Illinois Journal of Mathematics
Volume 52, Number 4, Winter 2008, Pages 1123–1145
S 0019-2082

A CLASS OF SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ ASSOCIATED TO HARMONIC FUNCTIONS AND A RELATION BETWEEN CMC-1/2 AND FLAT SURFACES

WALTERSON FERREIRA AND PEDRO ROITMAN

Abstract. We introduce a geometric motivated method to construct immersions into $\mathbb{H}^2 \times \mathbb{R}$ from a smooth function φ defined on an open set of the unit disc, and study the relation between the geometry of the immersion in terms of partial differential equations for φ. We give two applications of the method. First, we introduce the class of surfaces generated by harmonic functions and show that they have properties analogous to minimal surfaces in $\mathbb{R}^3$. We also exhibit an explicit local relation between CMC 1/2 and flat surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Introduction

In the past few years, there has been a considerable amount of work on surfaces in the so-called Thurston's model geometries, see for instance [2], [3], [5], [10], [11], and references therein. The present work is devoted to some aspects of surface theory in one such geometry, namely, the three dimensional Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$. That is, the product space (with the product metric) where the factors are $\mathbb{H}^2$, the hyperbolic plane, and the real line $\mathbb{R}$.

Our work is based on a natural geometric question involving a $\mathbb{H}^2$ valued map introduced in [5] for oriented surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for which the projection onto the first factor is not singular. We shall call this map the Fernandez–Mira map, or FM map for short.

Fernandez and Mira showed, among other things, that the FM map is harmonic for surfaces with constant mean curvature equal to 1/2, and devised a method to construct such surfaces starting with a harmonic map. Recently [6], they used it to solve the Bernstein problem for minimal surfaces in the

Received September 12, 2007; received in final form April 29, 2008.
Partially supported by CNPq/PADCT and CAPES/PROCAD
2000 *Mathematics Subject Classification.* 53A10.

Heisenberg space. Thus, the FM map seems to be an important notion to understand the geometry of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and in Heisenberg space.

The FM map resembles the hyperbolic Gauss map for surfaces in hyperbolic three space, $\mathbb{H}^3$. As in the case of surfaces in $\mathbb{H}^3$, the choice of an orientation for an immersed surface turns out to be important in the definition of the FM map. In other words, different orientations induce different maps that are not trivially related.

Thus, for an oriented immersed surface S, we have two $\mathbb{H}^2$ valued maps, denoted here by G_+ and G_-, see Section 1 for the definition.

The starting point of our work was the following basic question.

Let $\Omega \subset \mathbb{H}^2$ be an open set. How does one characterize and construct examples of smooth maps $w : \Omega \to \mathbb{H}^2$ such that there exists an oriented surface S in $\mathbb{H}^2 \times \mathbb{R}$ with the following property: $\forall z \in \Omega,\ \forall p \in S,\ z = G_-(p)$ and $w(z) = G_+(p)$?

In other words, we ask what maps w can be viewed as a composition of the FM maps: $w = G_+ \circ G_-^{-1}$. To our knowledge, this type of question was first considered, in the context of hyperbolic geometry for hyperbolic Gauss maps, by Bianchi. He showed, for instance, that flat surfaces are characterized by the composition of Gauss maps being holomorphic [4]. The authors have studied this composition of Gauss maps in detail for hypersurfaces in n-dimensional hyperbolic space [7]. This previous experience motivated the present work.

We show how the local answer to this question furnishes a new geometric method to generate immersions into $\mathbb{H}^2 \times \mathbb{R}$ starting with a smooth real valued function $\varphi : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{H}^2$ is an open set. We call this method to generate immersions the FM-composition method.

We present two applications of FM-composition. First, we show that if the function φ is harmonic, then the associated immersion satisfies the following relation

$$2H + \nu^2 - 2K_- + 1 = 0, \tag{1}$$

where H is the mean curvature (for a chosen orientation), ν^2 is minus the sectional curvature with respect to the vectors that span the tangent plane and K_- is roughly the ratio between the area elements of the surface and of $\mathbb{H}^2$ (at corresponding points via the pullback under G_-). For a definition of K_-, see Section 1.

The class of surfaces satisfying (1) has some interesting properties, such as variational characterization, associated family and a Schwarz's reflection principle, that reminds us of minimal surfaces in $\mathbb{R}^3$. In particular, this class of surfaces is relevant in the study of Bonnet families in $\mathbb{H}^2 \times \mathbb{R}$, as a nontrivial example of surfaces admitting a smooth 1-parameter isometric deformation such that the mean curvature is preserved along the deformation. We shall discuss the above properties in detail and present some examples.

As a second application, we present an explicit local relation between constant mean curvature equal to 1/2 and flat surfaces in $\mathbb{H}^2 \times \mathbb{R}$. This relation is roughly the following. If we consider a smooth real valued function $\varphi : \Omega \to \mathbb{R}$ as before, then we may use it to generate a surface by FM-composition. But it is also natural to consider the graph of φ. It turns out that the PDE for φ associated to $H = -1/2$ for surfaces generated by FM-composition is the same PDE we get for flat surfaces generated as graphs. We remark that the sign of the mean curvature is not essential in the above discussion. An argument to show this is the one used in [5] to justify that the choice of sign of the mean curvature. Essentially, they argue that if the mean curvature is $-1/2$ for their canonical choice of orientation, then by "reflection" with respect to a horizontal hyperbolic plane (an orientation reversing ambient isometry) the reflected surface has mean curvature 1/2 with respect to the canonical orientation.

This work is organized as follows. In Section 1, we establish notation and recall the definition of FM maps, we also briefly discuss the invariants associated with these maps.

In Section 2, we give the answer to the question posed above, which allows us to consider the method to generate immersions that is discussed in Section 3. In Section 4, we exhibit expressions for the fundamental forms of the immersions generated by FM-composition. Finally, Section 5 is devoted to the two applications mentioned above.

1. Preliminaries

1.1. Notation and FM maps. In this work, we shall use two models of $\mathbb{H}^2 \times \mathbb{R}$. The first model identifies $\mathbb{H}^2 \times \mathbb{R}$ with a submanifold of Lorentzian space $\mathbb{L}^4$. More precisely, we consider $\mathbb{L}^4$, that is, $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, endowed with the quadratic form $\langle \mathbf{x}, \mathbf{x} \rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2$, and $\mathbb{H}^2 \times \mathbb{R} = \{\mathbf{x} \in \mathbb{L}^4 \mid -x_0^2 + x_1^2 + x_2^2 = -1,\ x_0 > 0\}$. The second model, which we call the cylinder model, identifies $\mathbb{H}^2 \times \mathbb{R}$ with the subset of $\mathbb{R}^3$, given by $\mathbb{D} \times \mathbb{R}$, where $\mathbb{D}$ is the unit disc centered at the origin in $\mathbb{R}^2$ with coordinates (y_1, y_2), endowed with the metric $ds^2 = \frac{4}{(1-y_1^2-y_2^2)^2}(dy_1^2 + dy_2^2) + dy_3^2$.

Let $\psi : M \to \mathbb{H}^2 \times \mathbb{R} \subset L^4$ be an immersion of an abstract smooth surface M into $\mathbb{H}^2 \times \mathbb{R}$. In the Lorentzian model, we write $\psi(p) = (N(p), h(p))$, with $N(p) \in \mathbb{H}^2 = \{\mathbf{x} \in \mathbb{L}^4 \mid -x_0^2 + x_1^2 + x_2^2 = -1,\ x_0 > 0,\ x_3 = 0\}$, and $h(p) \in \mathbb{R}$.

We also consider $\eta : M \to S_1^3$, where $S_1^3 = \{\mathbf{x} \in \mathbb{L}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$, such that $\langle \eta, \eta \rangle = 1$, $\langle \eta, N \rangle = 0$ and $\langle d\psi, \eta \rangle = 0$, and write $\eta = (\hat{N}, \nu)$, where $\nu \in \mathbb{R}$. In this way, N and η span the normal bundle of ψ.

In all that follows, we shall assume that the immersion ψ is nonvertical, meaning that $\nu \neq 0$.

Without loss of generality, we shall consider $\nu > 0$ and ψ with the orientation given by the unit normal field η.

Now, we recall the definition of the FM map, introduced in [5]. Consider first

$$\xi_\pm = \frac{1}{\nu}(N \pm \eta).$$

It is easy to check that $\langle \xi_\pm, \xi_\pm \rangle = 0$. We define the positive and negative FM maps, denoted respectively by G_+ and G_-, by the following

$$\xi_\pm = (G_\pm, \pm 1).$$

In this way, $G_\pm$ are both $\mathbb{H}^2$ valued maps.

