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THE BOUNDEDNESS OF MARCINKIEWICZ INTEGRAL WITH VARIABLE KERNEL

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ABSTRACT. In this article, we study the fractional Marcinkiewicz integral with variable kernel defined by

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)\,dy \right|^2 \frac{dt}{t^{3-\alpha}} \right)^{1/2},$$

where $0 < \alpha \le 2$. We first prove that $\mu_{\Omega,\alpha}$ is bounded from $L^{2n/n+\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ without any smoothness assumption on the kernel Ω. Then we show that, if the kernel Ω satisfies a class of Dini condition, $\mu_{\Omega,\alpha}$ is bounded from $H^p(\mathbb{R}^n)$ $(p \le 1)$ to $H^q(\mathbb{R}^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2n}$. As corollary of the above results, we obtain the $L^p - L^q$ $(1 < p < 2)$ boundedness of this fractional Marcinkiewicz integral.

1. Introduction

In order to give an analogue of the Littlewood–Paley g-function without going into the interior of the unit disk, in 1938, Marcinkiewicz [M] introduced the function

$$\mu(f)(\theta) = \left(\int_0^{2\pi} |F(\theta+t) + F(\theta-t) - 2F(\theta)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F(\theta) = \int_0^\theta f(s)\,ds + C$. He conjectured that, for $1 < p < \infty$, $\|\mu(f)\|_{L^p} \le C_p \|f\|_{L^p}$, and if $\int_0^{2\pi} f(\theta)\,d\theta = 0$, then $\|f\|_{L^p} \le C'_p \|\mu(f)\|_{L^p}$. In 1944, using a complex variable method, Zygmund [Z] proved Marcinkiewicz's conjecture.

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In 1958, Stein [S] introduced the Marcinkiewicz integral of higher dimensions. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω_0 be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega_0 \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega_0(x')\, d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral of higher dimension is defined by

$$\mu_{\Omega_0}(f)(x) = \left(\int_0^\infty |F_{\Omega_0,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2}$$

and

$$F_{\Omega_0,t}(x) = \int_{|x-y|\leq t} \frac{\Omega_0(x-y)}{|x-y|^{n-1}} f(y)\, dy.$$

Stein proved that:

(a) if Ω_0 is odd, then μ_{Ω_0} is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$;
(b) if $\Omega_0 \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then μ_{Ω_0} is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$, and is of weak type $(1,1)$.

Stein also introduced a device to linearize $\mu_{\Omega_0}(f)(x)$ as follows. Let $\phi(x,t)$ be a function defined for $x \in \mathbb{R}^n$ and $0 < t < \infty$ such that

$$\int_0^\infty \frac{|\phi(x,t)|^2}{t^3}\, dt \leq 1 \quad \text{for all } x \in \mathbb{R}^n. \tag{1.1}$$

Define a linear operator T by

$$T(f)(x) = \int_0^\infty F_{\Omega_0,t}(x) \phi(x,t) \frac{dt}{t^3}.$$

Because of (1.1), by Schwarz's inequality, we have $|Tf(x)| \leq |\mu_{\Omega_0}(f)(x)|$ and

$$\mu_{\Omega_0}(f)(x) = \sup_{\phi(x,t)} |Tf(x)| \quad \text{for all } \phi(x,t) \text{ satisfying (1.1)}.$$

We point out that the Marcinkiewicz integral is essentially a Littlewood–Paley g-function. In fact, if let $\phi(x) = \Omega_0(x)|x|^{-n+1}\chi_B(x)$ and $\phi_t(x) = t^{-n} \times \phi(x/t)$, where B denotes the unit ball of $\mathbb{R}^n$ and χ_B denotes the characteristic function of B, then

$$\mu_{\Omega_0}(f)(x) = \left(\int_0^\infty |\phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} = g_\phi(f)(x).$$

It is well known that the Littlewood–Paley operators, such as the Littlewood–Paley g-function, the area integral S, and the Littlewood–Paley g_λ^*-function, play very important role in harmonic analysis. Therefore, many authors has been interested in studying the Marcinkiewicz integral μ_{Ω_0} since it was introduced by Stein (see [BCP], [DLeL], [H], [LeL], [LL], [SY]).

In order to consider nonsmoothness partial differential equations, mathematicians pay more attention to the singular integral with variable kernels (cf. [CZ1], [CZ2], [LLLY]). Specially, in 2004, Ding, Lin, and Shao [DLS] (also see [DLL]) considered the boundedness of Marcinkiewicz integral with variable kernel defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

and they proved the L^p $(1<p\le 2)$ boundedness, $H^1 - L^1$ boundedness, and $H^{1,\infty} - L^{1,\infty}$ boundedness of μ_Ω under certain conditions.

In this article, we consider the following fractional Marcinkiewicz integral with variable kernel defined by

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^{3-\alpha}}\right)^{1/2}, \quad 0<\alpha\le 2.$$

Furthermore, we may interpret $\mu_{\Omega,\alpha}$ by using Hilbert-valued function. Denote the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \left\{h(t) : \|h\|_{\mathcal{H}} = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t}\right)^{1/2} < +\infty\right\}$$

and let $h_f(t,x) = t^{\alpha/2-1} F_{\Omega,t}(x)$, where

$$F_{\Omega,t}(x) = \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)\,dy.$$

Then we have $\mu_{\Omega,\alpha}(f)(x) = \|h_f(\cdot,x)\|_{\mathcal{H}}$.

Before stating our main results, we recall the definition and results about the variable kernel $\Omega(x,z)$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \ge 1$, if $\Omega(x,z)$ satisfies the following conditions:

(i) $\Omega(x,\lambda z) = \Omega(x,z)$ for any $x,z \in \mathbb{R}^n$ and $\lambda > 0$;

(ii) $\int_{S^{n-1}} \Omega(x,z')\,d\sigma(z') = 0$ for any $x \in \mathbb{R}^n$, where $z' = z/|z|$ for $z \in \mathbb{R}^n \setminus \{0\}$;

(iii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} := \sup_{\substack{x\in\mathbb{R}^n \\ r\ge 0}} \left(\int_{S^{n-1}} |\Omega(x+rz',z')|^q\,d\sigma(z')\right)^{1/q} < \infty$.

In 1955, Calderón and Zygmund [CZ1] investigated the L^p boundedness of the singular integral operator T_Ω with variable kernel defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^n} f(y)\,dy.$$

They proved that if Ω satisfies (i), (ii), and

(iii′) $\sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x,z')|^q\,d\sigma(z')\right)^{1/q} < \infty$,

then T_Ω is bounded on L^2 provided $q > 2(n-1)/n$. They also found that these operators connect closely with the problem about the second order linear elliptic equations with variable coefficients. We note that the condition (iii) implies (iii′), so the L^2 boundedness of T_Ω still holds if we take (iii) in place of (iii′) (cf. [DLL], [DLS]).

For $q \ge 1$ and $0 < \beta \le 1$, a function $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ is said to satisfy the $L^{q,\beta}$-*Dini condition* if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\beta}}\, d\delta < \infty,$$

where

$$\omega_q(\delta) := \sup_{\substack{x\in\mathbb{R}^n\\ r\ge 0}} \left(\int_{S^{n-1}} \sup_{\substack{y'\in S^{n-1}\\ |y'-z'|\le\delta}} |\Omega(x+rz', y') - \Omega(x+rz', z')|^q \, d\sigma(z') \right)^{1/q}.$$

For the special case $\beta = 0$, it reduces to the L^q-Dini condition. It is clear that, for $\beta > 0$, $L^{q,\beta}$-Dini condition is stronger than L^q-Dini condition. As a contrast, we call a function Ω to satisfy the $\overline{L}^{q,\beta}$-*Dini condition* if it satisfies (i), (ii), (iii′), and

$$\int_0^1 \frac{\overline{\omega}_q(\delta)}{\delta^{1+\beta}}\, d\delta < \infty,$$

where

$$\overline{\omega}_q(\delta) = \sup_{\substack{x\in\mathbb{R}^n\\ \|\rho\|\le\delta}} \left(\int_{S^{n-1}} |\Omega(x, \rho z') - \Omega(x, z')|^q \, d\sigma(z') \right)^{1/q}$$

and ρ is a rotation in $\mathbb{R}^n$ with $\|\rho\| = \sup_{z'\in S^{n-1}} |\rho z' - z'|$. It is obvious that $\overline{\omega}_q(\delta) \le \omega_q(\delta)$ for all $\delta > 0$ and $q \ge 1$. Therefore, if Ω satisfies the $L^{q,\beta}$-Dini condition, then it also satisfies the $\overline{L}^{q,\beta}$-Dini condition.

