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DOUBLING MEASURES AND NONQUASISYMMETRIC MAPS ON WHITNEY MODIFICATION SETS IN EUCLIDEAN SPACES

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ABSTRACT. Let E be a closed set in $\mathbb{R}^n$ and $\mathcal{W}$ a Whitney decomposition of $\mathbb{R}^n \setminus E$. Choosing one point from the interior of each cube in $\mathcal{W}$ we obtain a set F and then we say that the set $E \cup F$ is a Whitney modification of E. The Whitney modification of a measure μ on $\mathbb{R}^n$ to $E \cup F$ is a measure ν defined on $E \cup F$ by $\nu \equiv \mu$ on E and by $\nu(\{x\}) = \mu(I_x)$ for every $x \in F$, where $I_x \in \mathcal{W}$ is the cube containing the point x. We prove that a measure on $E \cup F$ is doubling if and only if it is the Whitney modification of a doubling measure on $\mathbb{R}^n$. As its application, we show that there are metric spaces X, Y and a nonquasisymmetric homeomorphism f of X onto Y such that a measure μ on X is doubling if and only if its image $\mu \circ f^{-1}$ is doubling on Y.

1. Definitions and main results

We start by the definition of doubling measures. Let (X,d) be a metric space and denote by $B(x,r)$ the ball of radius $r > 0$ centered at $x \in X$. A Borel measure μ on X is called doubling if there is a constant $C \geq 1$ such that

$$0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < +\infty$$

for all balls $B(x,r)$ in X. In this case, μ is also called C-doubling.

A cube in $\mathbb{R}^n$ is a set of the type $\prod_{i=1}^{n}[x_i - r, x_i + r]$, or more precisely, it is called the cube of sidelength $2r$ centered at $x = (x_1, \ldots, x_n)$. We denote it by $Q(x,r)$. For Borel measures on $\mathbb{R}^n$, the doubling condition can be equivalently

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defined by cubes in place of balls because

$$Q\big(x, r/\sqrt{n}\big) \subseteq B(x,r) \subseteq Q(x,r).$$

Now, we introduce Whitney's decomposition. Let E be a nonempty closed set in $\mathbb{R}^n$. A family $\mathcal{W}$ of cubes in $\mathbb{R}^n$ is called a Whitney decomposition of $\mathbb{R}^n \setminus E$, if it satisfies the conditions:

(W1) $\bigcup_{I \in \mathcal{W}} I = \mathbb{R}^n \setminus E$.

(W2) The cubes in $\mathcal{W}$ have mutually disjoint interiors.

(W3) There is a constant $K \geq 1$ such that for every cube $I \in \mathcal{W}$

$$K^{-1} \operatorname{dist}(I, E) \leq |I| \leq K \operatorname{dist}(I, E),$$

where $\operatorname{dist}(I,E) = \inf_{x \in I, y \in E} |x-y|$ denotes the distance to E from I and $|I|$ denotes the diameter of I.

Definition 1. Let E be a closed set in $\mathbb{R}^n$ and $\mathcal{W}$ a Whitney decomposition of $\mathbb{R}^n \setminus E$. Choosing one point from the interior of each cube in $\mathcal{W}$, we obtain a set F and then we say that the set $E \cup F$ is a Whitney modification of E.

It is clear that the set E has infinitely many different Whitney modifications for the same Whitney decomposition $\mathcal{W}$ of $\mathbb{R}^n \setminus E$. For clarity, write $F(\mathcal{W})$ for the set F in Definition 1. For each $x \in F(\mathcal{W})$, denote by I_x the cube in $\mathcal{W}$ containing x.

Definition 2. Let $X = E \cup F(\mathcal{W})$ be a Whitney modification of the set E. Let ν be a measure on X and μ be a measure on $\mathbb{R}^n$. We say that ν is the Whitney modification of μ to X, if $\nu \equiv \mu$ on E and $\nu(\{x\}) = \mu(I_x)$ for each $x \in F(\mathcal{W})$.

Given a Whitney modification X of the set E, every measure on $\mathbb{R}^n$ has a unique Whitney modification to X. However, it is possible that the Whitney modifications of two different measures coincide.

Let X be a Whitney modification of a closed set $E \subset \mathbb{R}^1$ and μ a doubling measure on $\mathbb{R}^1$. Kaufman–Wu [2] proved that the Whitney modification of μ to X is a doubling measure on X. As an application, they showed that some metric spaces carry purely atomic as well as nonpurely-atomic doubling measures. We continue the study of doubling measures on Whitney modification sets and prove the following theorem.