We shall call G_+ the positive FM map and G_- the negative FM map of the immersion ψ.

We note that the isometry group of $\mathbb{H}^2 \times \mathbb{R}$ is the product of the isometry group of $\mathbb{H}^2$ with $\mathbb{R}$. Thus, an isometry of $\mathbb{H}^2$ can also be considered as an ambient isometry of $\mathbb{H}^2 \times \mathbb{R}$ in a natural way. We shall adopt this convention freely without any further comment.

Finally, a few comments about notation. Throughout this work, we shall use (x, y) coordinates in the unit disc $\mathbb{D}$. The symbols ∇, $|\cdot|$ denote, respectively, the Euclidean gradient and norm. And $\nabla_{\mathbb{H}}$ is defined by $\nabla_{\mathbb{H}} = \frac{\Lambda}{2}\nabla$, where $\Lambda = 1 - x^2 - y^2$.

1.2. Invariants associated with the FM maps. We define below two invariants that are naturally induced by the FM maps.

Definition 1.1. Let $\psi : M \to \mathbb{H}^2 \times R$ be a nonvertical immersion. Let $dA_{\mathbb{H}^2}$ be the canonical area element of $\mathbb{H}^2$ and dA_M be the area element of M in the orientation defined by the normal field $\eta = (\hat{N}, \nu)$, with $\nu > 0$. Let $G_\pm : M \to \mathbb{H}^2$ be the FM maps of M. We define $K_\pm$ by the relation

$$G_\pm^*(dA_{\mathbb{H}^2}) = K_\pm dA_M.$$

We mention two facts to justify the importance of $K_\pm$ as relevant geometric quantities. First, we note that they're related to the extrinsic curvature K_{ext} of ψ by the simple relation

$$K_{\text{ext}} = \nu^2 \left(\frac{K_+ + K_-}{2} - 1 \right).$$

Second, the critical points of the functional

$$\int_M H \, dA_M,$$

for smooth variations with compact support are characterized by

$$6H = K_+ - K_-.$$

For proofs, see [8].

2. A natural question

We shall now discuss the question mentioned in the Introduction.

Let $\Omega \subset \mathbb{H}^2$ be an open set. How does one characterize and construct examples of smooth maps $w : \Omega \to \mathbb{H}^2$ such that there exists an oriented surface S in $\mathbb{H}^2 \times \mathbb{R}$ with the following property: $\forall z \in \Omega$, $\forall p \in S$, $z = G_-(p)$ and $w(z) = G_+(p)$?

Our goal is to answer this question and then try to relate analytical properties of the map w with geometric features of the corresponding surface S. The simple lemma that follows shows what has to be done.

LEMMA 2.1. *Let $\psi : M \to \mathbb{H}^2 \times \mathbb{R}$ be an immersion given by $\psi = (N, h)$, and $\eta : M \to S^3_1$ be such that $\eta = (\hat{N}, \nu)$, $\nu > 0$, satisfies $\langle \eta, \eta \rangle = 1$, $\langle \eta, N \rangle = 0$, and $\langle d\psi, \eta \rangle = 0$. Let $G_\pm : M \to \mathbb{H}^2$ be defined by the equation*

$$\frac{1}{\nu}(N \pm \eta) = (G_\pm, \pm 1). \tag{2}$$

Then

$$N = \left(\frac{\nu}{2}(G_+ + G_-), 0\right),$$
$$\eta = \frac{\nu}{2}\big((G_+ - G_-), 2\big),$$

and

$$\nu^2 = \frac{2}{1 - \langle G_+, G_- \rangle}.$$

Moreover, the function $h : M \to \mathbb{R}$ satisfies

$$dh = -\frac{1}{\nu}\langle dN, \hat{N} \rangle. \tag{3}$$

Proof. Straightforward computation. □

From Lemma 2.1, we see that if one starts with a map $w : \Omega \subset \mathbb{H}^2 \to \mathbb{H}^2$, and one supposes that there exists a surface S in $\mathbb{H}^2 \times \mathbb{R}$ such that $\forall z \in \Omega$ $z = G_-(p)$, and $w(z) = G_+(p)$, then N, the horizontal part of the immersion, is determined algebraically and the height function h satisfies (3). Thus, if one wishes to construct a surface with the above properties, a necessary condition is that the system (3) is integrable in the classical Frobenius sense.

To simplify things, and understand the integrability condition of (3), we shall consider $z = x + iy$ as a complex variable in the Poincaré disc model of $\mathbb{H}^2$ and a map $z \to w(z) \in \mathbb{D}$. The variable z in the disc model will represent G_- and $w(z)$ will represent G_+.

From the standard change of models (from the disc model to the hyperboloid model), we write

$$G_-(z) = \left(\frac{1+|z|^2}{\Lambda}, \frac{2\operatorname{Re}(z)}{\Lambda}, \frac{2\operatorname{Im}(z)}{\Lambda}\right),$$
$$G_+(z) = \left(\frac{1+|w|^2}{\Delta}, \frac{2\operatorname{Re}(w)}{\Delta}, \frac{2\operatorname{Im}(w)}{\Delta}\right),$$

where

$$\Delta = 1-|w|^2,$$
$$\Lambda = 1-|z|^2.$$

Now, in local coordinates $z = x+iy$, (3) can be written in terms of $G_\pm$ as:

$$h_x = \frac{\nu}{4}(\langle G_{+,x}, G_-\rangle - \langle G_{-,x}, G_+\rangle),$$
$$h_y = \frac{\nu}{4}(\langle G_{+,y}, G_-\rangle - \langle G_{-,y}, G_+\rangle).$$

The proposition below shows how w looks like locally.

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{D}$ be an open set and $\varphi : \Omega \to \mathbb{R}$ be a smooth function. Let*

$$\zeta = \frac{\nabla_{\mathbb{H}}\varphi}{\sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2}}, \tag{4}$$

where $\nabla_{\mathbb{H}} = \frac{\Lambda}{2}\nabla$ and ∇, $|\cdot|$ denote, respectively, the Euclidean gradient and norm. Let

$$w(z) = \frac{z+\zeta(z)}{1+\bar{z}\zeta(z)}. \tag{5}$$

Then if we define

$$G_-(z) = \left(\frac{1+|z|^2}{\Lambda}, \frac{2\operatorname{Re}(z)}{\Lambda}, \frac{2\operatorname{Im}(z)}{\Lambda}\right),$$
$$G_+(z) = \left(\frac{1+|w|^2}{\Delta}, \frac{2\operatorname{Re}(w)}{\Delta}, \frac{2\operatorname{Im}(w)}{\Delta}\right),$$

and $\nu > 0$, such that

$$\nu^2 = \frac{2}{1-\langle G_+, G_-\rangle}, \tag{6}$$

the system (3) is integrable.

Conversely, if (3) is integrable, there exists a smooth function φ defined on a simply-connected open set such that the above relations are verified. In terms of the function φ, the solution of (3) is given by

$$h = -\sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2} - \varphi + c, \tag{7}$$

where $c \in \mathbb{R}$.

