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MONOMIAL SEQUENCES OF LINEAR TYPE

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ABSTRACT. Let R be a Noetherian commutative ring, $\langle a_1,\dots,a_n\rangle$ a sequence of elements of R, $I=(a_1,\dots,a_n)$ the ideal generated by the elements a_i and $I_i=(a_1,\dots,a_i)$, $i=0,1,\dots,n$, the ideal generated by the first i elements of the sequence. A c-sequence is a sequence $\langle a_1,\dots,a_n\rangle$ which satisfies the condition

$$[I_{i-1}I^k : a_i]\cap I^k = I_{i-1}I^{k-1}$$

for every $i\in\{1,\dots,n\}$ and every $k\geq 1$. It generates an ideal of linear type. We characterize c-sequences in terms of the corresponding sequences in the Rees algebra of the ideal generated by the elements of the sequence. We then characterize monomial c-sequences of three terms.

1. Introduction

Let R be a Noetherian commutative ring, $\langle \mathbf{a}\rangle = \langle a_1,\dots,a_n\rangle$ a sequence of elements of R, $I=(a_1,\dots,a_n)$ the ideal generated by the a_i's and $I_i=(a_1,\dots,a_i)$, $i=0,1,\dots,n$, the ideal generated by the first i elements of the sequence.

Let $S(I)=\bigoplus_{i\geq 0} S^i(I)$ be the symmetric algebra of the ideal I, $R[It]=\bigoplus_{i\geq 0} I^i t^i$ its Rees algebra and $\alpha : S(I)\to R[It]$ the canonical map, which maps $a_i\in S^1(I)$ to a_it. The ideal I is said to be of *of linear type* if α is an isomorphism. There are also the canonical maps $\rho : R[T_1,\dots,T_n]\to R[It]$, mapping T_i to a_it, and $\sigma : R[T_1,\dots,T_n]\to S(I)$, mapping T_i to $a_i\in S^1(I)$. Let $Q_\infty=\ker(\rho)$ and $Q=\ker(\sigma)$. Then $Q\subset Q_\infty$ and $\mathcal{A}:=\ker(\alpha)$ can be identified with Q_∞/Q.

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Now, we list various types of sequences related to the notion of ideals of linear type. The notion of *regular sequence* is one of the most important notions in Commutative Algebra (see [10]) and there are various generalizations of it.

We say that $\langle \mathbf{a} \rangle$ is a *relative regular* or *d-sequence* [8] if

$$[I_{i-1} : a_i] : a_j = I_{i-1} : a_j$$

for every $i, j \in \{1, 2, \ldots, n\}$ with $j \geq i$. Equivalently,

$$[I_{i-1} : a_i] \cap I = I_{i-1}$$

for every $i \in \{1, 2, \ldots, n\}$.

We say that $\langle \mathbf{a} \rangle$ is a *weakly relative regular* sequence [2] if

$$[I_{i-1} I : a_i] \cap I = I_{i-1}$$

for every $i \in \{1, 2, \ldots, n\}$.

We say that $\langle \mathbf{a} \rangle$ is a *proper* sequence [5] if

$$a_i \cdot H_j(a_1, \ldots, a_{i-1}) = 0,$$

for $i = 1, \ldots, n$, $j \geq 1$, where $H_j(a_1, \ldots, a_{i-1})$ denotes the jth homology module of the Koszul complex on $a_1, \ldots, a_{i-1}$. (Actually, it is enough to have this property for $j = 1$, and it is then true for all $j \geq 1$ by [9].)

We say that $\langle \mathbf{a} \rangle$ is a *sequence of linear type* [1] if each of the ideals $I_i = (a_1, \ldots, a_i)$, $i = 1, \ldots, n$, is of linear type.

It is well known that the ideals generated by d-sequences are of linear type ([6], [12]), in fact that the d-sequences are sequences of linear type. Every d-sequence is weakly relative regular and every weakly relative regular sequence is proper [5].

For any type of sequences, we say that a sequence is an *unconditioned* sequence of that type if it is a sequence of that type in any order.