Theorem 1. *Let $X = E \cup F(\mathcal{W})$ be a Whitney modification of a closed set $E \subset \mathbb{R}^n$ and ν be a measure on X. Then ν is doubling on X if and only if there exists a doubling measure μ on $\mathbb{R}^n$ such that ν is the Whitney modification of μ to X.*

Many important theorems in classic analysis, such as Lebesgue's differential theorem, Hardy–Littlewood's maximal function theorem, etc., also hold in doubling metric measure spaces (see [1]). This phenomenon leads to the study of doubling measures on metric spaces. For the existence of doubling measures, it is known that every complete doubling metric space carries doubling measures (see [4] and [7]). Especially, every closed set in $\mathbb{R}^n$ carries doubling measures. However, there exist Jordan open domains in $\mathbb{R}^n$ which carry no doubling measures (see [5]). For the description of doubling measures, it is known that doubling measures on the real line are those obtained by quasi-symmetric maps. However, a similar result does not hold for higher dimension (see [3] and [6]). Up to now, there are quite few examples of metric spaces for which doubling measures have been characterized. The Whitney modification set considered in this paper is closed, so it carries doubling measures. Theorem 1 provides us a useful description for these doubling measures.

Now, we show an application of Theorem 1. Let X, Y be metric spaces which are bilipschitzly equivalent under f. It is clear that a measure μ on X is doubling if and only if its image $\mu \circ f^{-1}$ is doubling on Y. In other words, bilipschitz maps preserve the doubling property of measures. It is also known that quasisymmetric homeomorphisms of $\mathbb{R}^1$ preserve the doubling property of measures on $\mathbb{R}^1$. It is natural to ask: is there a nonquasisymmetric homeomorphism of metric spaces which preserves the doubling property of measures? The answer to this question is positive.

THEOREM 2. *There are metric spaces X and Y for which there is a homeomorphism f of X onto Y such that a measure μ on X is doubling if and only if its image $\mu \circ f^{-1}$ is doubling on Y, but f is not quasisymmetric.*

Thus, f has the property that was stated above for bilipschitz maps. We recall that a homeomorphism $f : X \to Y$ is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{|f(x) - f(a)|}{|f(x) - f(b)|} \le \eta\left(\frac{|x - a|}{|x - b|}\right)$$

for any triple x, a, b of distinct points in X, where $| \cdot - \cdot |$ denotes metrics.

2. Preliminary lemmas

We adopt the following notation throughout the paper. Denote by E a nonempty closed set in $\mathbb{R}^n$, by $\mathcal{W}$ a Whitney decomposition of $\mathbb{R}^n \setminus E$, by K a constant for which $\mathcal{W}$ satisfies the condition $(W3)$, and by $X = E \cup F(\mathcal{W})$ a Whitney modification of the set E. For each $x \in F(\mathcal{W})$, denote by I_x the cube in $\mathcal{W}$ containing x. We say that two cubes in $\mathcal{W}$ are adjacent if their boundaries meet.

We need the following simple properties of the Whitney modification sets and doubling measures on them.

LEMMA 1. *$|I| \le 2K^2|J|$ for any pair of adjacent cubes I, J in $\mathcal{W}$. In other words, the diameters of two adjacent cubes in $\mathcal{W}$ are comparable.*

Proof. Let I, J be two adjacent cubes in $\mathcal{W}$. Then

$$|I| \le K \operatorname{dist}(I,E) \le K\big(\operatorname{dist}(J,E) + |J|\big) \le (K^2+K)|J| \le 2K^2|J|$$

by the condition (W3). □

LEMMA 2. *Let $x \in X$, $r > 0$, $\lambda > 0$, and $A = \{I \in \mathcal{W} : I \cap Q(x,r) \neq \emptyset\}$. Then the following propositions hold.*

(a) *If $x \in E$, then $\bigcup_{I\in A} I \subset Q(x, 2Knr)$.*

(b) *If $x \in F$ and $|I_x| \le \lambda r$, then $\bigcup_{I\in A} I \subset Q(x, 3K^2n(1+\lambda)r)$.*

Proof. Without loss of generality, assume that $A \neq \emptyset$.