We now present our main results as follows.

THEOREM 1. *Let $n \ge 2$ and $0 < \alpha < 1$. If Ω satisfies* (i), (ii), *and* (iii′) *for $q = 2$, then there exists a constant C independent of f such that*

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n)}.$$

THEOREM 2. *Let $n \ge 2$ and $0 < \alpha < 1$. Also let $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and set $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2n}$. Suppose that p and Ω satisfy one of the following conditions*:

(a) $\frac{2n}{2n+\alpha} \le p \le 1$, *$\Omega$ satisfies the $L^{q,\frac{\alpha}{2}}$-Dini condition*;

(b) $\max\{\frac{2n}{2n+1}, \frac{2n}{2n+\beta}\} < p < \frac{2n}{2n+\alpha}$ *for some β with $\alpha < \beta \le 2$, Ω satisfies the $L^{1,\frac{\beta}{2}}$-Dini condition.*

Then there exists a constant C independent of f such that

$$\|\mu_{\Omega,\alpha}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{H^p(\mathbb{R}^n)}.$$

We note that $0 < q \le 2$ in Theorem 2, so the $L^{2,\alpha}$-Dini condition implies the $L^{q,\alpha}$-Dini condition and the $L^{2,\beta}$-Dini condition implies the the $L^{2,\alpha}$-Dini condition for $0<\alpha<\beta\le 1$. By Theorems 1 and 2, using the interpolation theorem of sublinear operators, we get the L^p-L^q boundedness of $\mu_{\Omega,\alpha}$.

COROLLARY 3. *Suppose $n\ge 2$ and $0<\alpha<1$. Let $1<p<\frac{2n}{n+\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{2n}$. If $\Omega\in L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})$ and satisfies the $L^{2,\frac{\alpha}{2}}$-Dini condition, then there exists a constant C independent of f such that $\|\mu_{\Omega,\alpha}(f)\|_{L^q(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}$.*

REMARK 1. For $n=2$, the range of α in Theorem 2 and Corollary 3 can be enlarged to $0<\alpha<2$. We will discuss this issue in the last section.

REMARK 2. It is worthy noting that Theorem 1.1 in [DLS] can be regarded as the limit case of the above Theorem 1 by letting $\alpha\to 0$. Similarly, if we consider the limit case for $\alpha=0$, Theorem 1.3 and Corollary 1.7 in [DLS] (also cf. [DLL]) are the special cases of the above Theorem 2 and Corollary 3, respectively.

REMARK 3. It is easy to check that, for $0<\alpha\le 1$,

$$L^\infty(\mathbb{R}^n)\times \mathrm{Lip}_\alpha(S^{n-1})\subsetneq L^\infty(\mathbb{R}^n)\times L^2(S^{n-1}).$$

Moreover, if $\Omega(x,z')$ satisfies the Lip_α condition in variable $z'\in S^{n-1}$, then $\Omega(x,z')$ satisfies the $L^{2,0}$-Dini condition. So, the conclusion of the above Corollary 3 can be regarded as an extension of Stein's results about μ_Ω with convolution kernel in [S].

2. Proof of Theorem 1

We recall a classical result (see [SW], p. 158, Theorem 3.10]).

THEOREM A. *Suppose $n\ge 2$ and $f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ has the form $f(x)=f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree m. Then the Fourier transform of f has the form $\hat f(x)=F_0(|x|)P(x)$, where*

$$F_0(r)=2\pi i^{-m}r^{-(n+2m-2)/2}\int_0^\infty f_0(s)J_{(n+2m-2)/2}(2\pi rs)s^{(n+2m)/2}\,ds,$$

$r=|x|$ and $J_\nu(r)$ is the Bessel function of order ν.

The proof of Theorem 1 needs the following lemmas.

LEMMA 4. *For $\lambda\ge 0$, there exists a constant $C>0$ depending only on λ such that*

$$\left|\int_0^t \frac{J_{m+\lambda}(\rho)}{\rho^\lambda}\,d\rho\right|\le\frac{C}{m^\lambda}\quad \text{for } 0<t<\infty \text{ and } m=1,2,\ldots.$$

Proof. Let us write $\nu=m+\lambda$. In case $0<t\le\nu$, since $J_\nu(\rho)>0$ for $0<\rho<\nu$, it follows from [CZ1, (6.1)] that

$$\left|\int_0^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right|\le\nu\left|\int_0^t \frac{J_\nu(\rho)}{\rho^{1+\lambda}}\,d\rho\right|\le\frac{C\nu}{m^{1+\lambda}}\le\frac{C}{m^\lambda}.$$

In case $\nu < t \le 2\nu$, the second mean value theorem and [CZ1, (6.2)] yield

$$\begin{aligned}\int_\nu^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho &= \nu^{-\lambda}\int_\nu^{h'} J_\nu(\rho)\,d\rho \quad (\nu < h' \le t < 2\nu)\\ &= \nu^{-\lambda+1}\int_1^h J_\nu(\nu\rho)\,d\rho \quad (1 < h < 2)\\ &= O(\nu^{-\lambda}),\end{aligned}$$

where the big oh is an absolute one. Thus,

$$\left|\int_0^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| \le \left|\int_0^\nu \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| + \left|\int_\nu^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| \le \frac{C}{m^\lambda}.$$

In case $t > 2\nu$, we use the following differential equation (see [CZ1, p. 221])

$$\frac{J_\nu(\rho)}{\rho^\lambda} = -\frac{J_\nu'(\rho)}{\rho^{\lambda-1}(\rho^2-\nu^2)} - \frac{J_\nu''(\rho)}{\rho^{\lambda-2}(\rho^2-\nu^2)}.$$

Since $|J_\nu(\rho)| \le 1, |J_\nu'(\rho)| \le 1$, and the fact that $[\rho^{\lambda-1}(\rho^2-\nu^2)]^{-1}$ and $[\rho^{\lambda-2}(\rho^2-\nu^2)]^{-1}$ are decreasing functions for $\rho \ge 2\nu$, we apply the second mean value theorem again and obtain

$$\begin{aligned}\left|\int_{2\nu}^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| &\le \left|\int_{2\nu}^t \frac{J_\nu'(\rho)}{\rho^{\lambda-1}(\rho^2-\nu^2)}\,d\rho\right| + \left|\int_{2\nu}^t \frac{J_\nu''(\rho)}{\rho^{\lambda-2}(\rho^2-\nu^2)}\,d\rho\right|\\ &\le \frac{4}{3(2\nu)^{\lambda+1}}\left|\int_{2\nu}^{t_1} J_\nu'(\rho)\,d\rho\right| + \frac{4}{3(2\nu)^\lambda}\left|\int_{2\nu}^{t_2} J_\nu''(\rho)\,d\rho\right|\\ &= \frac{4}{3(2\nu)^{\lambda+1}}|J_\nu(t_1) - J_\nu(2\nu)| + \frac{4}{3(2\nu)^\lambda}|J_\nu'(t_2) - J_\nu'(2\nu)|\\ &\le \frac{C}{\nu^\lambda},\end{aligned}$$

where $2\nu < t_1$ and $t_2 \le t$. Thus,

$$\left|\int_0^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| \le \left|\int_0^{2\nu} \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| + \left|\int_{2\nu}^t \frac{J_\nu(\rho)}{\rho^\lambda}\,d\rho\right| \le \frac{C}{m^\lambda}. \qquad \square$$

