Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 419–444
S 0019-2082

ON THE UNIQUENESS OF THE FINITE SIMPLE GROUPS WITH A GIVEN CENTRALIZER OF A 2-CENTRAL INVOLUTION

GERHARD O. MICHLER

In commemoration of my teacher Reinhold Baer

ABSTRACT. Let H be a finite group having center $Z(H)$ of even order. By the classical Brauer-Fowler theorem there can be only finitely many non isomorphic simple groups G which contain a 2-central involution t for which $C_G(t) \cong H$.

In this article we give several conditions which together suffice to prove that up to isomorphism there is a unique simple group G having a 2-central involution z with centralizer $C_G(z) \cong H$. Together they yield a practical uniqueness criterion (Theorem 2.1). This is demonstrated by giving a new short uniqueness proof for Janko's sporadic simple group J_1. Similar, uniform uniqueness proofs are outlined for ten other sporadic simple groups.

1. Introduction

Throughout this article H denotes a finite group with center $Z(H)$ of even order. Although the classical Brauer-Fowler theorem asserts that there are only finitely many non-isomorphic simple groups G which possess a 2-central involution t such that $C_G(t) \cong H$, it does not provide an algorithm for the construction of all these simple groups G.

In [17] we gave a deterministic algorithm for the construction of all the finite simple groups G having a 2-central involution $t \neq 1$ with the following properties:

(1) There exists an isomorphism $\tau : C_G(t) \longrightarrow H$.
(2) There is some elementary abelian normal subgroup A of a fixed Sylow 2-subgroup S of H of maximal order $|A| \geq 4$ such that

$$G = \langle C_G(t), N = N_G(\tau^{-1}(A))\rangle.$$

Received February 6, 2002.
2000 *Mathematics Subject Classification.* 20D05, 20C15.

©2003 University of Illinois

(3) For some prime $p < |H|^2 - 1$ not dividing $|H||N|$ the group G has an irreducible p-modular representation M with multiplicity-free restriction $M_{|H}$.

A finite group G having a 2-central involution t with centralizer $C_G(t) \cong H$ is said to be of H*-type*. In [17] examples of groups H have been given for which the algorithm has been used to construct several non-isomorphic groups G of H-type. In any case, it constructs a faithful permutation representation, an irreducible matrix representation over a finite field, and the complete character table of G from a given faithful permutation representation of H. In the meantime the author and collaborators have shown that this algorithm provides uniform existence proofs for all but three sporadic groups.

In this article a sufficient criterion is given which ensures that for a given group H having a center $Z(H)$ of even order there is up to isomorphism only one group G of H-type.

An element $x \neq 1$ of a finite group G is called *p-central*, provided x belongs to the center $Z(P)$ of some Sylow p-subgroup P of G.

The criterion is proved in Section 2 (see Theorem 2.1). Its hypothesis requires that some finite simple group $\mathfrak{G}$ of H-type has been constructed with the following properties:

(1) For some prime $p > 1$ $\mathfrak{G}$ has one conjugacy class of p-central elements $\mathcal{G}$ of order p.
(2) Let $\mathcal{G}$ be a central element of order p of a fixed Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Let $\mathfrak{N} = N_{\mathfrak{G}}(\langle \mathcal{G} \rangle)$. Then there is a non-cyclic elementary abelian normal subgroup $\mathfrak{A}$ of $\mathfrak{P}$ containing $\mathcal{G}$ such that $\mathfrak{D} = N_{\mathfrak{N}}(\mathfrak{A})$ is maximal among the normalizers $N_{\mathfrak{N}}(\mathfrak{B}) \neq \mathfrak{N}$ of non-cyclic elementary abelian normal subgroups $\mathfrak{B}$ of $\mathfrak{P}$ with $\mathcal{G} \in \mathfrak{B}$, and $\mathfrak{D}$ is uniquely determined up to isomorphism by these properties.
(3) $\mathfrak{G} = \langle \mathfrak{N}, \mathfrak{M} \rangle$, where $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{A})$.
(4) The automorphism group $\mathrm{Aut}(\mathfrak{D})$ of $\mathfrak{D}$ has only one $\mathfrak{N}_1 - \mathfrak{M}_1$ double coset, where $\mathfrak{N}_1 = \{\gamma \in \mathrm{Aut}(\mathfrak{N}) \mid \gamma(\mathfrak{D}) = \mathfrak{D}\}$, and $\mathfrak{M}_1 = \{\mu \in \mathrm{Aut}(\mathfrak{M}) \mid \mu(\mathfrak{D}) = \mathfrak{D}\}$.
(5) There is a smallest integer $n > 1$ and a finite splitting field F for $\mathfrak{G}$ with characteristic r prime to $|\mathfrak{N}||\mathfrak{M}|$ such that the free product $\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M}$ of the groups $\mathfrak{N} = N_{\mathfrak{G}}(\langle \mathcal{G} \rangle)$ and $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{A})$ with amalgamated subgroup $\mathfrak{D} = \mathfrak{N} \cap \mathfrak{M}$ has up to algebraic conjugacy or duality exactly one representation $\kappa : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to \mathrm{GL}_n(F)$ with faithful restrictions to $\mathfrak{N}$ and $\mathfrak{M}$ such that $|\kappa(\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M})|$ divides $|\mathfrak{G}|$.

Theorem 2.1 asserts then the uniqueness of the finite simple group $\mathfrak{G}$ of H-type (up to isomorphism) provided it can be shown that each finite simple group G of H-type satisfies the following conditions:

(a) G has only one conjugacy class of p-central elements of order p.
(b) $|G| = |\mathfrak{G}|$.

(c) G has a faithful irreducible FG-module W of dimension $\dim_F(W) = n$.

(d) A Sylow p-subgroup P of G contains a central element g of order p and a non cyclic elementary abelian normal subgroup A with $g \in A$ such that

$$L = N_G(\langle g\rangle) \cong N_{\mathfrak{G}}(\langle \mathcal{G}\rangle), N_G(A) \cong N_{\mathfrak{G}}(\mathfrak{A}) \text{ and } D = N_L(A) \cong \mathfrak{D}.$$

The uniqueness criterion is practical because it builds on the fact that one simple group $\mathfrak{G}$ of H-type has already been constructed by means of the author's algorithm 4.6 of [17] or by other tools. Furthermore, the only technical conditions to be checked are conditions (4) and (5). For that task a deterministic algorithm is given in Section 3 (see Algorithm 3.6). In order to verify conditions (a), (b) and (c) theoretically it suffices to show that all finite simple groups G of H-type have the same character table. It is then used to show that each of the simple groups of H-type has either an irreducible ordinary character χ of degree $\chi(1) = n = \deg(\kappa)$ or an r-block B with cyclic defect group R of order r which has an irreducible r-modular character φ of degree $\varphi(1) = n$. This is demonstrated in Section 4, where a new and short uniqueness proof is given for Janko's sporadic simple group J_1 [10].

In Section 5 we outline similar uniqueness proofs for ten other sporadic simple groups of H-type. In any case details for the verification of the technical conditions (4) and (5) of Theorem 2.1 are stated. These have been worked out by the author's collaborators M. Kratzer and M. Weller; full proofs will be published elsewhere.

Finally, it has to be mentioned that the 2-central involution centralizers H of the simple Mathieu groups M_{12}, M_{22}, M_{23} and M_{24} do not satisfy condition (4) of Theorem 2.1. Its failure has been used by W. Lempken in [16] to construct the non isomorphic simple groups $PSL_5(2)$, He of Held and M_{24} from the common 2-centralizer H.

Concerning notation and terminology we refer to the books by D. Gorenstein [7], W. Feit [5], H. Kurzweil and B. Stellmacher [14], and I.M. Isaacs [9].