Let $x_1, x_2, \ldots, x_d$ be an unconditioned regular sequence. A *monomial* in $x_1, x_2, \ldots, x_d$ is a product $x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$, where $n_1, n_2, \ldots, n_d$ are nonnegative integers. A *monomial ideal* is an ideal generated by monomials.

2. c-sequences

It was proved in [1] that d-sequences satisfy the following property:

$$[I_{i-1} I^k : a_i] \cap I^k = I_{i-1} I^{k-1}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$. It was also proved [1, Theorem 3] that, if a sequence satisfies this property, it generates an ideal of linear type. We call the sequences that satisfy this property c-sequences.

Definition 2.1. We say that $\langle \mathbf{a} \rangle$ is a *c-sequence* if

$$[I_{i-1} I^k : a_i] \cap I^k = I_{i-1} I^{k-1}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$.

In the next section, we will show that there are (monomial) sequences that are c-sequences but not d-sequences. We first establish an analogue for c-sequences of the following two statements:

(i) $\langle \mathbf{a} \rangle$ is a *proper sequence* if and only if the corresponding sequence of 1-forms $\langle \overline{\mathbf{a}} \rangle$ in $S_R(I)$ is a d-sequence [9, Theorem 2.2];

(ii) $\langle \mathbf{a} \rangle$ is a *d-sequence* if and only if the corresponding sequence of 1-forms $\langle \mathbf{a}^* \rangle$ in $gr_I(R)$ is a d-sequence ([7, Theorem 1.2] ($\Rightarrow$) and [5, Theorem 12.10] ($\Leftarrow$)).

PROPOSITION 2.2. *Let $a_1, \dots, a_n \in R$ and let $a_1 t, \dots, a_n t$ be the corresponding 1-forms in $R[It]$. Then $\langle a_1, \dots, a_n \rangle$ is a c-sequence in R if and only if $\langle a_1 t, \dots, a_n t \rangle$ is a d-sequence in $R[It]$.*

Proof. Denote $\mathcal{I} = (a_1 t, \dots, a_n t) = R[It]_+$, the ideal in $R[It]$ generated by $a_1 t, \dots, a_n t$. Also, $\mathcal{I}_{i-1} = (a_1 t, \dots, a_{i-1} t)$ and $I_{i-1} = (a_1, \dots, a_{i-1})$, $i = 1, 2, \dots, n$. Then $\langle \mathbf{a} t \rangle$ is a d-sequence in $R[It]$ if and only if

$$[\mathcal{I}_{i-1} : a_i t] \cap \mathcal{I} = \mathcal{I}_{i-1}, \quad i = 1, 2, \dots, n,$$

or equivalently,

$$[(I_{i-1} t + I_{i-1} I t^2 + \cdots + I_{i-1} I^{k-1} t^k + \cdots) : a_i t] \cap R[It]_+ = \mathcal{I}_{i-1}, \\ i = 1, 2, \dots, n.$$

This is further equivalent with

$$(c_1 t + c_2 t^2 + \cdots) a_i t \in \mathcal{I}_{i-1} \quad \Rightarrow \quad c_1 t + c_2 t^2 + \cdots \in \mathcal{I}_{i-1}, \\ i = 1, 2, \dots, n,$$

where $c_j \in I^j$, $j = 1, 2, \dots$ are arbitrary elements. This in turn is equivalent with

$$c_k \in I_{i-1} I^k : a_i \quad \Rightarrow \quad c_k \in I_{i-1} I^{k-1}, \quad i = 1, 2, \dots, n, \ k \geq 1,$$

where each $c_k \in I^k$. This is the same as

$$[I_{i-1} I^k : a_i] \cap I^k \subset I_{i-1} I^{k-1}, \quad i = 1, 2, \dots, n, \ k \geq 1,$$

which is the condition for $\langle \mathbf{a} \rangle$ to be a c-sequence. □

COROLLARY 2.3. *Let $\langle a_1, \dots, a_n \rangle$ be a sequence in R and let $I = (a_1, \dots, a_n)$. Then the following are equivalent*:

(i) $\langle \mathbf{a} \rangle$ *is a c-sequence*;
(ii) $\langle \mathbf{a} \rangle$ *is a weakly relative regular sequence and I is of linear type*;
(iii) $\langle \mathbf{a} \rangle$ *is a proper sequence and I is of linear type.*

Proof. (i) $\Rightarrow$ (ii): Follows from the definition of a weakly relatively regular sequence and [1, Theorem 3].