LEMMA 5. *Let $0 < \alpha < 2$ and $g_\alpha(f)(x) = (\int_0^\infty |N_t f(x)|^2 \frac{dt}{t^{1-\alpha}})^{1/2}$. If*

$$\widehat{N_t f}(\xi) = \frac{1}{t|\xi|}\int_0^{t|\xi|} \frac{J_{m+\lambda}(\rho)}{\rho^\lambda}\,d\rho \cdot \hat{f}(\xi), \quad \lambda \ge 0,$$

then there exists a constant $C > 0$, independent of m, such that

$$\|g_\alpha(f)\|_{L^2} \le \frac{C}{m^{\lambda+1-\alpha/2}}\|f\|_{L^{\frac{2n}{n+\alpha}}}.$$

Proof. The idea of the proof comes from [AH, Lemma 2.1(b)]. Using Fubini's theorem and Plancherel's formula, we get

$$\|g_\alpha(f)\|_{L^2}^2 = \int_0^\infty \int_{\mathbb{R}^n} |N_t f(x)|^2\,dx \frac{dt}{t^{1-\alpha}} = \int_0^\infty \int_{\mathbb{R}^n} |\widehat{N_t f}(\xi)|^2\,d\xi \frac{dt}{t^{1-\alpha}}.$$

For $\lambda \geq 0$ and $t \geq 0$, let $\eta(t) = \frac{1}{t}\int_0^t \frac{J_{m+\lambda}(\rho)}{\rho^\lambda}\,d\rho$. Applying Fubini's theorem again, we have

$$\begin{aligned}\|g_\alpha(f)\|_{L^2}^2 &= \int_0^\infty \int_{\mathbb{R}^n} |\eta(t|\xi|)\hat{f}(\xi)|^2\,d\xi\,\frac{dt}{t^{1-\alpha}}\\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \int_0^\infty |\eta(t|\xi|)|^2\,\frac{dt}{t^{1-\alpha}}\,d\xi\\ &= \bigg(\int_0^\infty |\eta(t)|^2\frac{dt}{t^{1-\alpha}}\bigg)\bigg(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\frac{d\xi}{|\xi|^\alpha}\bigg).\end{aligned}$$

Hardy–Littlewood–Sobolev's theorem yields

$$\|g_\alpha(f)\|_{L^2}^2 \leq C\bigg(\int_0^\infty |\eta(t)|^2\frac{dt}{t^{1-\alpha}}\bigg)\|f\|^2_{L^{\frac{2n}{n+\alpha}}}.$$

We now estimate the integral $I = \int_0^\infty |\eta(t)|^2\frac{dt}{t^{1-\alpha}} = \int_0^{m/2} + \int_{m/2}^\infty := I_1 + I_2$. By Lemma 4,

$$I_2 \leq \frac{C}{m^{2\lambda}}\int_{m/2}^\infty \frac{dt}{t^{3-\alpha}} = \frac{C}{m^{2\lambda+2-\alpha}}.$$

On the other hand, Aguilera and Harboure [AH, proof of Lemma 2.1(b)] proved the inequality

$$\eta(t) \leq \frac{J_{m+\lambda}(t)}{t^\lambda},$$

and showed the estimate [AH, p. 565]

$$|J_\nu(t)| \leq \frac{(t/2)^\nu}{\Gamma(\nu+1)}.$$

Thus, by Striling's formula, we have

$$\begin{aligned}I_1 &\leq \int_0^{m/2}\bigg(\frac{J_{m+\lambda}(t)}{t^\lambda}\bigg)^2\frac{dt}{t^{1-\alpha}} \leq \int_0^{m/2}\frac{t^{2m-1+\alpha}}{(2^{m+\lambda}\Gamma(m+\lambda+1))^2}\,dt\\ &\leq C\bigg(\frac{(m/2)^{m+\alpha/2}}{2^m\Gamma(m+\lambda+1)}\bigg)^2 \leq \frac{C}{m^{2\lambda+2-\alpha}}.\end{aligned}$$

□

Proof of Theorem 1. The basic idea of the proof comes from [CZ2]. We denote the space of surface spherical harmonics of degree m on S^{n-1} by $\mathcal{H}_m$ and its dimension by D_m. By a limit argument, we may reduce the proof of Theorem 1 to the case that $f \in C_0^\infty(\mathbb{R}^n)$ and

$$\Omega(x,z') = \sum_{m\geq 0}\sum_{j=1}^{D_m} a_{m,j}(x)Y_{m,j}(z')$$

is a finite sum, where $\{Y_{m,j} : m \geq 0, 1 \leq j \leq D_m\}$ denotes the complete system of normalized surface spherical harmonics. Notice that $\Omega(x,z')$ satisfies (ii), so we have $a_{0,j} \equiv 0$. Let

$$a_m(x) = \left(\sum_{j=1}^{D_m} |a_{m,j}(x)|^2 \right)^{1/2} \quad \text{and} \quad b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}.$$

Then

$$\sum_{j=1}^{D_m} b_{m,j}^2(x) = 1 \tag{2.1}$$

and

$$\Omega(x,z') = \sum_{m\ge 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) Y_{m,j}(z').$$

If we write

$$\mu_{m,j} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{Y_{m,j}(\frac{x-y}{|x-y|})}{|x-y|^{n-1}} f(y)\,dy \right|^2 \frac{dt}{t^{3-\alpha}} \right)^{1/2},$$

then, by using Hölder's inequality twice and (2.1),

$$\begin{aligned}
&(\mu_{\Omega,\alpha} f(x))^2 \\
&\quad = \int_0^\infty \left| \int_{|x-y|\le t} \sum_{m\ge 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) \frac{Y_{m,j}(\frac{x-y}{|x-y|})}{|x-y|^{n-1}} f(y)\,dy \right|^2 \frac{dt}{t^{3-\alpha}} \\
&\quad \le \left(\sum_{m\ge 1} a_m^2(x) \right) \cdot \sum_{m\ge 1} \int_0^\infty \left| \int_{|x-y|\le t} \sum_{j=1}^{D_m} b_{m,j}(x) \frac{Y_{m,j}(\frac{x-y}{|x-y|})}{|x-y|^{n-1}} f(y)\,dy \right|^2 \frac{dt}{t^{3-\alpha}} \\
&\quad \le \left(\sum_{m\ge 1} a_m^2(x) \right) \cdot \sum_{m\ge 1} \int_0^\infty \left(\sum_{j=1}^{D_m} b_{m,j}^2(x) \right) \\
&\qquad \cdot \sum_{j=1}^{D_m} \left| \int_{|x-y|\le t} \frac{Y_{m,j}(\frac{x-y}{|x-y|})}{|x-y|^{n-1}} f(y)\,dy \right|^2 \frac{dt}{t^{3-\alpha}} \\
&\quad = \left(\sum_{m\ge 1} a_m^2(x) \right) \cdot \left(\sum_{m\ge 1} \sum_{j=1}^{D_m} (\mu_{m,j} f(x))^2 \right).
\end{aligned}$$

By Bessel's inequality,

$$\sum_{m\ge 1} a_m^2(x) \le \int_{S^{n-1}} |\Omega(x,z')|^2\,d\sigma(z').$$

Applying Minkowski's inequality and condition (iii′) for $q = 2$, we get

$$\|\mu_{\Omega,\alpha} f\|_{L^2}^2 \le C \sum_{m\ge 1} \left(\sum_{j=1}^{D_m} \|\mu_{m,j} f\|_{L^2}^2 \right).$$