Proof. It is a long computation, we will indicate the main steps. The first derivatives of $G_\pm$ are given by

$$G_{-,x} = \frac{2}{\Lambda^2}\big(2\,\mathrm{Re}(z), \mathrm{Re}(z^2+1), \mathrm{Re}[i(1-z^2)]\big), \tag{8}$$

$$G_{-,y} = \frac{2}{\Lambda^2}\big(2\,\mathrm{Im}(z), \mathrm{Im}(z^2+1), \mathrm{Im}[i(1-z^2)]\big), \tag{9}$$

$$G_{+,x} = \frac{2}{\Delta^2}\big(2\,\mathrm{Re}(w\bar{w}_x), \mathrm{Re}\big((1+w^2)\bar{w}_x\big), \mathrm{Re}\big([i(1-w^2)]\bar{w}_x\big)\big), \tag{10}$$

$$G_{+,y} = \frac{2}{\Delta^2}\big(2\,\mathrm{Re}(w\bar{w}_y), \mathrm{Re}\big((1+w^2)\bar{w}_y\big), \mathrm{Re}\big([i(1-w^2)]\bar{w}_y\big)\big). \tag{11}$$

Now, we list some inner products

$$\begin{aligned}
\langle G_{+,x}, G_-\rangle &= \frac{4}{\Lambda\Delta^2}\,\mathrm{Re}\big(\bar{w}_x[(z-w)(1-\bar{z}w)]\big),\\
\langle G_{+,y}, G_-\rangle &= \frac{4}{\Lambda\Delta^2}\,\mathrm{Re}\big(\bar{w}_y[(z-w)(1-\bar{z}w)]\big),\\
\langle G_{-,x}, G_+\rangle &= \frac{4}{\Delta\Lambda^2}\,\mathrm{Re}[(w-z)(1-z\bar{w})],\\
\langle G_{-,y}, G_+\rangle &= \frac{4}{\Delta\Lambda^2}\,\mathrm{Im}[(w-z)(1-z\bar{w})],\\
\langle G_+, G_-\rangle &= 1 - \frac{2}{\Lambda\Delta}|1-\bar{z}w|^2.
\end{aligned}$$

And using (6), we obtain

$$\nu = \frac{\Lambda^{1/2}\Delta^{1/2}}{|1-\bar{z}w|}.$$

To abbreviate, we define

$$\begin{aligned}
A &= 1-\bar{z}w,\\
B &= w-z.
\end{aligned}$$

Thus, we may rewrite (3) as

$$\begin{aligned}
h_x &= -\frac{\nu}{\Lambda^2\Delta^2}\big(\Delta\,\mathrm{Re}(\bar{A}B) + \Lambda\,\mathrm{Re}(\bar{A}\bar{B}w_x)\big),\\
h_y &= -\frac{\nu}{\Lambda^2\Delta^2}\big(\Delta\,\mathrm{Im}(\bar{A}B) + \Lambda\,\mathrm{Re}(\bar{A}\bar{B}w_y)\big).
\end{aligned}$$

Now, suppose that

$$w(z) = \frac{z+\zeta(z)}{1+\bar{z}\zeta(z)},$$

where $\zeta(z) = u(z) + iv(z)$ is a complex valued function such that $|\zeta(z)| < 1$. This assures that $|w(z)| < 1$.

In terms of ζ, after some computations, we have, for instance

$$\begin{aligned}\operatorname{Re}(\bar{A}B) &= \frac{u\Lambda^2}{|1+\bar{z}\zeta|^2},\\ \Delta &= \frac{\Lambda(1-|\zeta|^2)}{|1+\bar{z}\zeta|^2},\\ A &= \frac{\Lambda}{1+\bar{z}\zeta},\\ \nu &= \sqrt{1-u^2-v^2}.\end{aligned}$$

If we write

$$h_x = T_1 + T_2,$$

where

$$T_1 = \frac{-\nu\operatorname{Re}(\bar{A}B)}{\Lambda^2\Delta},$$

and

$$T_2 = \frac{-\nu\operatorname{Re}(\bar{A}\bar{B}w_x)}{\Lambda\Delta^2},$$

after some manipulation we obtain

$$\begin{aligned}T_1 &= -\frac{u}{\Lambda(1-|\zeta|^2)^{1/2}},\\ T_2 &= -\frac{uu_x+vv_x}{(1-u^2-v^2)^{3/2}} - \frac{u}{(1-x^2-y^2)(1-u^2-v^2)^{1/2}}.\end{aligned}$$

Thus,

$$h_x = -\frac{\partial}{\partial x}[(1-u^2-v^2)^{-1/2}] - \frac{2u}{(1-x^2-y^2)(1-u^2-v^2)^{1/2}},$$

and, in the same way, we have

$$h_y = -\frac{\partial}{\partial y}[(1-u^2-v^2)^{-1/2}] - \frac{2v}{(1-x^2-y^2)(1-u^2-v^2)^{1/2}}.$$

Therefore, system (3) is integrable if and only if the one form

$$\omega = \frac{2}{\Lambda(1-|\zeta|^2)^{1/2}}(u\,dx + v\,dy),$$

is closed.

For a simply connected open set, we may write $\omega = d\varphi$, where φ is an arbitrary smooth function.

The expression for ζ in terms of $\nabla_{\mathbb{H}}\varphi$ is simply:

$$\zeta = \frac{\nabla_{\mathbb{H}}\varphi}{\sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2}}.$$

To check that φ can really be an arbitrary smooth function, note that $|\zeta|$ is given by

$$|\zeta| = \frac{|\nabla_{\mathbb{H}}\varphi|}{\sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2}} < 1,$$

so that $|w| < 1$ and everything makes sense.

It is then easy to check that h is given in terms of φ by the following expression

$$h = -\sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2} - \varphi + c,$$

where c is an arbitrary real constant. □

3. FM composition

The result in the previous section suggests a method to construct maps into $\mathbb{H}^2 \times \mathbb{R}$ starting from a smooth function $\varphi : \Omega \subset \mathbb{D} \to \mathbb{R}$ defined on an open set. More precisely, from φ we define ζ and w using expressions (4), (5), and, from these, we define the horizontal part of the map. The vertical part, or height function, is defined in terms of φ by (7). Of course, the interesting case is when the above cited map is actually an immersion.

To abbreviate, we shall call the above procedure the FM-composition method.

For future reference, we list below some expressions relative to a surface generated by FM-composition.

A straightforward computation shows that the expression of the immersion is given, in the cylinder model, by

$$\begin{aligned} y_1(z) &= \frac{\Lambda((1-y^2+x^2)\frac{\varphi_x}{2} + xy\varphi_y) + 2x\Sigma}{\Lambda + \Sigma(x^2+y^2) + \Lambda(x\varphi_x + y\varphi_y) + \Sigma}, \\ y_2(z) &= \frac{\Lambda((1-x^2+y^2)\frac{\varphi_y}{2} + xy\varphi_x) + 2y\Sigma}{\Lambda + \Sigma(x^2+y^2) + \Lambda(x\varphi_x + y\varphi_y) + \Sigma}, \\ y_3(z) &= -\Sigma - \varphi, \end{aligned} \tag{12}$$

where $\Sigma = \sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2}$.

The coordinates of the horizontal part of the immersion, in the Lorentzian model, are

$$\begin{aligned} N_0 &= \Lambda^{-1}\big(\Sigma(1+x^2+y^2) + \Lambda(x\varphi_x + y\varphi_y)\big), \\ N_1 &= \Lambda^{-1}\left(\frac{\Lambda}{2}\big((x^2-y^2+1)\varphi_x + 2xy\varphi_y\big) + 2x\Sigma\right), \\ N_2 &= \Lambda^{-1}\left(\frac{\Lambda}{2}\big((y^2-x^2+1)\varphi_y + 2xy\varphi_x\big) + 2y\Sigma\right). \end{aligned} \tag{13}$$

Finally, the coordinates of the horizontal part, in the Lorentzian model, of the unit normal η are

$$\hat{N}_0 = \Sigma^{-1}\left(\frac{\Lambda}{4}(\varphi_x^2+\varphi_y^2)(x^2+y^2+1)+\Sigma(x\varphi_x+y\varphi_y)\right),\tag{14}$$
$$\hat{N}_1 = \Sigma^{-1}\left(x\frac{\Lambda}{2}(\varphi_x^2+\varphi_y^2)+\Sigma(x^2-y^2+1)\frac{\varphi_x}{2}+\Sigma xy\varphi_y\right),$$
$$\hat{N}_2 = \Sigma^{-1}\left(y\frac{\Lambda}{2}(\varphi_x^2+\varphi_y^2)+\Sigma(y^2-x^2+1)\frac{\varphi_y}{2}+\Sigma xy\varphi_x\right).$$

The proposition that follows shows that FM-composition has some geometric meaning, in the sense that it is well behaved with respect to the isometry group of $\mathbb{H}^2\times\mathbb{R}$.

Proposition 3.1. *Let $\Omega\subset\mathbb{D}$ be an open set and $\varphi:\Omega\subset\mathbb{D}\to\mathbb{R}$ be a smooth function and consider $\mathbf{X}_\varphi(z)=(y_1(z),y_2(z),y_3(z))$, given by (12). Let $M:\mathbb{H}^2\to\mathbb{H}^2$ be an isometry of $\mathbb{H}^2$ and let $\tilde{\varphi}=\varphi\circ M^{-1}$. Then $\mathbf{X}_{\tilde{\varphi}}=M(\mathbf{X}_\varphi)$.*

Proof. The verification that the third coordinate y_3 satisfies $\mathbf{X}_{\tilde{\varphi}}=M(\mathbf{X}_\varphi)$ is immediate. We shall indicate the verification for the first coordinate y_1, the second coordinate y_2 can be handled in the same manner.