2. Criterion for uniqueness

The following criterion for uniqueness builds on ideas derived from J.G. Thompson's uniqueness theorem [20, p. 148] and assertion (2.7) of D.M. Goldschmidt [6, p. 381]. Thompson's theorem alone does not suffice. In [16] W. Lempken has shown that for $H = 2^{1+6} : L_3(2)$ the group $D = 2^6 : [2 \times L_3(2)]$ has two non equivalent embeddings into $M \cong 2^6 : [S_3 \times L_3(2)]$. For the first embedding the author's algorithm 4.6 of [17] constructs up to isomorphism the simple group M_{24}, and for the second it gives the simple group $L_5(2)$ with the same centralizer H of a 2-central involution.

Theorem 2.1. *Let H be a finite group with center $Z(H)$ of even order. Suppose that some finite simple group $\mathfrak{G}$ of H-type has been constructed with the following properties:*

(1) *For some prime $p > 1$ $\mathfrak{G}$ has one conjugacy class of p-central elements $\mathcal{G}$ of order p.*
(2) *Let $\mathcal{G}$ be a central element of order p of a fixed Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Let $\mathfrak{N} = N_{\mathfrak{G}}(\langle \mathcal{G} \rangle)$. Then there is a non-cyclic elementary abelian normal subgroup $\mathfrak{A}$ of $\mathfrak{P}$ containing $\mathcal{G}$ such that $\mathfrak{D} = N_{\mathfrak{N}}(\mathfrak{A})$ is maximal among the normalizers $N_{\mathfrak{N}}(\mathfrak{B}) \neq \mathfrak{N}$ of non-cyclic elementary abelian normal subgroups $\mathfrak{B}$ of $\mathfrak{P}$ with $\mathcal{G} \in \mathfrak{B}$, and $\mathfrak{D}$ is uniquely determined up to isomorphism by these properties.*
(3) $\mathfrak{G} = \langle \mathfrak{N}, \mathfrak{M} \rangle$*, where* $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{A})$.
(4) *The automorphism group* $\mathrm{Aut}(\mathfrak{D})$ *of* $\mathfrak{D}$ *has only one* $\mathfrak{N}_1 - \mathfrak{M}_1$ *double coset, where* $\mathfrak{N}_1 = \{\gamma \in \mathrm{Aut}(\mathfrak{N}) \mid \gamma(\mathfrak{D}) = \mathfrak{D}\}$*, and* $\mathfrak{M}_1 = \{\mu \in \mathrm{Aut}(\mathfrak{M} \mid \mu(\mathfrak{D}) = \mathfrak{D}\}$.
(5) *There is a smallest integer $n > 1$ and a finite splitting field F for $\mathfrak{G}$ with characteristic r prime to $|\mathfrak{N}||\mathfrak{M}|$ such that the the free product $\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M}$ of the groups $\mathfrak{N} = N_{\mathfrak{G}}(\langle \mathcal{G} \rangle)$ and $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{A})$ with amalgamated subgroup $\mathfrak{D} = \mathfrak{N} \cap \mathfrak{M}$ has up to algebraic conjugacy or duality exactly one representation $\kappa : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to \mathrm{GL}_n(F)$ with faithful restrictions to $\mathfrak{N}$ and $\mathfrak{M}$ of degree n such that $|\kappa(\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M})|$ divides $|\mathfrak{G}|$.*

Then $\mathfrak{G}$ is (up to isomorphism) the unique finite simple group of H-type provided it can be shown that each finite simple group G of H-type has the following properties:

(a) *G has one conjugacy class of p-central elements of order p.*
(b) $|G| = |\mathfrak{G}|$.
(c) *G has a faithful irreducible FG-module W of dimension* $\dim_F(W) = n$.
(d) *A Sylow p-subgroup P of G contains a central element g of order p and a non cyclic elementary abelian normal subgroup A such that*

$$L = N_G(\langle g \rangle) \cong N_{\mathfrak{G}}(\langle \mathfrak{G} \rangle), N_G(A) \cong N_{\mathfrak{G}}(\mathfrak{A}), \text{ and } N_L(A) \cong \mathfrak{D}.$$