To complete the proof of Theorem 1, it suffices to show that

$$\sum_{j=1}^{D_m}\|\mu_{m,j}f\|_{L^2}^2\le\frac{C}{m^{2-\alpha}}\|f\|^2_{L^{\frac{2n}{n+\alpha}}}. \tag{2.2}$$

Denote $\varphi_t^{m,j}(x)=t^{-1}Y_{m,j}(x')|x|^{-n+1}\chi_{\{|x|\le t\}}(x)$. Plancherel's theorem implies

$$\begin{aligned}\|\mu_{m,j}f\|_{L^2}^2&=\int_0^\infty\int_{\mathbb{R}^n}|\varphi_t^{m,j}*f(x)|^2\,dx\frac{dt}{t^{1-\alpha}}\\&=\int_{\mathbb{R}^n}\int_0^\infty|(\widehat{\varphi_t^{m,j}*f})(\xi)|^2\frac{dt}{t^{1-\alpha}}\,d\xi.\end{aligned}$$

Now set $P_{m,j}(x)=Y_{m,j}(x')|x|^m$. Then $P_{m,j}$ is a solid spherical harmonic of degree m and we have $\varphi_t^{m,j}(x)=t^{-1}P_{m,j}(x)|x|^{-n-m+1}\chi_{\{|x|\le t\}}(x)$. Obviously, $\psi_0(|x|):=t^{-1}|x|^{-n-m+1}\chi_{\{|x|\le t\}}(x)$ is a radial function in x for fixed $t>0$. Using Theorem A, we have

$$(\widehat{\varphi_t^{m,j}})(\xi)=F_0(|\xi|)P_{m,j}(\xi)=Y_{m,j}(\xi')|\xi|^mF_0(|\xi|), \tag{2.3}$$

where

$$\begin{aligned}F_0(r)&=2\pi i^{-m}r^{-(n+2m-2)/2}\int_0^\infty\psi_0(s)J_{(n+2m-2)/2}(2\pi rs)s^{(n+2m)/2}\,ds\\&=2\pi i^{-m}t^{-1}r^{-(n+2m-2)/2}\int_0^t s^{-n-m+1}J_{(n+2m-2)/2}(2\pi rs)s^{(n+2m)/2}\,ds\\&=i^{-m}(2\pi)^{(n-2)/2}t^{-1}r^{-m-1}\int_0^{2\pi rt}\frac{J_{(n+2m-2)/2}(\rho)}{\rho^{(n-2)/2}}\,d\rho\\&=i^{-m}(2\pi)^{n/2}r^{-m}\frac{1}{2\pi rt}\int_0^{2\pi rt}\frac{J_{(n+2m-2)/2}(\rho)}{\rho^{(n-2)/2}}\,d\rho.\end{aligned}$$

From this and (2.3), we have

$$(\widehat{\varphi_t^{m,j}*f})(\xi)=i^{-m}(2\pi)^{n/2}Y_{m,j}(\xi')\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t}\frac{J_{(n+2m-2)/2}(\rho)}{\rho^{(n-2)/2}}\,d\rho\cdot\hat f(\xi).$$

Hence,

$$\begin{aligned}\sum_{j=1}^{D_m}\|\mu_{m,j}f\|_{L^2}^2&=\int_{\mathbb{R}^n}\int_0^\infty\sum_{j=1}^{D_m}|(\widehat{\varphi_t^{m,j}*f})(\xi)|^2\frac{dt}{t^{1-\alpha}}\,d\xi\\&=(2\pi)^n\int_{\mathbb{R}^n}\int_0^\infty\sum_{j=1}^{D_m}|Y_{m,j}(\xi')|^2\\&\qquad\cdot\left|\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t}\frac{J_{(n+2m-2)/2}(\rho)}{\rho^{(n-2)/2}}\,d\rho\cdot\hat f(\xi)\right|^2\frac{dt}{t^{1-\alpha}}\,d\xi.\end{aligned}$$

Since $\sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 = \omega^{-1} D_m \sim m^{n-2}$ (cf. [CZ2, p. 225, (2.6)]), where ω denotes the area of S^{n-1}, we have

$$\begin{aligned}(2.4)\quad &\sum_{j=1}^{D_m} \|\mu_{m,j} f\|_{L^2}^2 \\ &\le C m^{n-2} \int_{\mathbb{R}^n} \int_0^\infty \left| \frac{1}{2\pi|\xi|t} \int_0^{2\pi|\xi|t} \frac{J_{(n+2m-2)/2}(\rho)}{\rho^{(n-2)/2}} \, d\rho \cdot \hat{f}(\xi) \right|^2 \frac{dt}{t^{1-\alpha}} \, d\xi.\end{aligned}$$

By (2.4) and Lemma 5, we obtain (2.2). Hence, the proof of Theorem 1 is completed. □

3. Proof of Theorem 2

We recall that $a \in L^q(\mathbb{R}^n)$ is said to be a (p,q)-atom, $0 < p \le 1 \le q \le \infty$ and $p \ne q$, if a is supported in a ball B with mean value 0 and $\|a\|_{L^q} \le |B|^{\frac{1}{q} - \frac{1}{p}}$. It is known that, in general, one cannot conclude from the uniform boundedness

$$(3.1)\qquad \|Ta\|_{L^r(\mathbb{R}^n)} \le C \quad \text{for any } (1,\infty)\text{-atoms } a \in H^1(\mathbb{R}^n)$$

that the (sub)linear operator T is bounded from $H^1(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, i.e.,

$$(3.2)\qquad \|Tf\|_{L^r(\mathbb{R}^n)} \le C \|f\|_{H^1(\mathbb{R}^n)} \quad \text{for any } f \in H^1(\mathbb{R}^n).$$

A counterexample of functional for the case $r = 1$ was given by Bownik [B1]. Fortunately, for many operators such as Calderón–Zygmund operators, the uniform boundedness (3.1) of T implies the boundedness (3.2). See [B2, Chapter 1, Section 9], [GR, Chapter III, Section 7], [G, Chapter 1, Section 6.7.a], and [MC, Chapter 7, Section 3].

In 1993, Yabuta [Y] derived (3.2) from the uniform boundedness (3.1) provided the linear operator T satisfies a weak (q,s) estimate with some $1 \le q \le \infty$ and $1 \le s < \infty$, and the same conclusion also holds for H^p spaces for $0 < p < 1$. Actually, Miyachi presented this fact in the Harmonic Analysis Satellite Conference of ICM 1990 in Kyoto, Japan.

Recently Meda, Sjögren, and Vallarino [MSV, Theorem 3.1, Remark 3.3, and Corollary 3.4] also gave a similar result.

THEOREM B. *Let $0 < p \le 1 < q < \infty$ and Y be a Banach space. Denote by $H^{p,q}_{fin}(\mathbb{R}^n)$ the vector space of all finite linear combinations of (p,q)-atoms. If $T : H^{p,q}_{fin}(\mathbb{R}^n) \mapsto Y$ is a linear operator satisfying*

$$\sup\{\|Ta\|_Y : a \text{ is a } (p,q)\text{-atom}\} < \infty,$$

then there exists a unique bounded linear operator $\widetilde{T}$ from $H^p(\mathbb{R}^n)$ to Y which extends T.

Having these remarks in mind, we may start proving Theorem 2. Before proving, we need an estimate for the variable kernel Ω as well.

LEMMA C ([DLL]). *Let $0 \le \alpha < n$, $q \ge 1$, and $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$. If there exists a constant $0 < \beta < \frac{1}{2}$ such that $|y| < \beta R$, then for any $x_0 \in \mathbb{R}^n$,*

$$\left(\int_{R<|x|\le 2R} \left|\frac{\Omega(x_0+x, x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x_0+x, x)}{|x|^{n-\alpha}}\right|^q dx\right)^{1/q} \le CR^{n/q-(n-\alpha)}\left(\frac{|y|}{R} + \int_{2|y|/R}^{4|y|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta\right),$$

where the constant $C > 0$ is independent of R and y.