By composition, it is sufficient to prove for a rotation

$$R_\theta(z)=e^{i\theta}z,$$

and a Mobius transformation given by

$$M_a(z)=\frac{z-a}{1-az},$$

where $a\in\mathbb{R}$ and $|a|<1$.

For $\tilde{z}=\tilde{x}+i\tilde{y}=R_\theta(z)$, we have

$$\tilde{x}=x\cos\theta-y\sin\theta,\tag{15}$$
$$\tilde{y}=x\sin\theta+y\cos\theta,$$

and

$$\tilde{\varphi}_{\tilde{x}}=\varphi_x\cos\theta-\varphi_y\sin\theta,\tag{16}$$
$$\tilde{\varphi}_{\tilde{y}}=\varphi_x\sin\theta+\varphi_y\cos\theta.$$

Let $\tilde{y}_1$ be the first coordinate of $\mathbf{X}_{\tilde{\varphi}}$ and Y_1 be the first coordinate of $R_\theta(\mathbf{X}_\varphi)$. We have

$$\tilde{y}_1=\frac{\tilde{\Lambda}((1-\tilde{y}^2+\tilde{x}^2)\frac{\tilde{\varphi}_{\tilde{x}}}{2}+\tilde{x}\tilde{y}\tilde{\varphi}_{\tilde{y}})+2\tilde{x}\tilde{\Sigma}}{\tilde{\Lambda}+\tilde{\Sigma}(\tilde{x}^2+\tilde{y}^2)+\tilde{\Lambda}(\tilde{x}\tilde{\varphi}_{\tilde{x}}+\tilde{y}\tilde{\varphi}_{\tilde{y}})+\tilde{\Sigma}},$$
$$Y_1=\left(\left(\Lambda\left((1-y^2+x^2)\frac{\varphi_x}{2}+xy\varphi_y\right)+2x\Sigma\right)\cos\theta\right.$$

$$- \left(\Lambda \left((1 - x^2 + y^2) \frac{\varphi_y}{2} + xy\varphi_x \right) + 2y\Sigma \right) \sin\theta \Bigg) \times \left(\Lambda + \Sigma(x^2 + y^2) + \Lambda(x\varphi_x + y\varphi_y) + \Sigma \right)^{-1}.$$

After some computation, using (15) and (16), one checks that $Y_1 = \tilde{y}_1$.

To verify the statement for M_a a long computation is necessary, we shall limit ourselves to indicate the main steps. It is convenient to represent M_a as a linear operator $M_a : L^3 \to L^3$. The matrix associated to M_a, for the standard basis of L^3, is given by

$$[M_a] = \begin{bmatrix} -\frac{a^2+1}{a^2-1} & \frac{2a}{a^2-1} & 0 \\ \frac{2a}{a^2-1} & -\frac{a^2+1}{a^2-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $N = (N_0, N_1, N_2)$ represent the horizontal part of the immersion $\mathbf{X}_\varphi$, and denote the horizontal part of $M_a(\mathbf{X}_\varphi)$ by $\breve{N} = M_a(N)$. The action of M_a on $\mathbf{X}_\varphi$, in the Lorentzian model, is linear and this fact simplifies the computation. For instance,

$$\breve{N}_0 = -\frac{a^2+1}{a^2-1} N_0 + \frac{2a}{a^2-1} N_1.$$

We have to compare $\breve{N}$ with $\tilde{N}$, the horizontal part of the immersion $\mathbf{X}_{\tilde{\varphi}}$.

Let $\tilde{z} = \tilde{x} + i\tilde{y} = M_a(z)$, then

$$\tilde{x} = \frac{(1 + a^2)x - a(1 + x^2 + y^2)}{(1 - ax)^2 + a^2y^2}, \qquad (17)$$

$$\tilde{y} = \frac{y(1 - a^2)}{(1 - ax)^2 + a^2y^2},$$

and, using the partial derivatives of the above expressions with respect to x and y, we obtain

$$\tilde{\varphi}_{\tilde{x}} = \frac{1}{1 - a^2} \left((1 - ax)^2 - a^2y^2 \right) \varphi_x + \frac{2ay}{1 - a^2} (ax - 1) \varphi_y, \qquad (18)$$

$$\tilde{\varphi}_{\tilde{y}} = \frac{-2ay}{1 - a^2} (ax - 1) \varphi_x + \frac{1}{1 - a^2} \left((1 - ax)^2 - a^2y^2 \right) \varphi_y.$$

Substitution of (17) and (18) into the expression for $\tilde{N}$ yields the desired result. □

4. The fundamental forms

To relate analytical properties of the potential function $\varphi : \Omega \to \mathbb{R}$ and the geometry of $\mathbf{X}_\varphi$, as in Proposition 3.1, we need to compute the first and second fundamental forms of $\mathbf{X}_\varphi$. It turns out that the expressions for the coefficients of these forms are rather complicated. However, due to Proposition 3.1, it will be sufficient for our purposes to suppose that $0 \in \Omega$ and to compute the value of the coefficients at 0.

In the sequel, we shall consider an open set $\Omega \subset \mathbb{D}$, a smooth map $w : \Omega \to \mathbb{D}$ given by $w(z) = \dfrac{z + \zeta(z)}{1 + \overline{z}\zeta(z)}$, where $\zeta : \Omega \to \mathbb{H}^2$ is a smooth map, and a map $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$\psi = \left(\frac{\nu}{2}(G_+ + G_-), h\right), \tag{19}$$

where

$$\nu^2 = \frac{2}{1 - \langle G_+, G_- \rangle}, \quad \nu > 0,$$
$$G_-(z) = \left(\frac{1 + |z|^2}{\Lambda}, \frac{2\,\mathrm{Re}(z)}{\Lambda}, \frac{2\,\mathrm{Im}(z)}{\Lambda}\right),$$
$$G_+(z) = \left(\frac{1 + |w|^2}{\Delta}, \frac{2\,\mathrm{Re}(w)}{\Delta}, \frac{2\,\mathrm{Im}(w)}{\Delta}\right),$$

and

$$\Delta = 1 - |w|^2,$$
$$\Lambda = 1 - |z|^2.$$

We will also suppose that the function h is a solution of the integrable system

$$h_x = -\frac{\partial}{\partial x}[(1 - |\zeta|^2)^{-1/2}] - \frac{2\,\mathrm{Re}(\zeta)}{\Lambda(1 - |\zeta|^2)^{1/2}}, \tag{20}$$

$$h_y = -\frac{\partial}{\partial y}[(1 - |\zeta|^2)^{-1/2}] - \frac{2\,\mathrm{Im}(\zeta)}{\Lambda(1 - |\zeta|^2)^{1/2}}. \tag{21}$$

When ψ is an immersion, we shall consider the unit normal field η, given by

$$\eta = \left(\frac{\nu}{2}(G_+ - G_-), \nu\right).$$

We shall write $\zeta = u + iv$ and note that $\nu = (1 - |\zeta|^2)^{1/2}$.

We also introduce the following notation: given two complex numbers z and w, we define $z \cdot w = \mathrm{Re}(\bar{z}w)$.

PROPOSITION 4.1. *Let $\Omega \subset \mathbb{D}$ be an open set and $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ an immersion given by* (19). *Then*

$$E = \langle \psi_x, \psi_x \rangle = \frac{1}{\nu^2}\left(\frac{\nu_x^2(\nu^2 + 1)}{\nu^2} - \frac{4u\nu_x}{\Lambda\nu} + \frac{|\Lambda\zeta_x + i\Lambda_y\zeta + 2|^2}{\Lambda^2}\right),$$
$$F = \langle \psi_x, \psi_y \rangle = \frac{1}{\nu^2}\left(\frac{\nu_x\nu_y(\nu^2 + 1)}{\nu^2} - \frac{2(u\nu_y + v\nu_x)}{\Lambda\nu} + \frac{(\Lambda\zeta_x + i\Lambda_y\zeta + 2) \cdot (\Lambda\zeta_y - i\Lambda_x\zeta + 2i)}{\Lambda^2}\right),$$

$$
\begin{aligned}
G &= \langle \psi_y, \psi_y \rangle = \frac{1}{\nu^2}\left(\frac{\nu_y^2(\nu^2+1)}{\nu^2} - \frac{4v\nu_y}{\Lambda\nu} + \frac{|\Lambda\zeta_y - i\Lambda_x\zeta + 2i|^2}{\Lambda^2}\right),\\
e &= -\langle \psi_x, \eta_x \rangle = -\frac{1}{\Lambda^2\nu^2}\big(\Lambda\nu_x(\Lambda\nu_x - 2u\nu) + |\Lambda\zeta_x + i\Lambda_y\zeta + 1 - \zeta^2|^2 - \nu^4\big),\\
f &= -\langle \psi_x, \eta_y \rangle = -\frac{1}{\Lambda^2\nu^2}\big(\Lambda\big(\Lambda\nu_x\nu_y - \nu(v\nu_x + u\nu_y)\big)\\
&\qquad + (\Lambda\zeta_x + i\Lambda_y\zeta + 1 - \zeta^2)\cdot\big(\Lambda\zeta_y - i\Lambda_x\zeta + i(1+\zeta^2)\big)\big),\\
g &= -\langle \psi_y, \eta_y \rangle = -\frac{1}{\Lambda^2\nu^2}\big(\Lambda\nu_y(\Lambda\nu_y - 2v\nu) + |\Lambda\zeta_y - i\Lambda_x\zeta + i(1+\zeta^2)|^2 - \nu^4\big).
\end{aligned}
$$