(ii) $\Rightarrow$ (iii): Follows from [5, p. 113].

(iii) $\Rightarrow$ (i): By [9, Theorem 2.2], if $\langle \mathbf{a} \rangle$ is a proper sequence, then the corresponding sequence of 1-forms $\langle \overline{\mathbf{a}} \rangle$ is a d-sequence in $S_R(I)$. Since I is assumed to be of linear type, $S_R(I)$ is canonically isomorphic to $R[It]$. Hence, $\langle a_1 t, \dots, a_n t \rangle$ is a d-sequence in $R[It]$. Now, by Proposition 2.2, $\langle a_1, \dots, a_n \rangle$ is a c-sequence in R. □

Thus, if $I = (\mathbf{a})$ is an ideal of linear type, where $\langle \mathbf{a} \rangle$ is a proper or weakly relative regular sequence, then $\langle \mathbf{a} \rangle$ is necessarily a c-sequence. Note that neither proper nor weakly relative regular sequences are necessarily sequences of linear type.

We will use Corollary 2.3 in the proof of Theorem 3.1.

3. Monomial c-sequences

THEOREM 3.1. *Let $\langle g_1, g_2, g_3 \rangle$ be a monomial sequence. Then the following are equivalent*:

(i) *$\langle g_1, g_2, g_3 \rangle$ is a weakly relative regular sequence*;

(ii) *$\langle g_1, g_2, g_3 \rangle$ is a c-sequence*;

(iii) *$[g_1, g_2] | g_3$ and $([g'_1, g'_3] = [g'_1, g'^2_3]$ or $[g'_2, g'_3] = [g'_2, g'^2_3])$, where $g'_i = \frac{g_i}{[g_1, g_2]}$, $i = 1, 2, 3$.*

Proof. By [11, Theorem 3.1], a monomial sequence $\langle g_1, g_2, g_3 \rangle$ is proper if and only if $[g_1, g_2] | g_3$. Also, every weakly relative regular and every c-sequence are proper. Hence, in the statement of the theorem we can assume that $\langle g_1, g_2, g_3 \rangle$ is a proper monomial sequence. Then (because of [11, Theorem 3.1]) its elements can be written as products of monomials in the following way:

$$\begin{aligned} g_1 &= f_1 f_{13} f_{123}, \\ g_2 &= f_2 f_{23} f_{123}, \\ g_3 &= f_3 f_{13} f_{23} f_{123}, \end{aligned} \tag{1}$$

with the condition that the following pairs of monomials are relatively prime:

$$(f_1, f_2), (f_1, f_{23}), (f_{13}, f_2), (f_{13}, f_{23}), (f_1, f_3), (f_2, f_3). \tag{2}$$

Note that then

$$[g_1, g_2] = f_{123}.$$

(i) $\Leftrightarrow$ (iii): Since the ideals $I_{i-1} I : g_i$ and I are monomial and the intersection of monomial ideals is a monomial ideal [3, Theorem 3.10], the condition (i) is equivalent with the following conditions:

$$\begin{gathered} g_1(g_1, g_2, g_3) : g_2^2 = (g_1) : g_2, \\ g_1(g_1, g_2, g_3) : g_2 g_3 = (g_1) : g_3, \\ (g_1, g_2)(g_1, g_2, g_3) : g_3^2 = (g_1, g_2) : g_3. \end{gathered}$$

Using [4, p. 485], this is equivalent with:

$$\left(\frac{g_1^2}{[g_1^2,g_2^2]},\frac{g_1}{[g_1,g_2]},\frac{g_1g_3}{[g_1g_3,g_2^2]}\right)=\left(\frac{g_1}{[g_1,g_2]}\right),$$
$$\left(\frac{g_1^2}{[g_1^2,g_2g_3]},\frac{g_1}{[g_1,g_3]},\frac{g_1}{[g_1,g_2]}\right)=\left(\frac{g_1}{[g_1,g_3]}\right),$$
$$\left(\frac{g_1^2}{[g_1^2,g_3^2]},\frac{g_1}{[g_1,g_3]},\frac{g_1g_2}{[g_1g_2,g_3^2]},\frac{g_2}{[g_2,g_3]},\frac{g_2^2}{[g_2^2,g_3^2]}\right)=\left(\frac{g_1}{[g_1,g_3]},\frac{g_2}{[g_2,g_3]}\right).$$