Proof of (a). If $x \in E$, then by the condition (W3), for every $I \in A$

$$|I| \le K \operatorname{dist}(I,E) \le Kr\sqrt{n} \le Knr,$$

which implies $\bigcup_{I\in A} I \subset Q(x, r+Knr) \subset Q(x, 2Knr)$.

Proof of (b). If $x \in F$ and $|I_x| \le \lambda r$, then by (W3), for every $I \in A$

$$\begin{aligned} |I| &\le K \operatorname{dist}(I,E) \le K\big(\operatorname{dist}(I_x,E) + |I_x| + r\sqrt{n}\big) \\ &\le K(2K|I_x| + nr) \le 2K^2n(1+\lambda)r, \end{aligned}$$

which implies $\bigcup_{I\in A} I \subset Q(x, r + 2K^2n(1+\lambda)r) \subset Q(x, 3K^2n(1+\lambda)r)$. □

LEMMA 3. *Let $x \in \mathbb{R}^n \setminus E$. Let y be a point in F such that $x \in I_y$. Then the following propositions hold.*

(a) *If $r > 2K|I_y|$, then $Q(x,r) \cap E \neq \emptyset$.*

(b) *If $r < (3Kn)^{-2}|I_y|$, then $Q(x,r)$ can be covered by no more than 2^n cubes in $\mathcal{W}$.*

Proof. (a) It is immediate since the assumption $r > 2K|I_y|$, together with the condition (W3), implies

$$\operatorname{dist}(x,E) \le \operatorname{dist}(I_y,E) + |I_y| \le (K+1)|I_y| \le 2K|I_y| < r.$$

(b) We first prove $E \cap Q(x,r) = \emptyset$. In fact, if this does not hold then the assumption $r < (3Kn)^{-2}|I_y|$, together with the condition (W3), yields

$$|I_y| \le K \operatorname{dist}(I_y,E) \le K \operatorname{dist}(x,E) \le K\sqrt{n}r < (9Kn)^{-1}|I_y|,$$

a contradiction. Therefore, $Q(x,r)$ can be covered by cubes in $\mathcal{W}$.

Let $A = \{I \in \mathcal{W} : I \cap Q(x,r) \neq \emptyset\}$. To complete this proof, it suffices to prove that the cardinality $\operatorname{card}(A)$ of A is at most 2^n. If not, since $Q(x,2r)$ has only 2^n vertices, there is a cube I in A whose interior does not contain any vertex of $Q(x,2r)$. Therefore, the sidelength of the cube I is no more than $4r$, which implies

$$|I| \le 4r\sqrt{n}. \tag{1}$$

Since $x \in I_y$ and $I \cap Q(x,r) \neq \emptyset$, the inequality (1) together with the condition (W3) yields

$$\begin{aligned}|I_y| &\le K\,\mathrm{dist}(I_y,E) \le K\,\mathrm{dist}(x,E) \le K\big(\mathrm{dist}(I,E)+|I|+r\sqrt{n}\big)\\ &\le K(K+1)|I|+Kr\sqrt{n} \le 9K^2r\sqrt{n} \le (3Kn)^2 r,\end{aligned}$$

which contradicts with the assumption $r < (3Kn)^{-2}|I_y|$. □

LEMMA 4. *Let ν be a C-doubling measure on the Whitney modification set X. Let x,y be points in F such that I_x, I_y are adjacent. Then*

$$H^{-1}\nu(\{y\}) \le \nu(\{x\}) \le H\nu(\{y\}),$$

where and below H is a constant that may vary from line to line.

Proof. Suppose that x,y are points in F such that I_x, I_y are adjacent. Then $y \in Q(x,(2K^2+1)|I_x|)$ by Lemma 1. It follows from the doubling property of ν that

$$\nu(\{y\}) \le \nu\big(Q\big(x,(2K^2+1)|I_x|\big)\big) \le H\nu(Q(x,(4Kn)^{-2}|I_x|)). \tag{2}$$

Since, by Lemma 3(2), $Q(x,(4Kn)^{-2}|I_x|)$ can be covered by at most 2^n cubes in $\mathcal{W}$, it contains at most 2^n points of F, say $x, y_1, \ldots, y_m$, where $m < 2^n$. Therefore,

$$\nu(Q(x,(4Kn)^{-2}|I_x|)) = \nu(\{x,y_1,\ldots,y_m\}). \tag{3}$$

Let $d_i = |x-y_i|$, $i=1,\ldots,m$. Without loss of generality, assume that

$$d_1 \le d_2 \le \cdots \le d_m.$$

By the doubling property of ν,

$$\nu(\{x,y_1,\ldots,y_i\}) \le H\nu(\{x,y_1,\ldots,y_{i-1}\}), \quad i=1,\ldots,m.$$