Proof of Theorem 2. By the atomic decomposition, for each $f \in H^p$, there exist a sequence $\{a_j\}$ of (p,ℓ)-atoms ($\ell = \frac{2n}{n+\alpha} > 1$) and a sequence $\{\lambda_j\}$ of real numbers with $\sum |\lambda_j|^p \le C\|f\|_{H^p}^p$ such that $f = \sum \lambda_j a_j$ both in the sense of distributions and in the H^p norm.

We have written the Marcinkiewicz integral as a Hilbert-valued linear operator; that is,

$$\mu_{\Omega,\alpha}(f)(x) = \|h_f(\cdot, x)\|_{\mathcal{H}},$$

where $\mathcal{H}$ is the Hilbert space mentioned in the first section,

$$h_f(t,x) = \int_{\mathbb{R}^n} K_t(x, x-y) f(y)\, dy$$

and

$$K_t(x,z) = t^{-1+\alpha/2} \frac{\Omega(x,z)}{|z|^{n-1}} \chi_{\{|z|\le t\}}.$$

Without loss of generality, we may assume that f is in a nice dense subset of $H^p(\mathbb{R}^n)$, say, $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$. Thus,

$$|h_f(t,x)| \le t^{-1+\alpha/2}\|f\|_{L^{r'}(\mathbb{R}^n)}\left(\int_{|x-y|\le t} \frac{|\Omega(x,x-y)|^r}{|x-y|^{(n-1)r}}\, dy\right)^{1/r} \le C_r t^{(\alpha/2)+(n/r)-n}\|f\|_{L^{r'}(\mathbb{R}^n)} \quad \text{for any } 1 \le r < n/(n-1).$$

Taking $r = 1$ and $r = \frac{n}{n-(\alpha+\varepsilon)/2}$, respectively, where $0 < \varepsilon < 2-\alpha$, we get

$$t^{-1}|h_f(t,x)|^2 \le C\big(t^{-1+\alpha}\chi_{(0,1]}(t) + t^{-1-\varepsilon}\chi_{(1,\infty)}(t)\big).$$

Hence, $h_f(\cdot, x) \in \mathcal{H}$ uniformly. We then have

$$\|h_f(\cdot,x)\|_{\mathcal{H}} \le \sum_j |\lambda_j| \cdot \|h_{a_j}(\cdot,x)\|_{\mathcal{H}}.$$

Note that $q > p$. For $0 < p \le 1$, the above inequality yields

$$\|\mu_{\Omega,\alpha}(f)\|_{L^q} \le \left\|\sum_j |\lambda_j||\mu_{\Omega,\alpha}(a_j)|\right\|_{L^q} = \left(\int_{\mathbb{R}^n} \left|\sum_j |\lambda_j||\mu_{\Omega,\alpha}(a_j)(x)|\right|^q dx\right)^{1/q}$$

$$\le \left(\int_{\mathbb{R}^n}\left(\sum_j |\lambda_j \mu_{\Omega,\alpha}(a_j)(x)|^p\right)^{q/p} dx\right)^{p/q\cdot 1/p}$$
$$\le C\left\{\sum_j\left(\int_{\mathbb{R}^n} |\lambda_j \mu_{\Omega,\alpha}(a_j)(x)|^q\, dx\right)^{p/q}\right\}^{1/p}$$
$$= C\left\{\sum_j |\lambda_j|^p\left(\int_{\mathbb{R}^n} |\mu_{\Omega,\alpha}(a_j)(x)|^q\, dx\right)^{p/q}\right\}^{1/p}.$$

Thus, it suffices to claim that there exists an absolute constant $C>0$ such that $\|\mu_{\Omega,\alpha}(a)\|_{L^q} \le C$ for any (p,ℓ)-atom a. The operator $\mu_{\Omega,\alpha}$ can be regarded as $Tf(x) = h_f(t,x)$ which is linear from $H^{p,\ell}_{fin}(\mathbb{R}^n)$ into $L^q(\mathcal{H})$, and we have proved in Theorem 1 that the operator $\mu_{\Omega,\alpha}$, and so T, is bounded from $L^{2n/(n+\alpha)}$ to L^2. Once the claim holds, Theorem 2 follows either by Yabuta's arguments or by Theorem B.

Let a be a (p,ℓ)-atom satisfying $\operatorname{supp}(a) \subset B = B(0,\rho)$, $\|a\|_{L^\ell} \le |B|^{\frac{1}{\ell}-\frac{1}{p}}$, and $\int a(x)\,dx = 0$. Since $n\ge 2$, $0<\alpha<1$, and $0<p\le 1$, we get $0<q\le 2$. Write

$$\|\mu_{\Omega,\alpha}(a)\|_{L^q} \le C\left(\int_{|x|\le 8\rho} |\mu_{\Omega,\alpha}(a)(x)|^q\, dx\right)^{1/q} + C\left(\int_{|x|>8\rho} |\mu_{\Omega,\alpha}(a)(x)|^q\, dx\right)^{1/q} := M_1 + M_2.$$

By Theorem 1, it is easy to get

$$M_1 \le C\|\mu_{\Omega,\alpha}(a)\|_2 |B|^{\frac{1}{q}-\frac{1}{2}} \le C\|a\|_{L^\ell}|B|^{\frac{1}{q}-\frac{1}{2}} \le C.$$

To estimate M_2, we write

$$\begin{aligned} M_2 &= C\left(\int_{|x|>8\rho}\left\{\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-1}}a(y)\,dy\right|^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\right)^{1/q}\\ &\le C\left(\int_{|x|>8\rho}\left\{\int_0^{|x|+2\rho}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)a(y)}{|x-y|^{n-1}}\,dy\right|^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\right)^{1/q}\\ &\quad + C\left(\int_{|x|>8\rho}\left\{\int_{|x|+2\rho}^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)a(y)}{|x-y|^{n-1}}\,dy\right|^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\right)^{1/q}\\ &:= U+V. \end{aligned}$$

For $|x|>8\rho$ and $|y|\le\rho$, we get $|x-y|\sim|x|\sim|x|+2\rho$, and hence

$$\left|\frac{1}{(|x|+2\rho)^{2-\alpha}} - \frac{1}{|x-y|^{2-\alpha}}\right|^{1/2} \le C\frac{\rho^{1/2}}{|x-y|^{(3-\alpha)/2}}, \tag{3.3}$$

$$\left|\frac{1}{|x|^{n-1}}-\frac{1}{|x-y|^{n-1}}\right|\le C\frac{|y|}{|x-y|^{n}}, \tag{3.4}$$

$$\left|\frac{x}{|x|}-\frac{x-y}{|x-y|}\right|\le 2\frac{|y|}{|x|}. \tag{3.5}$$

For case (a), we have $1\le q\le 2$. By Minkowski's inequality for integrals and inequality (3.3),

$$\begin{aligned} U &\le C\bigg(\int_{|x|>8\rho}\bigg\{\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)||a(y)|}{|x-y|^{n-1}} \\ &\quad\times\bigg(\int_{|x-y|}^{|x|+2\rho}\frac{dt}{t^{3-\alpha}}\bigg)^{1/2}dy\bigg\}^q dx\bigg)^{1/q} \\ &\le C\rho^{1/2}\bigg(\int_{|x|>8\rho}\bigg\{\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)||a(y)|}{|x-y|^{n-1}} \\ &\quad\times\frac{1}{|x-y|^{(3-\alpha)/2}}\,dy\bigg\}^q dx\bigg)^{1/q}. \end{aligned} \tag{3.6}$$

Using Minkowski's inequality for integrals again, we get

$$\begin{aligned} U &\le C\rho^{1/2}\int_{\mathbb{R}^n}|a(y)|\bigg(\int_{|x-y|>7\rho}\frac{|\Omega(x,x-y)|^q}{|x-y|^{(2n-\alpha+1)q/2}}\,dx\bigg)^{1/q}dy \\ &\le C\rho^{1/2}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})}\int_{\mathbb{R}^n}|a(y)|\bigg(\int_{7\rho}^{\infty}r^{-(2n-\alpha+1)q/2+n-1}\,dr\bigg)^{1/q}dy. \end{aligned}$$