The condition $\frac{g_1}{[g_1,g_2]}|\frac{g_1^2}{[g_1^2,g_2^2]}$ and analogous conditions for g_1,g_3 and g_2,g_3 are automatically satisfied. The condition $\frac{g_1}{[g_1,g_3]}|\frac{g_1}{[g_1,g_2]}$ is equivalent with $[g_1,g_2]|g_3$, which is also satisfied by [11, Theorem 3.1], since we assumed that $\langle g_1,g_2,g_3\rangle$ is a proper sequence. Thus, $\langle g_1,g_2,g_3\rangle$ is a weakly relative regular sequence if and only if the following conditions hold:

$$\frac{g_1}{[g_1,g_2]}\bigg|\frac{g_1g_3}{[g_1g_3,g_2^2]}, \tag{3}$$

$$\frac{g_1}{[g_1,g_3]}\bigg|\frac{g_1^2}{[g_1^2,g_2g_3]}, \tag{4}$$

$$\frac{g_1}{[g_1,g_3]}\bigg|\frac{g_1g_2}{[g_1g_2,g_3^2]} \quad\text{or}\quad \frac{g_2}{[g_2,g_3]}\bigg|\frac{g_1g_2}{[g_1g_2,g_3^2]}. \tag{5}$$

Using (1) and the relative primeness (2), we conclude that the conditions (3) and (4) are also automatically satisfied. Thus, $\langle g_1,g_2,g_3\rangle$ is a weakly relative regular sequence if and only if the condition (5) holds. Using (1) and (2), we also conclude that (5) has the form

$$f_1\bigg|\frac{f_1}{[f_1,f_3^2f_{13}f_{23}]} \quad\text{or}\quad f_2\bigg|\frac{f_2}{[f_2,f_3^2f_{13}f_{23}]},$$

or, equivalently, at least one of the two pairs of monomials $(f_1,f_{13}),(f_2,f_{23})$ is relatively prime. Thus, we have shown that (i) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (ii): It is enough to show (i) $\Rightarrow$ (ii). Suppose that $\langle g_1,g_2,g_3\rangle$ a weakly relative regular sequence. By Corollary 2.3, a weakly relative regular monomial sequence $\langle g_1,g_2,g_3\rangle$ is a c-sequence if and only if the ideal I is of linear type. Since any monomial sequence of length 2 generates an ideal of linear type, we can use the Induction Theorem [1, Theorem 4]. So, by the Induction Theorem, $\langle g_1,g_2,g_3\rangle$ is a c-sequence if and only if the following conditions hold for every $k\geq 1$:

(a) $(g_1,g_2)(g_1,g_2,g_3)^k : g_3^{k+1}=(g_1,g_2):g_3$;

(b) for any $z_1,z_2\in(g_1,g_2,g_3)^k$ with $z_1g_1+z_2g_2=0$, there are $c_1,c_2\in(g_1,g_2,g_3)^{k-1}$ with $c_1g_1+c_2g_2=0$, such that $c_1g_3+z_1\in(g_1,g_2)^k$ and $c_2g_3+z_2\in(g_1,g_2)^k$.

We first show that the condition (a) holds. We can assume that g_1, g_2 are relatively prime, i.e., $f_{123} = 1$, since f_{123} is a nonzero divisor which is a factor in each of the three monomials. We can also assume, by symmetry and the fact that (i) $\Leftrightarrow$ (iii) is already proved, that the condition for weak relative regularity is $[g_1, g_3] = [g_1, g_3^2]$. Note that now for any $s \geq 1$, if $l + m = s, l \geq 1, m \geq 0$, then

$$
\begin{aligned}
[g_1^l g_2^m, g_3^s] &= [g_1^l, g_3^s][g_2^m, g_3^s] \\
&= [g_1^l, g_3^l][g_2^m, g_3^s] \\
&= [g_1, g_3]^l [g_2^m, g_3^s].
\end{aligned} \tag{6}
$$