Inductively, $\nu(\{x,y_1,\ldots,y_m\}) \le H\nu(\{x\})$, which, combined with (2) and (3), yields

$$\nu(\{y\}) \le H\nu(\{x\}).$$

By symmetry, $\nu(\{x\}) \le H\nu(\{y\})$. This completes the proof. □

3. The proof of Theorem 1

Proof of the necessity. Let $X = E \cup F(\mathcal{W})$ be a Whitney modification of E and ν be a C-doubling measure on X. We are going to construct a measure μ on $\mathbb{R}^n$ such that μ is doubling on $\mathbb{R}^n$ and ν is the Whitney modification measure of μ to X.

The measure μ is constructed by uniformly distributing the ν-measure of every point x in F to the cube I_x. Let

$$\mu(A) = \begin{cases}\nu(A), & \text{if } A \subset E,\\ \rho_x\mathcal{L}^n(A), & \text{if } x \in F \text{ and } A \subset I_x,\end{cases}$$

where A is a Borel set, $\rho_x = \nu(\{x\})/\mathcal{L}^n(I_x)$, and $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$. Clearly, this defines a measure on $\mathbb{R}^n$.

By the definition of μ, we have $\nu \equiv \mu$ on E and $\nu(\{x\}) = \mu(I_x)$ for every $x \in F$, so ν is the Whitney modification measure of μ to X. The rest is to prove that μ is doubling on $\mathbb{R}^n$. To this end, given a cube $Q(x,r) \subset \mathbb{R}^n$, we are going to show

$$\mu(Q(x,2r)) \le H\mu(Q(x,r)). \tag{4}$$

Case 1 ($x \in E$). Let $A = \{I \in \mathcal{W} : I \cap Q(x,2r) \neq \emptyset\}$. Since $\bigcup_{I\in A} I \subset Q(x, 4Knr)$ by Lemma 2(a), it follows from the definition of μ and the doubling property of ν that

$$\mu(Q(x,2r)) \le \nu(Q(x,4Knr)) \le H\nu(Q(x,r)). \tag{5}$$

On the other hand, let $B = \{I \in \mathcal{W} : I \cap Q(x,(2Kn)^{-1}r) \neq \emptyset\}$. Using Lemma 2(a), again, we get $\bigcup_{I\in B} I \subset Q(x,r)$, which combined with the definition of μ and the doubling property of ν, yields

$$\mu(Q(x,r)) \ge \nu(Q(x,(2Kn)^{-1}r)) \ge H\nu(Q(x,r)). \tag{6}$$

By (5) and (6), we get $\mu(Q(x,2r)) \le H\mu(Q(x,r))$.

Case 2 ($x \in F$). We consider three subcases.

(a) If $r > 4K|I_x|$ then $E \cap Q(x,r/2) \neq \emptyset$ by Lemma 3(a). Pick a point $x^* \in E \cap Q(x,r/2)$. Then $Q(x^*,r/2) \subset Q(x,r)$ and $Q(x,2r) \subset Q(x^*,4r)$. Using the conclusion of Case 1, we get

$$\mu(Q(x,2r)) \le \mu(Q(x^*,4r)) \le H\mu\big(Q(x^*,r/2)\big) \le H\mu(Q(x,r)).$$

(b) If $r < (3Kn)^{-2}|I_x|/2$, then by Lemma 3(b), $Q(x,2r)$ can be covered by at most 2^n cubes in $\mathcal{W}$, say $I_x, I_{y_1}, \dots, I_{y_m}$, where $x, y_1, \dots, y_m \in F, m < 2^n$. Moreover, these cubes can be arranged such that they are one by one adjacent. Therefore, their volumes are comparable due to Lemma 1 and the measures $\nu\{x\}, \nu\{y_1\}, \dots, \nu\{y_m\}$ are comparable due to Lemma 4. It then follows that $\rho_x, \rho_{y_1}, \dots, \rho_{y_m}$ are also comparable, and so

$$\frac{\mu(Q(x,2r))}{\mu(Q(x,r))} \le \frac{\max\{\rho_x, \rho_{y_1}, \dots, \rho_{y_m}\}\mathcal{L}^n(Q(x,2r))}{\min\{\rho_x, \rho_{y_1}, \dots, \rho_{y_m}\}\mathcal{L}^n(Q(x,r))} \le H. \tag{7}$$