Since $p\ge\frac{2n}{2n+\alpha}>\frac{2n}{2n+1}$, $-(2n-\alpha+1)q/2+n<0$, and hence

$$U\le C\rho^{-n+\frac{\alpha}{2}+\frac{n}{q}}\int_B|a(y)|\,dy\le C\rho^{-n+\frac{\alpha}{2}+\frac{n}{q}}\|a\|_{L^\ell}|B|^{1-\frac{1}{\ell}}\le C.$$

Applying the cancellation property of $a(x)$, Minkowski's inequality for integrals, and Lemma C, we get

$$\begin{aligned} V &= C\bigg(\int_{|x|>8\rho}\bigg\{\int_{|x|+2\rho}^{\infty}\bigg|\int_{|x-y|\le t}\bigg(\frac{\Omega(x,x-y)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x|^{n-1}}\bigg) \\ &\quad\cdot a(y)\,dy\bigg|^2\frac{dt}{t^{3-\alpha}}\bigg\}^{q/2}dx\bigg)^{1/q} \\ &\le C\bigg(\int_{|x|>8\rho}\bigg\{\int_{\mathbb{R}^n}\bigg|\frac{\Omega(x,x-y)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x|^{n-1}}\bigg||x|^{(\alpha-2)/2}|a(y)|\,dy\bigg\}^q dx\bigg)^{1/q} \\ &\le C\int_{\mathbb{R}^n}|a(y)|\bigg(\int_{|x|>8\rho}|x|^{(\alpha-2)q/2}\bigg|\frac{\Omega(x,x-y)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x|^{n-1}}\bigg|^q dx\bigg)^{1/q}dy \\ &= C\int_{\mathbb{R}^n}|a(y)|\bigg(\sum_{j=3}^{\infty}\int_{2^j\rho<|x|\le 2^{j+1}\rho}|x|^{(\alpha-2)q/2} \end{aligned}$$

$$\cdot \left|\frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}}\right|^q dx\Bigg)^{1/q} dy$$

$$\le C\int_{\mathbb{R}^n} |a(y)|\left(\sum_{j=3}^{\infty}(2^j\rho)^{-qn+\frac{q\alpha}{2}+n}\left\{\frac{1}{2^j} + \int_{\frac{|y|}{2^{j-1}\rho}}^{\frac{|y|}{2^{j-2}\rho}} \frac{\omega_q(\delta)}{\delta}\,d\delta\right\}^q\right)^{1/q} dy$$

$$\le C\int_{\mathbb{R}^n} |a(y)|\left(\sum_{j=3}^{\infty}(2^j\rho)^{-n+\frac{\alpha}{2}+\frac{n}{q}}\left\{\frac{1}{2^j} + \int_{\frac{|y|}{2^{j-1}\rho}}^{\frac{|y|}{2^{j-2}\rho}} \frac{\omega_q(\delta)}{\delta}\,d\delta\right\}\right) dy.$$

Since $p \ge \frac{2n}{2n+\alpha}$, we have $-n+\frac{n}{q} \le 0$ and $-n+\frac{\alpha}{2}+\frac{n}{q} < 1$. Thus,

$$V \le C\rho^{-n+\frac{\alpha}{2}+\frac{n}{q}}$$

$$\times \int_B |a(y)|\left(\sum_{j=3}^{\infty}\left\{2^{j(-n+\frac{\alpha}{2}+\frac{n}{q}-1)} + 2^{j(-n+\frac{n}{q})}\int_{\frac{|y|}{2^{j-1}\rho}}^{\frac{|y|}{2^{j-2}\rho}} \frac{\omega_q(\delta)}{\delta^{1+\frac{\alpha}{2}}}\,d\delta\right\}\right) dy$$

$$\le C\rho^{-n+\frac{\alpha}{2}+\frac{n}{q}} \int_{\mathbb{R}^n} |a(y)|\left(1+\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha/2}}\,d\delta\right) dy$$

$$\le C\rho^{-n+\frac{\alpha}{2}+\frac{n}{q}}\|a\|_{L^\ell}|B|^{1-\frac{1}{\ell}}$$

$$\le C.$$

For case (b), we have $q<1$. Since $p > \frac{2n}{2n+1}$, $2n+1-\alpha-\frac{2n}{q} > 0$ and hence we may choose ε satisfying $0<\varepsilon<2n+1-\alpha-\frac{2n}{q}$. It follows from (3.6) that

$$U \le C\rho^{1/2}\left(\int_{|x|>8\rho}\left\{\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)||a(y)|}{|x-y|^{n+\frac{\varepsilon}{2}}}\,dy\right\}^q |x|^{(\alpha+\varepsilon-1)q/2}\,dx\right)^{1/q}$$

$$\le C\rho^{1/2}\left(\int_{|x|>8\rho}\left|\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)||a(y)|}{|x-y|^{n+\frac{\varepsilon}{2}}}\,dy\right| dx\right)$$

$$\cdot\left(\int_{|x|>8\rho}|x|^{\frac{(\alpha+\varepsilon-1)q}{2(1-q)}}\,dx\right)^{(1-q)/q}$$

$$\le C\rho^{1/2}\left(\int_{\mathbb{R}^n}|a(y)|\int_{|x|>8\rho}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\varepsilon/2}}\,dx\,dy\right)$$

$$\cdot\left(\int_{|x|>8\rho}|x|^{\frac{(\alpha+\varepsilon-1)q}{2(1-q)}}\,dx\right)^{(1-q)/q}$$

$$\le C\rho^{1/2}\|\Omega\|_{L^\infty\times L^1}\int_{\mathbb{R}^n}|a(y)|\,dy\left(\int_{7\rho}^{+\infty} r^{-\frac{\varepsilon}{2}-1}\,dr\right)$$

$$\cdot\left(\int_{8\rho}^{+\infty} r^{\frac{(\alpha+\varepsilon-1)q}{2(1-q)}} r^{n-1}\,dr\right)^{(1-q)/q}$$

$$\le C\rho^{\frac{\alpha}{2}+\frac{n}{q}-n}\|a\|_{L^\ell}|B|^{1-\frac{1}{\ell}}$$

$$\le C.$$

As to the estimate of V, we decompose V^q as follows.

$$\begin{aligned}
V^q &= C\int_{|x|>8\rho}\left\{\int_{|x|+2\rho}^{\infty}\left|\int_{|x-y|\le t}\left(\frac{\Omega(x,x-y)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x|^{n-1}}\right)\right.\right.\\
&\qquad\left.\left.\cdot\, a(y)\,dy\right|^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\\
&\le C\int_{|x|>8\rho}\left\{\int_{|x|+2\rho}^{\infty}\left(\int_{|x-y|\le t}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x-y|^{n-1}}\right|\right.\right.\\
&\qquad\left.\left.\cdot\,|a(y)|\,dy\right)^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\\
&\qquad+C\int_{|x|>8\rho}\left\{\int_{|x|+2\rho}^{\infty}\left(\int_{|x-y|\le t}\left|\frac{\Omega(x,x)}{|x-y|^{n-1}}-\frac{\Omega(x,x)}{|x|^{n-1}}\right|\right.\right.\\
&\qquad\left.\left.\cdot\,|a(y)|\,dy\right)^2\frac{dt}{t^{3-\alpha}}\right\}^{q/2}dx\\
&:= V_1+V_2.
\end{aligned}$$