Hence,

$$\frac{g_1^l g_2^m}{[g_1^l g_2^m, g_3^s]} = \frac{g_1^l}{[g_1, g_3]^l} \frac{g_2^m}{[g_2^m, g_3^s]}.$$

Hence,

$$\frac{g_1}{[g_1, g_3]} \Big| \frac{g_1^l g_2^m}{[g_1^l g_2^m, g_3^s]}. \tag{7}$$

Also, note that for any $s \geq 1$,

$$\frac{g_2}{[g_2, g_3]} \Big| \frac{g_2^s}{[g_2^s, g_3^s]}. \tag{8}$$

The ideal $(g_1, g_2)(g_1, g_2, g_3)^k$ is generated by the monomials $g_1^l g_2^m g_3^n$, where $l + m + n = k + 1, l + m \geq 1, l, m, n \geq 0$. We show that each of the monomials

$$\frac{g_1^l g_2^m g_3^n}{[g_1^l g_2^m g_3^n, g_3^{k+1}]} \tag{9}$$

is divisible by one of the monomials $\frac{g_1}{[g_1, g_3]}, \frac{g_2}{[g_2, g_3]}$. If $n = 0$, this follows from the above relations (7), (8) since either $l \geq 1$ or $l = 0, m = k + 1$. Suppose $n \geq 1$. Then the monomial (9) is equal to

$$\frac{g_1^l g_2^m}{[g_1^l g_2^m, g_3^{k+1-n}]}, \tag{10}$$

with $k + 1 - n \geq 1$, so we can again apply (7), (8). Thus, the condition (a) holds.

We now show that the condition (b) holds. By [4, p. 485], from $z_1 g_1 + z_2 g_2 = 0$ and $c_1 g_1 + c_2 g_2 = 0$ we have $z_1 = a g_2, z_2 = -a g_1, c_1 = b g_2, c_2 = -b g_1$ for some $a, b \in R$. Hence, the condition (b) can be formulated in the following way:

(b′) for every $k \geq 1$, for every $a \in (g_1, g_2, g_3)^k : (g_1, g_2)$, there is a $b \in (g_1, g_2, g_3)^{k-1} : (g_1, g_2)$ such that $a + b g_3 \in (g_1, g_2)^k : (g_1, g_2)$.

It is easy to see that $(g_1,g_2)^k:(g_1,g_2)=(g_1,g_2)^{k-1}$. We show that since $\langle g_1,g_2,g_3\rangle$ is weakly relative regular, we have

$$(g_1,g_2,g_3)^k:(g_1,g_2)=(g_1,g_2,g_3)^{k-1}.$$

This is clear for $k=1$. Suppose that $k\geq 2$. Then $(g_1,g_2,g_3)^k:g_1$ is generated by the monomials $\frac{g_1^lg_2^mg_3^n}{[g_1^lg_2^mg_3^n,g_1]}$ $(l+m+n=k)$, that are equal to:

$$g_1^{l-1}g_2^mg_3^n, \quad \text{if } l\geq 1; \tag{11}$$

$$g_2^m\cdot\frac{g_3^n}{[g_3^n,g_1]}, \quad \text{if } l=0, m\geq 1; \tag{12}$$

$$\frac{g_3^k}{[g_3^k,g_1]}, \quad \text{if } l=m=0. \tag{13}$$

Similarly, $(g_1,g_2,g_3)^k:g_2$ is generated by the monomials $\frac{g_1^lg_2^mg_3^n}{[g_1^lg_2^mg_3^n,g_2]}$ $(l+m+n=k)$, that are equal to:

$$\begin{aligned} &g_1^lg_2^{m-1}g_3^n, && \text{if } m\geq 1;\\ &g_1^l\cdot\frac{g_3^n}{[g_3^n,g_2]}, && \text{if } m=0, l\geq 1;\\ &\frac{g_3^k}{[g_3^k,g_2]}, && \text{if } l=m=0. \end{aligned} \tag{14}$$