(c) For the case $(3Kn)^{-2}|I_x|/2 \le r \le 4K|I_x|$, let

$$A = \{I \in \mathcal{W} : I \cap Q(x,2r) \neq \emptyset\}.$$

Since $x \in F$, by Lemma 2(b), we have $\bigcup_{I\in A} I \subset Q(x,\xi r)$, where $\xi = 6K^2n(1 + (3Kn)^2)$, which combined with the definition of μ and the doubling property of ν, yields

$$\mu(Q(x,2r)) \le \nu(Q(x,\xi r)) \le H\nu(Q(x,r)). \tag{8}$$

On the other hand, by Lemma 3(b), $Q(x,(3Kn)^{-2}|I_x|/2)$ can be covered by at most 2^n cubes in $\mathcal{W}$, say $I_x, I_{y_1}, \dots, I_{y_m}$, where $x, y_1, \dots, y_m \in F, m < 2^n$. By the same argument as that in the above subcase (b), the volumes of these cubes are comparable, and so $\rho_x, \rho_{y_1}, \dots, \rho_{y_m}$ are comparable. Since ν is doubling, it then follows that

$$\begin{aligned}\mu(Q(x,r)) &\geq \mu\big(Q\big(x,(3Kn)^{-2}|I_x|/2\big)\big)\\ &\geq H|I_x|^n \min\{\rho_x, \rho_{y_1}, \dots, \rho_{y_m}\}\\ &\geq H\mathcal{L}^n\left(I_x \cup \bigcup_{i=1}^{m} I_{y_i}\right)\max\{\rho_x, \rho_{y_1}, \dots, \rho_{y_m}\}\\ &\geq H\nu\big(Q\big(x,(3Kn)^{-2}|I_x|/2\big)\big)\\ &\geq H\nu(Q(x,4K|I_x|)) \geq H\nu(Q(x,r)),\end{aligned}$$

which together with (8) yields $\mu(Q(x,2r)) \leq H\mu(Q(x,r))$.

CASE 3 ($x \notin X$ but $Q(x,r/2) \cap X \neq \emptyset$). Pick a point $x^* \in Q(x,r/2) \cap X$. Then one has $Q(x^*,r/2) \subset Q(x,r)$ and $Q(x,2r) \subset Q(x^*,4r)$. It follows from the conclusion of Case 1 or 2 that

$$\mu(Q(x,2r)) \leq \mu(Q(x^*,4r)) \leq H\mu\big(Q(x^*,r/2)\big) \leq H\mu(Q(x,r)).$$

CASE 4 ($Q(x,r/2) \cap X = \emptyset$). Let y be a point in F such that $x \in I_y$. Then we have

$$|I_y| > \sqrt{n}r/2 \quad \text{and} \quad \mathcal{L}^n\big(Q(x,r) \cap I_y\big) \geq (r/2)^n. \tag{9}$$

We consider two subcases.

(a) If $|I_y| > 2r(3Kn)^2$, then by Lemma 3(b), $Q(x,2r)$ can be covered by at most 2^n cubes in $\mathcal{W}$. By the same argument as that in Case 2(b), we have $\mu(Q(x,2r)) \leq H\mu(Q(x,r))$.

(b) In the case where $r\sqrt{n}/2 < |I_y| \leq 2r(3Kn)^2$, let

$$A = \{I \in \mathcal{W} : I \cap Q(x,2r) \neq \emptyset\}.$$

In this case, for every $I \in A$ we have

$$\begin{aligned}|I| &\leq K\,\mathrm{dist}(I,E) \leq K\big(\mathrm{dist}(I_y,E) + |I_y| + 2r\sqrt{n}\big)\\ &\leq K\big(2K|I_y| + 2r\sqrt{n}\big) \leq 38K^4n^2r\end{aligned}$$

by the condition (W3). Therefore,

$$\bigcup_{I\in A} I \subset Q(x,40K^4n^2r) \subset Q\big(y,40K^4n^2r + 2r(3Kn)^2\big).$$