By Minkowski's and Hölder's inequalities, we obtain

$$\begin{aligned}
V_1 &\le C\sum_{j=3}^{\infty}\int_{2^j\rho<|x|\le 2^{j+1}\rho}\left\{\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n-1}}\right.\\
&\qquad\left.\cdot\,|x|^{(\alpha-2)/2}|a(y)|\,dy\right\}^q dx\\
&\le C\sum_{j=3}^{\infty}(2^j\rho)^{n(1-q)}\left(\int_{2^j\rho<|x|\le 2^{j+1}\rho}\left\{\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n-1}}\right.\right.\\
&\qquad\left.\left.\cdot\,|x|^{(\alpha-2)/2}|a(y)|\,dy\right\}dx\right)^q\\
&= C\sum_{j=3}^{\infty}(2^j\rho)^{n(1-q)}\left(\int_B |a(y)|\left\{\int_{2^j\rho<|x|\le 2^{j+1}\rho}\frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n-1}}\right.\right.\\
&\qquad\left.\left.\cdot\,|x|^{(\alpha-2)/2}\,dx\right\}dy\right)^q.
\end{aligned}$$

We note that $\beta>\alpha$. By (3.5), the inner integral above can be estimated as follows.

$$\begin{aligned}
&\int_{2^j\rho<|x|\le 2^{j+1}\rho}\frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n-1}}|x|^{(\alpha-2)/2}\,dx\\
&\quad\le C\int_{2^j\rho}^{2^{j+1}\rho} r^{(\alpha-2)/2}\int_{S^{n-1}}\left|\Omega\left(rx',\frac{rx'-y}{|rx'-y|}\right)-\Omega(rx',x')\right|d\sigma(x')\,dr
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{2^j\rho}^{2^{j+1}\rho} r^{(\alpha-2)/2}\omega_1\left(2\frac{|y|}{r}\right)dr\\
&= C\int_{\frac{|y|}{2^j\rho}}^{\frac{|y|}{2^{j-1}\rho}}\left(\frac{|y|}{\delta}\right)^{(\alpha-2)/2}\omega_1(\delta)\frac{|y|}{\delta^2}\,d\delta\\
&\le C\rho^{\alpha/2}\int_{\frac{|y|}{2^j\rho}}^{\frac{|y|}{2^{j-1}\rho}}\frac{\omega_1(\delta)}{\delta^{1+\beta/2}}\delta^{(\beta-\alpha)/2}\,d\delta\\
&\le C\rho^{\alpha/2}\left(\frac{|y|}{2^j\rho}\right)^{(\beta-\alpha)/2}\int_{\frac{|y|}{2^j\rho}}^{\frac{|y|}{2^{j-1}\rho}}\frac{\omega_1(\delta)}{\delta^{1+\beta/2}}\,d\delta.
\end{aligned}
$$

Since $p > \frac{2n}{2n+\beta}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2n}$, and $\int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\beta/2}}\,d\delta < \infty$, we get

$$
\begin{aligned}
V_1 &\le C\rho^{\frac{\alpha q}{2}+n(1-q)}\sum_{j=3}^{\infty}2^{jn(1-q)-j\frac{(\beta-\alpha)q}{2}}\left(\int_B |a(y)|\int_{\frac{|y|}{2^j\rho}}^{\frac{|y|}{2^{j-1}\rho}}\frac{\omega_1(\delta)}{\delta^{1+\beta/2}}\,d\delta\,dy\right)^q\\
&\le C\rho^{\frac{\alpha q}{2}+n(1-q)}\left(\int_B |a(y)|\,dy\right)^q\left(\int_0^1\frac{\omega_1(\delta)}{\delta^{1+\beta/2}}\,d\delta\right)^q\sum_{j=3}^{\infty}2^{j(\frac{n}{p}-n-\frac{\beta}{2})q}\\
&\le C\rho^{\frac{\alpha q}{2}+n(1-q)}\|a\|_{L^\ell}^q|B|^{(1-1/\ell)q}\\
&\le C.
\end{aligned}
$$

Since $p > \frac{2n}{2n+1} > \frac{n}{n+1}$ implies $(\frac{\alpha}{2}-n-1)q < -n$, inequality (3.4), the size condition of $a(x)$, and Hölder's inequality yield

$$
\begin{aligned}
V_2 &\le C\int_{|x|>8\rho}\left\{\int_{|x|+2\rho}^{\infty}\left(\int_{|x-y|\le t}|\Omega(x,x)||y||x|^{-n}|a(y)|\,dy\right)^2\frac{dt}{t^{3-\alpha}}\right\}^{\frac{q}{2}}dx\\
&\le C\rho^q\left(\int_{|x|>8\rho}|\Omega(x,x)|^q|x|^{(\frac{\alpha}{2}-n-1)q}\,dx\right)\left(\int_B |a(y)|\,dy\right)^q\\
&\le C\rho^q\left(\int_{|x|>8\rho}|\Omega(x,x)||x|^{(\frac{\alpha}{2}-n-1)q}\,dx\right)^q\left(\int_{|x|>8\rho}|x|^{(\frac{\alpha}{2}-n-1)q}\,dx\right)^{1-q}\\
&\quad\cdot\left(\int_B |a(y)|\,dy\right)^q\\
&\le C\rho^q|B|^{(1-\frac{1}{p})q}\rho^{(\frac{\alpha}{2}-n-1)q}\rho^n\|\Omega\|_{L^\infty\times L^1}^q\\
&\le C.
\end{aligned}
$$

Thus, the proof of Theorem 2 is finished. □

4. The case of $\mathbb{R}^2$

As mentioned in Remark 1, for the case of $\mathbb{R}^2$, we may enlarge the range of α in Theorem 2 and Corollary 3 to $0 < \alpha < 2$.

THEOREM 1′. *Let $n=2$ and $0<\alpha<2$. If Ω satisfies* (i), (ii), (iii′) *for $q=2$, and the $\overline{L}^{2,\frac{\alpha}{2}}$-Dini condition, then there exists a constant C independent of f such that*

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)}.$$

To prove Theorem 1′, we need the following lemma.

LEMMA D ([DCF]). *Let $0<\beta\le 1$. If Ω satisfies* (i), (ii), *and* (iii′) *for $q\ge 1$, then*

$$\int_0^1 \frac{\overline{\omega}_q(\delta)}{\delta^{1+\beta}}\,d\delta<\infty \quad \textit{if and only if} \quad \sum_{j=1}^{\infty}\frac{\overline{\omega}_q(\frac{1}{j})}{j^{1-\beta}}<\infty.$$

REMARK 4. Using the same argument as the proof of [DCF, Lemma 3.1], the result also holds if we restrict Ω to be in $L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})$ and replace $\overline{\omega}_q$ by ω_q.

Proof of Theorem 1′. For $m\in\mathbb{N}\cup\{0\}$ and $z'=(\cos\theta,\sin\theta)\in S^1$, let

$$Y_{m,1}(z')=\frac{1}{\sqrt{\pi}}\cos(m\theta); \qquad Y_{m,2}(z')=\frac{1}{\sqrt{\pi}}\sin(m\theta).$$

By [SW], $\{Y_{m,1},Y_{m,2}\}_{m=1}^{\infty}$ forms a complete system of normalized surface spherical harmonics and we may decompose $\Omega(x,z')$ into

$$\Omega(x,z')=\sum_{m=0}^{\infty}\sum_{j=1}^{2}a_{m,j}(x)Y_{m,j}(z'),$$

where

$$a_{m,j}(x)=\int_{S^1}\Omega(x,z')\overline{Y_{m,j}(z')}\,d\sigma(z'), \quad j=1,2.$$

Note that $a_{0,j}(x)\equiv 0$ since Ω satisfies the condition (ii).