In each of (11) and (14) we have all of the monomials $g_1^pg_2^qg_3^r$ with $p+q+r=k-1$, so each of the ideals $(g_1,g_2,g_3)^k:g_1$ and $(g_1,g_2,g_3)^k:g_2$ contains $(g_1,g_2,g_3)^{k-1}$. Since the sequence $\langle g_1,g_2,g_3\rangle$ is weakly relative regular, then for all $k\geq 2$ either $\frac{g_3^k}{[g_3^k,g_1]}=\frac{g_3^k}{[g_3,g_1]}$ or $\frac{g_3^k}{[g_3^k,g_2]}=\frac{g_3^k}{[g_3,g_2]}$. Without loss of generality, we assume $\frac{g_3^k}{[g_3^k,g_1]}=\frac{g_3^k}{[g_3,g_1]}$, then note that $g_3^{k-1}|\frac{g_3^k}{[g_3^k,g_1]}$ for all $k\geq 2$. It follows that (12) and (13) are elements of $(g_1,g_2,g_3)^{k-1}$, and hence that $(g_1,g_2,g_3)^k:g_1=(g_1,g_2,g_3)^{k-1}$. We conclude that

$$(g_1,g_2,g_3)^k:(g_1,g_2)=\big((g_1,g_2,g_3)^k:g_1\big)\cap\big((g_1,g_2,g_3)^k:g_2\big)$$

equals $(g_1,g_2,g_3)^{k-1}$ as required.

Now, the condition (b′) is equivalent with:

(b″) for every $k\geq 2$, for every $a\in(g_1,g_2,g_3)^{k-1}$ there is a $b\in(g_1,g_2,g_3)^{k-2}$ such that $a+bg_3\in(g_1,g_2)^{k-1}$.

Any $a\in(g_1,g_2,g_3)^{k-1}$ can be written as $a=F(g_1,g_2)+g_3^lG(g_1,g_2,g_3)$, $l\geq 1$, where F and G are homogeneous polynomials of degrees $k-1$ and $k-l-1$. Now if we put $b=-g_3^{l-1}G(g_1,g_2,g_3)$, we will have $a+bg_3\in(g_1,g_2)^{k-1}$. So the condition (b) holds.

Thus, (i) ⇔ (ii).

This finishes the proof of the theorem. □

Example 3.2. Consider the sequence $\langle x, y^2, yz\rangle$, where x, y, z are variables. It is not a d-sequence (by [11, Theorem 2.1]), but it is a c-sequence (by Theorem 3.1). However, if we change the order, the sequence $\langle x, yz, y^2\rangle$ is a d-sequence.

Example 3.3. Consider the sequence $\langle xy, z^2u, yzu^2\rangle$, where x, y, z, u are variables. As before, it is a c-sequence, but not a d-sequence. Moreover, it is not a d-sequence in any order.

Remark 3.4. A monomial sequence $\langle g_1, g_2, g_3\rangle$ is a regular sequence if and only if g_i and g_j are relatively prime whenever $i \neq j$. It is then an unconditioned regular sequence.

Corollary 3.5. *Let $\langle g_1, g_2, g_3\rangle$ be a monomial sequence. The following are equivalent*:

(i) $\langle g_1, g_2, g_3\rangle$ *is an unconditioned c-sequence*;
(ii) $\langle g_1, g_2, g_3\rangle$ *is an unconditioned proper sequence*;
(iii) $g_i = df_i$ $(i = 1, 2, 3)$, *where d is a monomial and $\langle f_1, f_2, f_3\rangle$ is a regular monomial sequence.*

Proof. (ii) ⇒ (iii): From (1) and the assumption that $\langle g_1, g_3, g_2\rangle$ is proper, i.e., that $[g_1, g_3]|g_2$, we get $f_{13} = 1$. Similarly, $f_{23} = 1$.
(iii) ⇒ (i) and (i) ⇒ (ii): clear. □

Remark 3.6. By [11, Corollary 3.3], a monomial sequence $\langle g_1, g_2, g_3\rangle$ is an unconditioned d-sequence if and only if $g_i = df_i$ $(i = 1, 2, 3)$, where d is a monomial, $\langle f_1, f_2, f_3\rangle$ a regular monomial sequence and d relatively prime with f_1, f_2, f_3. Hence, an unconditioned c-sequence is not necessarily a d-sequence.

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