Proof. Let $N = \frac{\nu}{2}(G_+ + G_-)and\hat{N} = \frac{\nu}{2}(G_+ - G_-)$. Note that

$$\langle dN, dN \rangle = \frac{(d\nu)^2}{\nu^2} + \frac{\nu^2}{4}(\langle dG_+, dG_+\rangle + 2\langle dG_+, dG_-\rangle + \langle dG_-, dG_-\rangle), \tag{22}$$

$$\langle dN, d\hat{N} \rangle = \frac{\nu^2}{4}(\langle dG_+, dG_+\rangle - \langle dG_-, dG_-\rangle). \tag{23}$$

From (8)–(11), we obtain

$$\langle dG_-, dG_- \rangle = \frac{4}{\Lambda^2}(dx^2 + dy^2), \tag{24}$$

$$\langle dG_+, dG_+ \rangle = \frac{4}{\Delta^2}(|w_x|^2\,dx^2 + 2w_x\cdot w_y\,dx\,dy + |w_y|^2\,dy^2), \tag{25}$$

$$
\begin{aligned}
\langle dG_+, dG_- \rangle = \frac{4}{\Lambda^2\Delta^2}\big(w_x\cdot(A^2+B^2)\,dx^2 + \big(w_x\cdot i(A^2-B^2)\\
+ w_y\cdot(A^2+B^2)\big)\,dx\,dy + w_y\cdot i(A^2-B^2)\,dy^2\big),
\end{aligned} \tag{26}
$$

where $A = 1 - \overline{z}w$ and $B = w - z$

Substitution of (24)–(26) into (22), (23) (using (20) and (21)) completes the proof. ☐

Remark 4.2. The Christoffel symbols for the Poincaré disc model of $\mathbb{H}^2$ are given by

$$\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = -\frac{\Lambda_x}{\Lambda} \quad \text{and} \quad \Gamma_{22}^2 = \Gamma_{12}^1 = -\Gamma_{11}^2 = -\frac{\Lambda_y}{\Lambda}.$$

Let $\varphi : \Omega \subset \mathbb{D} \to \mathbb{R}$ be a smooth function, then the matrix for the Hessian of φ is given by $Hess_H(\varphi) = [h_{ij}]$, where

$$
\begin{aligned}
h_{11} &= \frac{\Lambda^2}{4}\left(\varphi_{xx} + \frac{\Lambda_x\varphi_x}{\Lambda} - \frac{\Lambda_y\varphi_y}{\Lambda}\right),\\
h_{12} &= \frac{\Lambda^2}{4}\left(\varphi_{xy} + \frac{\Lambda_y\varphi_x}{\Lambda} + \frac{\Lambda_x\varphi_y}{\Lambda}\right),\\
h_{22} &= \frac{\Lambda^2}{4}\left(\varphi_{yy} - \frac{\Lambda_x\varphi_x}{\Lambda} + \frac{\Lambda_y\varphi_y}{\Lambda}\right).
\end{aligned}
$$

PROPOSITION 4.3. *Let $\Omega \subset \mathbb{D}$ be an open set and $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ an immersion given by (19). Suppose that $\varphi : \Omega \to \mathbb{R}$ is a smooth function, such that $\nabla_{\mathbb{H}}\varphi = \dfrac{\zeta}{\nu}$. Then the matrix associated to the first fundamental form of ψ is given by*

$$I = \frac{4}{\Lambda^2}\big(Hess_H(\varphi) + \Sigma Id\big)^2,$$

where $Hess_H(\varphi)$ is the Hessian of φ with respect to the hyperbolic metric $ds^2 = \frac{4}{\Lambda^2}|dz|^2$, $\Sigma = \sqrt{1+|\nabla_{\mathbb{H}}\varphi|^2}$ and Id is the 2×2 identity matrix.

Proof. From $\zeta = u + iv = \nu\nabla_{\mathbb{H}}\varphi$ and $\nu = \dfrac{1}{\Sigma}$, we obtain

$$|\Lambda\zeta_x + i\Lambda_y\zeta + 2|^2 = \left(\Lambda\left(\nu\frac{\Lambda}{2}\varphi_x\right)_x - \nu\frac{\Lambda}{2}\Lambda_y\varphi_y + 2\right)^2 + \left(\Lambda\left(\nu\frac{\Lambda}{2}\varphi_y\right)_x + \nu\frac{\Lambda}{2}\Lambda_y\varphi_x\right)^2. \tag{27}$$

We also have

$$\nu_x = -\nu^2\Sigma_x, \tag{28}$$

$$\Sigma_x = \frac{\nu\Lambda}{4}\big(\Lambda_x|\nabla\varphi|^2 + \Lambda(\varphi_x\varphi_{xx} + \varphi_y\varphi_{xy})\big). \tag{29}$$

Substitution of (27), (28), and (29) into the expression for E given by Proposition 4.1 yields

$$E = \frac{\Lambda^2}{4}\left(\left(\varphi_{xy} + \frac{\Lambda_y\varphi_x}{\Lambda} + \frac{\Lambda_x\varphi_y}{\Lambda}\right)^2 + \left(\varphi_{xx} + \frac{\Lambda_x\varphi_x}{\Lambda} - \frac{\Lambda_y\varphi_y}{\Lambda} + \frac{4\Sigma}{\Lambda^2}\right)^2\right). \tag{30}$$

In the same manner,

$$F = \frac{\Lambda^2}{4}\left(\varphi_{xy} + \frac{\Lambda_y\varphi_x}{\Lambda} + \frac{\Lambda_x\varphi_y}{\Lambda}\right)\left(\varphi_{xx} + \varphi_{yy} + \frac{8\Sigma}{\Lambda^2}\right), \tag{31}$$

$$G = \frac{\Lambda^2}{4}\left(\left(\varphi_{xy} + \frac{\Lambda_y\varphi_x}{\Lambda} + \frac{\Lambda_x\varphi_y}{\Lambda}\right)^2 + \left(\varphi_{yy} - \frac{\Lambda_x\varphi_x}{\Lambda} + \frac{\Lambda_y\varphi_y}{\Lambda} + \frac{4\Sigma}{\Lambda^2}\right)^2\right). \tag{32}$$

Therefore, the proposition follows from (30), (31), and (32). □

Remark 4.4. In Proposition 4.3, we have assumed that ψ is an immersion. Suppose that we drop this assumption and define a map ψ in the following manner. Let $\Omega \subset \mathbb{D}$ be an open set and $\varphi : \Omega \to \mathbb{R}$ a smooth function and define ζ by the expression $\nabla_{\mathbb{H}}\varphi = \frac{\zeta}{\nu}$, where $\nu = (1-|\zeta|^2)^{1/2}$. Finally, we define $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ as the map given by (19). Then it follows from computations as in Proposition 4.3 that ψ, defined in the above manner, is an immersion at $z \in \Omega$ if and only if

$$det\big(Hess_H(\varphi) + \Sigma Id\big) = det\big((Hess_H(\varphi)) + \Sigma(\triangle_H\varphi + \Sigma)\big) \neq 0,$$

at z, where $\triangle_H$ denotes the Laplacian in the hyperbolic metric.