It then follows from the definition of μ and the doubling property of ν that

$$\mu(Q(x,2r)) \leq \nu\big(Q\big(y,40K^4n^2r + 2r(3Kn)^2\big)\big) \leq H\nu(Q(y,r)). \tag{10}$$

On the other hand, by Lemma 3(b), $Q(y,(4Kn)^{-2}|I_y|)$ can be covered by at most 2^n cubes in $\mathcal{W}$, say $I_y, I_{y_1},\dots,I_{y_m}$, where $m<2^n$. Similar to Case 2(c), we get from the inequality (9) and the doubling property of ν that

$$\begin{aligned}\mu(Q(x,r)) &\ge \mu\big(Q(x,r)\cap I_y\big) \ge (r/2)^n\rho_y \ge H|I_y|^n \max\{\rho_y,\rho_{y_1},\dots,\rho_{y_m}\}\\ &\ge H\nu(Q(y,(4Kn)^{-2}|I_y|)) \ge H\nu(Q(y,r)),\end{aligned}$$

which together with (10) yields $\mu(Q(x,2r))\le H\mu(Q(x,r))$.

The proof of the necessity of Theorem 1 is now completed.

Proof of the sufficiency. Let $X=E\cup F(\mathcal{W})$ be a Whitney modification of E, μ be a C-doubling measure on $\mathbb{R}^n$ and ν be the Whitney modification measure of μ to X. We are going to prove that ν is doubling on X. Given a cube $Q(x,r)\subset\mathbb{R}^n$, we are going to show

$$\nu(Q(x,2r))\le H\nu(Q(x,r)). \tag{11}$$

Case 1 $(x\in E)$. Let $A=\{I\in\mathcal{W}: I\cap Q(x,2r)\ne\emptyset\}$. Then $\bigcup_{I\in A} I\subset Q(x,4Knr)$ by Lemma 2(a). Since ν is the Whitney modification of μ to X, it follows from the doubling property of μ that

$$\nu(Q(x,2r))\le\mu(Q(x,4Knr))\le H\mu(Q(x,r)). \tag{12}$$

On the other hand, let $B=\{I\in\mathcal{W}: I\cap Q(x,(2Kn)^{-1}r)\ne\emptyset\}$. Then $\bigcup_{I\in B} I\subset Q(x,r)$ by Lemma 2(a), and so

$$\nu(Q(x,r))\ge\mu(Q(x,(2Kn)^{-1}r))\ge H\mu(Q(x,r)),$$

which together with (12) implies $\nu(Q(x,2r))\le H\nu(Q(x,r))$.

Case 2 $(x\in F)$. We consider three subcases.

(a) If $r>4K|I_x|$, then $E\cap Q(x,r/2)\ne\emptyset$ by Lemma 3(a). Pick a point $x^*\in E\cap Q(x,r/2)$. Then $Q(x^*,r/2)\subset Q(x,r)$ and $Q(x,2r)\subset Q(x^*,4r)$. Using the conclusion of Case 1, we have $\nu(Q(x,2r))\le H\nu(Q(x,r))$.

(b) If $r<(3Kn)^{-2}|I_x|/2$, then by Lemma 3(b), $Q(x,2r)$ can be covered by at most 2^n cubes in $\mathcal{W}$, say $I_x, I_{y_1},\dots,I_{y_m}$, where $x,y_1,\dots,y_m\in F, m<2^n$. These cubes can be arranged such that they are one by one adjacent. By Lemma 1, their diameters are comparable, and so $\mu\{I_x\},\mu\{I_{y_1}\},\dots,\mu\{I_{y_m}\}$ are comparable by the doubling property of μ. As ν is the Whitney modification measure of μ to X, we see that $\nu\{x\},\nu\{y_1\},\dots,\nu\{y_m\}$ are comparable, which implies

$$\frac{\nu(Q(x,2r))}{\nu(Q(x,r))}\le\frac{\nu\{x\}+\nu\{y_1\}+\cdots+\nu\{y_m\}}{\nu\{x\}}\le H.$$