Choosing a 2×2 matrix

$$R_m=\begin{pmatrix}\cos(\pi/m) & -\sin(\pi/m)\\ \sin(\pi/m) & \cos(\pi/m)\end{pmatrix},$$

we have

$$Y_{m,1}(R_m z')=-Y_{m,1}(z'), \qquad Y_{m,2}(R_m z')=-Y_{m,2}(z');$$

and

$$a_{m,j}(x)=-\frac{1}{2}\int_{S^1}\big(\Omega(x,R_m z')-\Omega(x,z')\big)\overline{Y_{m,j}(z')}\,d\sigma(z'), \quad j=1,2.$$

Minkowski's and Hölder's inequalities yield

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2} \le \sum_{m=1}^{\infty}\Bigg(\int_{\mathbb{R}^2}\int_0^{\infty}\Bigg|\int_{S^1}\big(\Omega(x,R_m z')-\Omega(x,z')\big)\int_{|x-y|\le t}\frac{f(y)}{|x-y|}$$

$$\times \sum_{j=1}^{2} \overline{Y_{m,j}(z')}\, Y_{m,j}\left(\frac{x-y}{|x-y|}\right) dy\, d\sigma(z')\Bigg|^2 \frac{dt}{t^{3-\alpha}}\, dx\Bigg)^{\frac{1}{2}}$$

$$\leq \sum_{m=1}^{\infty}\Bigg(\int_{\mathbb{R}^2}\int_0^\infty\Bigg\{\int_{S^1}|\Omega(x,R_m z')-\Omega(x,z')|^2\, d\sigma(z')$$

$$\times \int_{S^1}\Bigg|\int_{|x-y|\le t}\frac{f(y)}{|x-y|}\sum_{j=1}^{2}\overline{Y_{m,j}(z')}$$

$$\times Y_{m,j}\left(\frac{x-y}{|x-y|}\right) dy\Bigg|^2 d\sigma(z')\Bigg\}\frac{dt}{t^{3-\alpha}}\, dx\Bigg)^{\frac{1}{2}}$$

$$\leq \sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg(\int_{\mathbb{R}^2}\int_0^\infty\int_{S^1}\Bigg|\int_{|x-y|\le t}\sum_{j=1}^{2}\overline{Y_{m,j}(z')}$$

$$\times Y_{m,j}\left(\frac{x-y}{|x-y|}\right)\frac{f(y)\, dy}{|x-y|}\Bigg|^2\frac{d\sigma(z')\, dt}{t^{3-\alpha}}\, dx\Bigg)^{\frac{1}{2}}$$

$$= \sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg(\int_0^\infty\int_{S^1}\int_{\mathbb{R}^2}\Bigg|\int_{|x-y|\le t}\sum_{j=1}^{2}\overline{Y_{m,j}(z')}$$

$$\times Y_{m,j}\left(\frac{x-y}{|x-y|}\right)\frac{f(y)\, dy}{|x-y|}\Bigg|^2 dx\frac{d\sigma(z')\, dt}{t^{3-\alpha}}\Bigg)^{\frac{1}{2}}.$$

Let

$$P_{m,j}(x) = Y_{m,j}(x')|x|^m \quad \text{and} \quad \varphi_{m,j}(x) = P_{m,j}(x)|x|^{-m-1}\chi_{\{|x|\le t\}}.$$

By Theorem A, it is easy to get

$$\widehat{\varphi}_{m,j}(\xi) = G_0(|\xi|)P_{m,j}(\xi) = Y_{m,j}(\xi')|\xi|^m G_0(|\xi|).$$

Also, similar to the argument in the proof of Theorem 1, we have

$$G_0(r) = i^{-m} r^{-m-1}\int_0^{2\pi r t} J_m(\rho)\, d\rho.$$

Applying Plancherel formula, we get

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2}$$

$$\leq \sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg\{\int_0^\infty\int_{S^1}\int_{\mathbb{R}^2}\Bigg|\sum_{j=1}^{2}\overline{Y_{m,j}(z')}\varphi_{m,j}*f(x)\Bigg|^2 dx\frac{d\sigma(z')\, dt}{t^{3-\alpha}}\Bigg\}^{\frac{1}{2}}$$

$$\leq C\sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg\{\int_0^\infty\int_{S^1}\int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\left(\sum_{j=1}^{2}\overline{Y_{m,j}(z')}Y_{m,j}(\xi')\right)^2\frac{1}{|\xi|^2}$$

$$\times\left(\int_0^{2\pi|\xi|t} J_m(\rho)\,d\rho\right)^2 d\xi \frac{d\sigma(z')\,dt}{t^{3-\alpha}}\Bigg\}^{\frac{1}{2}}$$

$$\le C\sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg\{\int_{\mathbb{R}^2}\int_0^{\infty}\int_{S^1}\left(\sum_{j=1}^{2}|Y_{m,j}(z')|^2|Y_{m,j}(\xi')|^2\right)d\sigma(z')$$

$$\times\left|\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t} J_m(\rho)\,d\rho\cdot\hat{f}(\xi)\right|^2\frac{dt}{t^{1-\alpha}}\,d\xi\Bigg\}^{\frac{1}{2}}.$$

Thus,

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2}$$

$$= C\sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg\{\int_{\mathbb{R}^2}\int_0^{\infty}\sum_{j=1}^{2}|Y_{m,j}(\xi')|^2$$

$$\times\left|\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t} J_m(\rho)\,d\rho\cdot\hat{f}(\xi)\right|^2\frac{dt}{t^{1-\alpha}}\,d\xi\Bigg\}^{\frac{1}{2}}$$

$$= C\sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)\Bigg\{\int_{\mathbb{R}^2}\int_0^{\infty}\left|\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t} J_m(\rho)\,d\rho\cdot\hat{f}(\xi)\right|^2\frac{dt}{t^{1-\alpha}}\,d\xi\Bigg\}^{\frac{1}{2}}.$$

Since Ω satisfies the $\overline{L}^{2,\frac{\alpha}{2}}$-Dini condition, it follows from Lemma 5 and Lemma D that

$$\|\mu_{\Omega,\alpha}(f)\|_{L^2}\le C\sum_{m=1}^{\infty}\overline{\omega}_2\left(\frac{\pi}{m}\right)m^{-1+\frac{\alpha}{2}}\|f\|_{L^{\frac{4}{2+\alpha}}}\le C\|f\|_{L^{\frac{4}{2+\alpha}}}.$$

This completes the proof of Theorem 1′. □

If we apply Theorem 1′ instead of Theorem 1 in the proof of Theorem 2, we immediately have the following result.

THEOREM 2′. *Let $n=2$ and $0<\alpha<2$. Also let $\Omega\in L^\infty(\mathbb{R}^2)\times L^2(S^1)$ and set $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{4}$. Suppose that p and Ω satisfy one of the following conditions*:

(a) $\max\{\frac{4}{5},\frac{4}{4+\alpha}\}<p\le 1$, *$\Omega$ satisfies the $L^{q,\frac{\alpha}{2}}$-Dini condition*;
(b) $0<\alpha<1$ *and* $p=\frac{4}{4+\alpha}$, *Ω satisfies the $L^{1,\frac{\alpha}{2}}$-Dini condition*;
(c) $\max\{\frac{4}{5},\frac{4}{4+\beta}\}<p<\frac{4}{4+\alpha}$ *for some β with $\alpha<\beta\le 2$, Ω satisfies the $L^{1,\frac{\beta}{2}}$-Dini condition.*

Then there exists a constant C independent of f such that

$$\|\mu_{\Omega,\alpha}(f)\|_{L^q(\mathbb{R}^2)}\le C\|f\|_{H^p(\mathbb{R}^2)}.$$

Applying the interpolation to Theorems 1′ and 2′, we get the following L^p-L^q boundedness of $\mu_{\Omega,\alpha}$.

Corollary 3′. *Suppose* $n = 2$ *and* $0 < \alpha < 1$. *Let* $1 < p < \frac{4}{2+\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{4}$. *If* $\Omega \in L^\infty(\mathbb{R}^2) \times L^2(S^1)$ *and satisfies the* $L^{2,\frac{\alpha}{2}}$*-Dini condition, then there exists a constant* C *independent of* f *such that* $\|\mu_{\Omega,\alpha}(f)\|_{L^q(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}$.

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