LEMMA 4.5. *Let $\Omega \subset \mathbb{D}$ be an open set such that $0 \in \Omega$, and $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ the map given by (19). Suppose that ψ is an immersion and that there is a smooth function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla_{\mathbb{H}}\varphi = \dfrac{\zeta}{\nu}$. Then the coefficients of the first and second fundamental forms at $z = 0$ are given by*

$$E = \frac{1}{4}\big((\varphi_{xx} + 4\Sigma)^2 + \varphi_{xy}^2\big),$$
$$F = \frac{1}{4}(\varphi_{xx} + \varphi_{yy} + 8\Sigma)\varphi_{xy},$$
$$G = \frac{1}{4}\big((\varphi_{yy} + 4\Sigma)^2 + \varphi_{xy}^2\big).$$

$$e = -\frac{1}{4\Sigma^2}\left(\frac{(\varphi_y\varphi_{xx} - \varphi_x\varphi_{xy})^2}{4} + \varphi_{xx}^2 + \varphi_{xy}^2\right) + \frac{1}{\Sigma}\left(\frac{\varphi_x\varphi_y\varphi_{xy}}{2} - \frac{\varphi_y^2\varphi_{xx}}{2} - \varphi_{xx}\right) - \varphi_y^2, \tag{33}$$

$$f = \frac{1}{4\Sigma^2}\left(\left(\frac{\varphi_{xy}^2 + \varphi_{xx}\varphi_{yy}}{4} + \Sigma(\varphi_{xx} + \varphi_{yy}) + 4\Sigma^2\right)\varphi_x\varphi_y - \left(\left(1 + \frac{\varphi_x^2}{4}\right)\varphi_{yy} + \left(1 + \frac{\varphi_y^2}{4}\right)\varphi_{xx} + 4\Sigma^3\right)\varphi_{xy}\right), \tag{34}$$

$$g = -\frac{1}{4\Sigma^2}\left(\frac{(\varphi_x\varphi_{yy} - \varphi_y\varphi_{xy})^2}{4} + \varphi_{yy}^2 + \varphi_{xy}^2\right) + \frac{1}{\Sigma}\left(\frac{\varphi_x\varphi_y\varphi_{xy}}{2} - \frac{\varphi_x^2\varphi_{yy}}{2} - \varphi_{yy}\right) - \varphi_x^2. \tag{35}$$

Proof. The coefficients of the first fundamental form are obtained by direct substitution. From Proposition 4.1, it follows that for $z = 0$ we have

$$\nu^2 e = -\nu_x(\nu_x - 2u\nu) + \nu^4 - \big((u_x + v^2 - u^2 + 1)^2 + (v_x - 2uv)^2\big). \tag{36}$$

Substitution of $u = \nu\frac{\varphi_x}{2}$, $v = \nu\frac{\varphi_y}{2}$ and

$$\nu_x = -\nu^2\Sigma_x, \quad \Sigma_x = \frac{\nu}{4}(\varphi_x\varphi_{xx} + \varphi_y\varphi_{yx}),$$

into (36) yields, after a long computation, that e is given by (33). In the same manner, one shows that f and g are given by (34) and (35). □

PROPOSITION 4.6. *Let $\Omega \subset \mathbb{D}$ be a connected open set and $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ the map given by (19). Suppose that ψ is an immersion and that there is a smooth function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla_{\mathbb{H}}\varphi = \frac{\zeta}{\nu}$. Then*

(a) *ψ has mean curvature $H = -\frac{1}{2}$ if and only if φ satisfies*

$$det(Hess_H(\varphi)) = \Sigma^2. \tag{37}$$

(b) ψ *satisfies the relation*

$$2H + \nu^2 - 2K_- + 1 = 0, \tag{38}$$

where $K_- = \frac{4\varepsilon}{\Lambda^2\sqrt{EG-F^2}}$, $\varepsilon = 1$ *if* $det(Hess_H(\varphi) + \Sigma Id) > 0$ *and* $\varepsilon = -1$ *if* $det(Hess_H(\varphi) + \Sigma Id) < 0$, *if only if* φ *is a harmonic function.*

Proof. Let $z \in \Omega$. Note that (37) and harmonicity are invariants by composition with Mobius automorphisms of the disc. Thus, by Proposition 3.1, without loss of generality, we may suppose that $z = 0$. From Lemma 4.5, at $z = 0$, we have

$$EG - F^2 = \left(\frac{\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2}{4} + \Sigma(\varphi_{xx} + \varphi_{yy}) + 4\Sigma^2\right)^2, \tag{39}$$

$$\begin{aligned} 2H + 1 &= \frac{eG - 2fF + gE + EG - F^2}{EG - F^2} \\ &= \frac{4\Sigma^2 - 4det(Hess_H(\varphi))}{\varepsilon\Sigma^2\sqrt{EG - F^2}}. \end{aligned}$$

To prove (b), we note that it follows from Lemma 4.5 that

$$det(Hess_H(\varphi)) = \frac{\varepsilon\sqrt{EG - F^2}}{4} - \Sigma\triangle_H\varphi - \Sigma^2. \tag{40}$$

Thus, combining (39) and (40) we finish the proof. □

5. Applications

5.1. A class of surfaces associated to harmonic functions. As a first application of the results in the previous section, we introduce and study some aspects of a class of surfaces that are critical points of the following functional

$$\int_S K_- \tan^2\theta \, dA,$$

where S is a nonvertical surface with upward orientation, K_- as in Definition 1.1, and θ the angle between the unit normal η and the unit Killing vector field in the vertical direction.

It can be shown a surface is a critical point of the above functional, for smooth variations with compact support, if and only if (1) is satisfied, for a proof see [8].

Thus, by Proposition (4.6), we can use FM-composition to construct explicit examples of surfaces that satisfy (1) and to study their properties.

Since our goal here is to show an application of FM-composition, we shall limit ourselves in this subsection to consider surfaces that satisfy (1) and are generated by FM-composition. To be precise, we shall consider in this subsection immersions $\mathbf{X}_\varphi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$, generated by FM-composition, where Ω is an open set $\varphi : \Omega \to \mathbb{R}$ is a harmonic function.

The drawback of our method is that it can only be applied locally to surfaces for which G_- is an immersion. We believe, however, that this is just a technical difficulty, and that the results below are also valid without assuming that G_- is an immersion.

5.1.1. *Associated family.* The theorem below shows that for the class of surfaces we are considering there is a notion of associated family similar to the usual notion for minimal surfaces in $\mathbb{R}^3$.

THEOREM 5.1. *Let $\Omega \subset \mathbb{D}$ be a simply-connected open set and $\varphi : \Omega \to \mathbb{R}$ be a harmonic function. Suppose that $\mathbf{X}_\varphi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ is an immersion given by (12). Let φ^* be a harmonic function conjugate to φ and $\varphi_t = \cos(t)\varphi + \sin(t)\varphi^*$, $t \in \mathbb{R}$. Then the family $\mathbf{X}_{\varphi_t} : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ associated to φ_t, are immersions. Furthermore, the elements of this family are locally isometric and have the same mean curvature at corresponding points.*

Proof. Let $z \in \Omega$. We shall show that the statement is true at z. By Proposition 3.1, without loss of generality, we may suppose $z = 0$. At $z = 0$, the coefficients of the first fundamental form of ψ_t are, by Lemma 4.5, given by

$$
\begin{aligned}
E_t &= \frac{1}{4}\big((\varphi_{t,xx} + 4\Sigma)^2 + \varphi_{t,xy}^2\big),\\
F_t &= 2\Sigma\varphi_{t,xy},\\
G_t &= \frac{1}{4}\big((-\varphi_{t,xx} + 4\Sigma)^2 + \varphi_{t,xy}^2\big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E_tG_t - F_t^2 &= \left(\frac{\varphi_{t,xx}^2 + \varphi_{t,xy}^2}{4} - 4\Sigma^2\right)^2\\
&= \left(\frac{\varphi_{xx}^2 + \varphi_{xy}^2}{4} - 4\Sigma^2\right)^2\\
&= EG - F^2.
\end{aligned}
$$

Therefore, by Proposition 4.6, if H_t is the mean curvature of ψ_t, then $\mathbf{X}_{\varphi_t}$ satisfies

$$2H_t + \nu_t^2 - 2(K_-)_t + 1 = 0.$$

But, since $\nu_t = \nu$, $(K_-)_t = K_-$, it follows that H_t does not depend on t.