(c) Now, consider the case $(3Kn)^{-2}|I_x|/2\le r\le 4K|I_x|$. Let

$$A=\{I\in\mathcal{W}: I\cap Q(x,2r)\ne\emptyset\}.$$

As $x \in F$ one has $\bigcup_{I\in A} I \subset Q(x,\xi r)$ by Lemma 2(b), where $\xi = 6K^2n(1+(3Kn)^2)$. Since ν is the Whitney modification measure of μ to X, it follows from the doubling property of μ that

$$\nu(Q(x,2r)) \le \mu(Q(x,\xi r)) \le H\mu(Q(x,r)). \tag{13}$$

On the other hand, by Lemma 3(b), $Q(x,(3Kn)^{-2}|I_x|/2)$ can be covered by at most 2^n cubes in $\mathcal{W}$, say $I_x, I_{y_1}, \dots, I_{y_m}$, where $x, y_1, \dots, y_m \in F, m < 2^n$. By the same argument as that in Case 2(b), $\nu\{x\}, \nu\{y_1\}, \dots, \nu\{y_m\}$ are comparable. It follows from the doubling property of μ that

$$\begin{aligned}\nu(Q(x,r)) \ge \nu\{x\} &\ge H(\nu\{x\} + \nu\{y_1\} + \cdots + \nu\{y_m\})\\ &\ge H\mu\left(I_x \cup \bigcup_{i=1}^{m} I_{y_i}\right) \ge H\mu\big(Q\big(x,(3Kn)^{-2}|I_x|/2\big)\big)\\ &\ge H\mu(Q(x,4K|I_x|)) \ge H\mu(Q(x,r)),\end{aligned}$$

which, combined with (13), yields $\nu(Q(x,2r)) \le H\nu(Q(x,r))$.

The proof of the sufficiency of Theorem 1 is now completed.

4. The proof of Theorem 2

Let E be a nonempty closed set in $\mathbb{R}^n$. For the same Whitney decomposition $\mathcal{W}$ of $\mathbb{R}^n \setminus E$, the set E has infinitely many different Whitney modifications. Let $X_i = E \cup F_i(\mathcal{W}), i = 1,2$, be two Whitney modifications of E. Then there is a natural homeomorphism $f : X_1 \to X_2$, which maps a point in E to itself and a point x in $F_1(\mathcal{W})$ to the unique point in $F_2(\mathcal{W}) \cap I_x$. It follows from Theorem 1 that a measure μ on X_1 is doubling if and only if its image $\mu \circ f^{-1}$ is doubling on X_2. To complete this proof, we are going to show that the Whitney modifications X_1 and X_2 can be chosen such that f is not quasisymmetric.

Let $E = \{0\}$ and $\mathcal{W} = \{[-2^{n+1}, -2^n], [2^n, 2^{n+1}] : n \in \mathbb{Z}\}$. It is obvious that $\mathcal{W}$ is a Whitney decomposition of $\mathbb{R}^1 \setminus E$. Let $F_1(\mathcal{W})$ be the set of midpoints of intervals in $\mathcal{W}$ and

$$F_2(\mathcal{W}) = \big((-\infty, 1] \cap F_1(\mathcal{W})\big) \cup \{2^n \pm 2^{-n} : n = 1,3,5,\dots\}.$$

Then $X_i = E \cup F_i(\mathcal{W}), i = 1,2$, are Whitney modifications of E. For this choice of X_1 and X_2, we claim that f is not quasisymmetric.

In fact, if f is quasisymmetric for a homeomorphism $\eta : [0,+\infty) \to [0,+\infty)$, then for any positive integer k

$$\frac{|f(x_k) - f(a_k)|}{|f(x_k) - f(b_k)|} \le \eta\left(\frac{|x_k - a_k|}{|x_k - b_k|}\right), \tag{14}$$

where a_k, x_k and b_k are respectively the midpoints of intervals $[2^{2k-1}, 2^{2k}]$, $[2^{2k}, 2^{2k+1}]$ and $[2^{2k+1}, 2^{2k+2}]$. Obviously,

$$\frac{|x_k - a_k|}{|x_k - b_k|} \leq 1.$$

Note that

$$f(a_k) = 2^{2k-1} + 2^{-(2k-1)}, \qquad f(x_k) = 2^{2k+1} - 2^{-(2k+1)},$$
$$f(b_k) = 2^{2k+1} + 2^{-(2k+1)}.$$

One has

$$\frac{|f(x_k) - f(a_k)|}{|f(x_k) - f(b_k)|} \to +\infty$$

as $k \to +\infty$. Since η is obviously increasing, it follows from the inequality (14) that $\eta(1) = +\infty$, a contradiction.

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