(\int_{\mathbb{T}} dm(t) \sum_{k=0}^\infty \int_{1-2^{-k}}^{1-2^{-k-1}} A_c(f)\big(t, (1+r)/2\big)|\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\right)$$

$$\le C\tau\left(\int_{\mathbb{T}} dm(t) \sum_{k=0}^\infty [\tilde{A}_c(f,\alpha)](t, 1-2^{-k-2}) \int_{1-2^{-k}}^{1-2^{-k-1}} |\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\right)$$

$$= C\tau\left(\int_{\mathbb{T}} dm(t) \sum_{k=0}^\infty \left(\sum_{j=0}^{k+1} d_j(t)\right) \int_{1-2^{-k}}^{1-2^{-k-1}} |\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\right)$$

$$= C\tau\left(\int_{\mathbb{T}} dm(t) \left(d_0(t) \int_0^1 + \sum_{k=1}^\infty d_k(t) \int_{1-2^{-k+1}}^1\right) |\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\right)$$

$$\le C\tau[d_0(1/2)]\left\|\int_{\mathbb{T}}\int_0^1 |\nabla g(rt)|^2 r \log\frac{1}{r}\, dr\, dm(t)\right\|$$

$$+ C\sum_{k=1}^\infty \sum_{j=1}^{2^k} \tau[d_k(e^{2\pi\theta_{jk}})]\left\|\int_{(j-1)2^{-k}}^{j2^{-k}} d\theta \int_{1-2^{-k+1}}^1 |\nabla g(re^{2\pi\theta})|^2 r \log\frac{1}{r}\, dr\right\|.$$

Hence, by Theorem 17(2) one concludes that

$$B^2 \leq C \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \tau[d_k(e^{2\pi\theta_{jk}})] 2^{-k} \|g\|_{\mathrm{BMO}_c}^2$$

$$= C\|g\|_{\mathrm{BMO}_c}^2 \tau\left(\sum_{k=0}^{\infty} \int_{\mathbb{T}} d_k(t)\, dm(t) \right)$$

$$\leq C\|g\|_{\mathrm{BMO}_c}^2 \tau\left(\int_{\mathbb{T}} [A_c(f, 5\pi\alpha)](t, 1)\, dm(t) \right)$$

$$\leq C\|g\|_{\mathrm{BMO}_c}^2 \|A_c(f, 5\pi\alpha)\|_{L^1}.$$

Combining the estimates of A and B yields that

$$\|f\|_{\mathrm{H}_c^1} = \sup_{\|g\|_{\mathrm{BMO}_c} \leq 1} \left| \tau\left(\int_{\mathbb{T}} fg^* dm \right) \right| \leq C\|g_c(f, \alpha)\|_{L^1}^{\frac{1}{2}} \|A_c(f, 5\pi\alpha)\|_{L^1}^{\frac{1}{2}}.$$

Then, by (34) we have that

$$\|A_c(f, 5\pi\alpha)\|_{L^1} \leq C\|g_c(f, \alpha)\|_{L^1}.$$

Therefore,

$$\|f\|_{\mathrm{H}_c^1} \leq C\|g_c(f, \alpha)\|_{L^1}.$$

This completes the proof.   □

REMARK 2. Let us consider the conformal map

$$t(w) = i\frac{1-w}{1+w}, \quad w \in \mathbb{T}, w \neq -1,$$

which maps $\mathbb{T}$ to the real line $\mathbb{R}$. If $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ and if $\psi(w) = \varphi(t(w))$, then we see that

$$\|\varphi\|_{\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})} \asymp \|\psi\|_{\mathrm{BMO}_c(\mathbb{T}, \mathcal{M})},$$

following arguments in the classical case [4]. Consequently, by the H^1-BMO duality we have

$$\|\varphi\|_{\mathrm{H}_c^1(\mathbb{R}, \mathcal{M})} \asymp \|\psi\|_{\mathrm{H}_c^1(\mathbb{T}, \mathcal{M})}.$$

Thus, under the mapping $w \mapsto t(w)$, $\mathrm{H}_c^1(\mathbb{R}, \mathcal{M})$ and $\mathrm{H}_c^1(\mathbb{T}, \mathcal{M})$ are transformed into each other.

COROLLARY 21. *If $f \in \mathcal{H}^1(\mathcal{M}, L_{cr}^2(\mathbb{T}))$, then $f \in \mathcal{H}_{cr}^1(\mathbb{T}, \mathcal{M})$ if and only if there exist $g \in \mathcal{H}^1(\mathcal{M}, L_c^2(\mathbb{T}))$ and $h \in \mathcal{H}^1(\mathcal{M}, L_r^2(\mathbb{T}))$ such that $f = g + h$ with $A_c(g) \in L^1(\mathbb{T}, L^1(\mathcal{M}))$ and $A_r(h) \in L^1(\mathbb{T}, L^1(\mathcal{M}))$.*

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