Now, we prove that the elements of the family are locally isometric. Suppose initially that $0 \in \Omega$ and consider an open disc D_0 centered at 0 and contained in Ω. By Proposition 4.3, the first fundamental forms of the family $\mathbf{X}_{\varphi_t}$ are given by

$$I_t = \frac{4}{\Lambda^2}\big(Hess_H(\varphi_t) + \Sigma_t Id\big)^2,$$

Since $\varphi_x = \varphi_y^*$ and $\varphi_y = -\varphi_x^*$, the Hessian matrix of φ is given by

$$Hess_H(\varphi) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where

$$a = \frac{\Lambda^2}{4}\left(\varphi_{xx} + \frac{\Lambda_x \varphi_x}{\Lambda} - \frac{\Lambda_y \varphi_y}{\Lambda}\right),$$
$$b = \frac{\Lambda^2}{4}\left(\varphi_{xy} + \frac{\Lambda_y \varphi_x}{\Lambda} + \frac{\Lambda_x \varphi_y}{\Lambda}\right).$$

It is easy to check that $\Sigma_t = \Sigma_0$ and

$$Hess_H(\varphi_t) = R_t Hess_H(\varphi)$$

where

$$R_t = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

On the other hand, by considering the change of variables defined on D_0 by

$$u = \cos(\theta)x - \sin(\theta)y,$$
$$v = \sin(\theta)x + \cos(\theta)y,$$

it follows easily that, $\hat{H}$, the Hessian of φ in the coordinates u and v, is given by

$$\hat{H} = R_{2\theta} Hess_H(\varphi).$$

Thus, for $\theta = t/2$ the immersions $\mathbf{X}_{\varphi_t}$ and $\mathbf{X}_{\varphi_0} \circ R_{\frac{-t}{2}}$ have the same coefficients of the first fundamental form defined on D_0. Therefore, we have an isometry between $\mathbf{X}_{\varphi_t}(D_0)$ and $\mathbf{X}_{\varphi_0}(D_0)$, for each $t \in [0, 2\pi]$.

Finally, let $z \in \Omega$, and consider an isometry of $\mathbb{H}^2$ $M : H^2 \to H^2$ such that $M(z) = 0$. Note that, by Proposition 3.1, the immersion $\mathbf{X}_{\hat{\varphi}_t}$ associated to $\hat{\varphi}_t = \varphi_t \circ M^{-1}$ is congruent to $\mathbf{X}_{\varphi_t}$. In this way, we reduce the general case to the one treated above, where $z = 0$. □

Remark 5.2. Theorem 5.1 shows that in $\mathbb{H}^2 \times \mathbb{R}$ there are non-trivial examples of a smooth one-parameter family of surfaces that are locally isometric and have the same mean curvature at corresponding points. This should be compared with the recent result in [9] on the Bonnet problem, where a classification of a one-parameter family of surfaces that are locally isometric and have the same principal curvatures at corresponding points was given.

Remark 5.3. For some choices of the harmonic function φ, the elements of the family ψ_t are congruent. But this is not true in general. For instance, if we consider $\varphi(z) = \mathrm{Re}(z^2)$, the elements of the associated family are congruent. On the other hand, elements of the associated family in the first example of Section 5.1.3 are not congruent.

5.1.2. *Schwarz's reflection principle.* We now show that a reflection principle, analogous to Schwarz's principle for minimal surfaces in Euclidean space, is valid for surfaces generated by harmonic functions via FM-composition.

Remark 5.4. Note that $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$, where Ω is an open set of $\mathbb{D}$, given by (19) is represented in the cylinder model of $\mathbb{H}^2 \times \mathbb{R}$ by

$$\psi = \left(\frac{N_1}{1+N_0}, \frac{N_2}{1+N_0}, h \right),$$

where $N = \frac{\nu}{2}(G_+ + G_-) = (N_0, N_1, N_2)$. In the same manner, η, the unit vector orthogonal to ψ in $\mathbb{H}^2 \times \mathbb{R}$, is given by

$$\eta = \left(\frac{\hat{N}_1(1+N_0) - N_1\hat{N}_0}{(1+N_0)^2}, \frac{\hat{N}_2(1+N_0) - N_2\hat{N}_0}{(1+N_0)^2}, \nu \right),$$

where $\hat{N} = \frac{\nu}{2}(G_+ - G_-) = (\hat{N}_0, \hat{N}_1, \hat{N}_2)$.

Proposition 5.5 (Schwarz reflection). *Let $\Omega \subset \mathbb{D}$ be an open set and $\varphi : \Omega \to R$ be a harmonic function. Let $\psi : \Omega \to H^2 \times R$ be an immersion, given by (19), associated to φ. Suppose that a vertical plane P intersects $\psi(\Omega)$ orthogonally. Then there exists an extension of $\psi(\Omega)$, say S, satisfying (1), and symmetric with respect to P.*

Proof. Consider the cylinder model $\mathbb{H}^2 \times \mathbb{R}$. Up to an ambient isometry we may suppose that $P = \{(y_1, y_2, y_3) \in \mathbb{H}^2 \times R; y_2 = 0\}$. Suppose further that Ω is symmetric with respect to the y_1 axis.

If we write $\eta = (\eta_1, \eta_2, \nu)$, and use (14), then η_2 is given by

$$\eta_2 = \frac{\hat{N}_2(1+N_0) - N_2\hat{N}_0}{(1+N_0)^2}. \tag{41}$$

Note also that if we write $\psi = (Y_1, Y_2, h)$, and use (13), the expressions for Y_1 and Y_2 are

$$Y_1 = \frac{1}{d}\left(\Lambda\left((x^2 - y^2 + 1)\frac{\varphi_x}{2} + xy\varphi_y \right) + 2x\Sigma \right), \tag{42}$$

$$Y_2 = \frac{1}{d}\left(\Lambda\left((y^2 - x^2 + 1)\frac{\varphi_y}{2} + xy\varphi_x \right) + 2y\Sigma \right), \tag{43}$$

where $d = \Sigma(1 + x^2 + y^2) + \Lambda(x\varphi_x + y\varphi_y + 1)$.

Now, along, $\psi(\Omega) \cap P$, we have $Y_2 = \eta_2 = 0$, and from (43), (41), and (14), we obtain

$$\begin{aligned} 0 &= \Sigma\left(\Lambda\left((y^2 - x^2 + 1)\frac{\varphi_y}{2} + xy\varphi_x \right) + 2y\Sigma \right) \\ &\quad - \Lambda\left(\Sigma(y^2 - x^2 + 1)\frac{\varphi_y}{2} + xy\Sigma\varphi_x + \frac{\Lambda y}{2}(\varphi_x^2 + \varphi_y^2) \right) \\ &= 2y. \end{aligned}$$

Substitution of $y = 0$ into (43) yields $\varphi_y(x, 0) = 0$ and, therefore, it follows from the classical Schwarz's reflection principle for harmonic functions that $\varphi(x, -y) = \varphi(x, y)$. From (42) and (43), we conclude that $\psi(x, -y) = (Y_1(x, y), -Y_2(x, y), h(x, y))$, and we have symmetry with respect to P.

Finally, if Ω is not symmetric with respect to the y_1 axis, we may reflect Ω in this axis and consider a new harmonic function defined on the union of Ω and it's reflection. This new harmonic function provides the extension S, having P as a symmetry plane. □

Remark 5.6. A stronger version of Schwarz's reflection can be proved by looking at the P.D.E. for a graph that satisfies (1). This P.D.E. is elliptic, [8], with analytic coefficients, so it's solutions are also analytic. Schwarz's principle follows from this fact and a symmetry of the equation with respect to reflection relative to a coordinate axis.

5.1.3. *Examples.* Figure 1 shows elements of the associated family generated by

$$\varphi_t = \cos(t) ln(\sqrt{x^2 + y^2}) + \sin(t) \arctan(y/x).$$

Figure 2 shows a piece of the surface associated to

$$\mathrm{Re}\left(\frac{1}{z^3} + z^3\right).$$

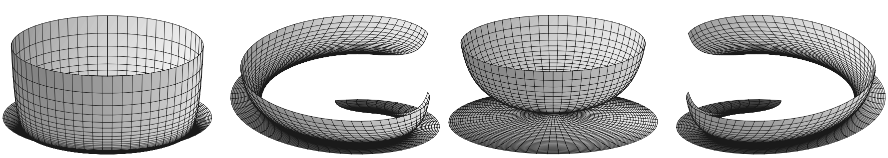

FIGURE 1. The sequence illustrates an example of an associated family.

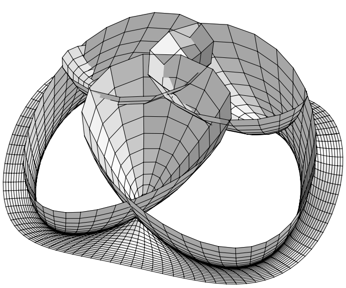

FIGURE 2. The surface associated to $\mathrm{Re}(\frac{1}{z^3} + z^3)$.

5.2. Relation between flat and CMC-1/2 surfaces. As we said in the Introduction, the map G_+ is harmonic for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with mean curvature $\frac{1}{2}$. It is also true that flat surfaces in $\mathbb{H}^2 \times \mathbb{R}$ have a harmonic horizontal projection, when viewed as Riemann surfaces with the conformal structure induced by the second fundamental form, [1]. In both cases, there is a harmonic $\mathbb{H}^2$ valued map, and one can imagine a relation between these two classes of surfaces. The FM-composition provides such relation in an explicit and simple way.

Let $\Omega \subset \mathbb{H}^2$ be an open set and consider a smooth function $\varphi : \Omega \to \mathbb{R}$. A natural way to generate a surface from φ is to consider it's graph: $\Gamma_\varphi = \{(z, \varphi(z)) \in \mathbb{H}^2 \times \mathbb{R} \mid z \in \Omega\}$. In this work, we have presented the FM-composition, which is another way to generate a surface starting with φ. It turns out that there are geometric relations between these surfaces.

Proposition 5.7. *Let $\Omega \subset \mathbb{D}$ be an open set and $\psi : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ an immersion given by (19). Suppose there is a smooth function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla_{\mathbb{H}}\varphi = \frac{\zeta}{\nu}$ and let $\Gamma_\varphi \subset \mathbb{H}^2 \times \mathbb{R}$ be the graph of φ. Then ψ has mean curvature $H = -1/2$ if and only if Γ_φ has zero intrinsic curvature.*

Proof. In the Lorentzian model, a natural parametrization for Γ_φ is given by

$$\Phi(x,y) = (N(x,y), \varphi(x,y)), \quad (x,y) \in \Omega,$$

where $N(x,y) = (\frac{1+x^2+y^2}{\Lambda}, \frac{2x}{\Lambda}, \frac{2y}{\Lambda})$, and $\Lambda(x,y) = 1 - x^2 - y^2$.

The unit vector field orthogonal to Φ is given by

$$\eta = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{H}}\varphi|^2}}\left(-\frac{\Lambda^2}{4}(\varphi_x N_x + \varphi_y N_y), 1\right),$$

where $\nabla_{\mathbb{H}}\varphi$ is the gradient of φ in the metric ds_H^2.

Note that

$$\langle N_x, N_x\rangle = \langle N_y, N_y\rangle = 4\Lambda^{-2} \quad \text{and} \quad \langle N_x, N_y\rangle = 0. \tag{44}$$

Thus, the coefficients of the first fundamental form of Φ are

$$E_\Phi = \frac{4}{\Lambda^2} + \varphi_x^2,$$
$$F_\Phi = \varphi_x \varphi_y,$$
$$G_\Phi = \frac{4}{\Lambda^2} + \varphi_y^2.$$

Using (44), we obtain the following expressions for the coefficients of the second fundamental form of Φ.

$$e_\Phi = \langle \eta, \Phi_{xx} \rangle = \frac{1}{\sqrt{1+|\nabla_H \varphi|^2}} \left(\varphi_{xx} + \frac{\Lambda_x \varphi_x}{\Lambda} - \frac{\Lambda_y \varphi_y}{\Lambda} \right),$$
$$f_\Phi = \langle \eta, \Phi_{xy} \rangle = \frac{1}{\sqrt{1+|\nabla_H \varphi|^2}} \left(\varphi_{xy} + \frac{\Lambda_y \varphi_x}{\Lambda} + \frac{\Lambda_x \varphi_y}{\Lambda} \right),$$
$$g_\Phi = \langle \eta, \Phi_{yy} \rangle = \frac{1}{\sqrt{1+|\nabla_H \varphi|^2}} \left(\varphi_{yy} - \frac{\Lambda_x \varphi_x}{\Lambda} + \frac{\Lambda_y \varphi_y}{\Lambda} \right).$$

Thus, the matrix of the second fundamental form of Φ is given by

$$II_\Phi = \frac{4}{\Lambda^2 \sqrt{1+|\nabla_H \varphi|^2}} Hess_H(\varphi).$$

The sectional curvature $K_{\rm sec}$ of $\mathbb{H}^2 \times \mathbb{R}$ with respect to the plane generated by Φ_x e Φ_y is given by

$$K_{\rm sec} = -\frac{1}{\Sigma^2} = -\frac{1}{1+|\nabla_H \hat{\varphi}|^2},$$

see [3]. We shall denote by $K_{\rm int}$ and $K_{\rm ext}$, respectively, the intrinsic and extrinsic curvatures of Φ. Gauss's equation is

$$K_{\rm int} + \frac{1}{\Sigma^2} = K_{\rm ext},$$

thus, $K_{\rm int} = 0$ if and only if $K_{\rm ext} = 1/\Sigma^2$.

Let $A = I_\Phi^{-1} II_\Phi$ be the shape operator of Φ. We have

$$K_{\rm ext} = det A = \frac{det(Hess_H(\varphi))}{\Sigma^4}$$

It follows that $K_{\rm ext} = 1/\Sigma^2$ if and only if $det(Hess_H(\varphi)) = \Sigma^2$. And, by Proposition 4.6, this happens if and only if ψ has constant mean curvature $H = -1/2$. □

Remark 5.8. Proposition 5.7 can be related to the comments in the beginning of this section in the following way. The domain Ω can naturally be viewed as a Riemann surface in two different ways. The first one by using the conformal structure induced by the first fundamental form of the immersion given by (19). The second one by using the conformal structure induced by the second fundamental form of Γ_φ. A simple computation shows that, under the conditions of Proposition 5.7, the two conformal structures actually coincide.

Thus, the identity map on Ω, which represents locally the map G_- in the first situation, and the horizontal projection in the second, is a harmonic map defined on Ω, viewed as a Riemann surface with the conformal structure discussed above. So one harmonic map defined on Ω, can be interpreted in two different manners: the map G_- for CMC $-1/2$ surfaces and the horizontal projection for flat surfaces.

Acknowledgment. The second author thanks Harold Rosenberg for inspiring conversations about surfaces in Thurston's geometries.

References

[1] J. A. Aledo, J. M. Espinar and J. A. Gálvez, *Complete surfaces of constant curvature in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Calc. Var. Partial Differential Equations **29** (2007), 347–363. MR 2321892

[2] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Acta Math. **193** (2004), 141–174. MR 2134864

[3] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007), 87–131. MR 2296059

[4] L. Bianchi, *Lezioni di Geometria Differenziale*, Terza Edizione, Nicola Zanichelli Editore, Bologna, 1927.

[5] I. Fernandez and P. Mira, *Harmonic maps and surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Amer. J. Math. **129** (2007), 1145–1181. MR 2343386

[6] I. Fernandez and P. Mira, *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space*, preprint, 2007.

[7] W. Ferreira and P. Roitman, *Hypersurfaces in hyperbolic space associated with the conformal scalar curvature equation* $\Delta u + ku^{\frac{n+2}{n-2}} = 0$, Differential Geom. Appl. **27** (2009), 279–295.

[8] W. Ferreira and P. Roitman, *Variations of geometric functionals in* $\mathbb{H}^2 \times \mathbb{R}$, work in progress.

[9] J. A. Gálvez, A. Martinez and P. Mira, *The Bonnet problem for surfaces in homogeneous 3-manifolds*, Comm. Anal. Geom. **16** (2008), 907–935.

[10] D. Hoffman, J. H. Lyra and H. Rosenberg, *Constant mean curvature in* $M^2 \times \mathbb{R}$, Trans. Amer. Math. Soc. **358** (2006), 491–507. MR 2177028

[11] W. H. Meeks and H. Rosenberg, *Stable minimal surfaces in* $M^2 \times \mathbb{R}$, J. Diff. Geometry **68** (2004), 515–535. MR 2144539

Walterson Ferreira, Departamento de Matematica, Universidade Federal de Goiás, 74001-970 Goiania GO, Brasil

E-mail address: walter@mat.ufg.br

Pedro Roitman, Departamento de Matematica, Universidade de Brasilia, 70910-900 Brasilia D.F., Brasil

E-mail address: roitman@mat.unb.br