Proof. Let g be a p-central element of order p contained in the Sylow p-subgroup P of the finite group G of H-type. Let $L = N_G(\langle g \rangle)$. By (d) there is a non-cyclic elementary abelian normal subgroup A of P such that $M = N_G(A) \cong N_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{M}$, and $D = N_L(A) \cong \mathfrak{D} = N_{\mathfrak{N}}(\mathfrak{A})$, where $\mathfrak{N} = N_{\mathfrak{G}}(\langle \mathfrak{G} \rangle) \cong L$. Hence the two amalgams $L \leftarrow D \rightarrow M$ and $\mathfrak{N} \leftarrow \mathfrak{D} \rightarrow \mathfrak{M}$ are of the same type in the sense of Goldschmidt [6, p. 381]. By condition (4) $\mathrm{Aut}(\mathfrak{D})$ has only one $\mathfrak{N}_1 - \mathfrak{M}_1$ double coset , where $\mathfrak{N}_1 = \{\gamma \in \mathrm{Aut}(\mathfrak{N}) \mid \gamma(\mathfrak{D}) = \mathfrak{D}\}$ and $\mathfrak{M}_1 = \{\mu \in \mathrm{Aut}(\mathfrak{M}) \mid \mu(\mathfrak{D}) = \mathfrak{D}\}$. Therefore statement (2.7) of [6, p. 381] asserts that there is an isomorphism

$$\tau : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to L *_D M$$

between the free products $\mathfrak{M} *_{\mathfrak{D}} \mathfrak{M}$ and $L *_D M$ with amalgamated subgroups $\mathfrak{D}$ and D, respectively.

Now consider the subgroup $U = \langle L, M\rangle$ of G. Then there is an epimorphism $\sigma : L *_D M \to U$. It induces an epimorphism $\sigma\tau : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to U$. Since by condition (c) each finite group G of H-type has at least one faithful irreducible representation $\rho : G \to \mathrm{GL}_n(F)$ of degree n over F, there is an n-dimensional representation

$$\rho\sigma\tau : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to \rho(U) \leq \mathrm{GL}_n(F)$$

of the free product $\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M}$ with amalgamated subgroup $\mathfrak{D}$. Since $\rho_{|U}$ is faithful, so are the restrictions of $\rho\sigma\tau$ to $\mathfrak{N}$ and $\mathfrak{M}$. Furthermore, $|\rho\sigma\tau(\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M})| = |\rho(U)|$ divides $|U|$ and therefore $|\mathfrak{G}|$ by (c). Now condition (5) implies that we may assume that $\rho\sigma\tau = \kappa : \mathfrak{N} *_{\mathfrak{D}} \mathfrak{M} \to \mathrm{GL}_n(F)$ is uniquely determined up to isomorphism.

As $\mathfrak{G} = \langle \mathfrak{N}, \mathfrak{M}\rangle$ by (3), another application of this argument yields that

$$\mathfrak{G} \cong \kappa(\mathfrak{N} *_{\mathfrak{D}} \mathfrak{M}) \cong \rho(U) \leq \mathrm{GL}_n(F).$$

Therefore condition (b) implies that

$$|\mathfrak{G}| = |\rho(U)| \leq |U| \leq |G| = |\mathfrak{G}|.$$

Hence each group G of H-type is isomorphic to $\mathfrak{G}$. □

3. Construction of suitable irreducible representations

Definition 3.1. Let N, E be a pair of finite groups having subgroups $D \leq N$ and $D_E \leq E$ such that there is a group isomorphism

$$\tau : D_E \longrightarrow D.$$

Let F be a finite splitting field for N and E of characteristic $r > 0$ not dividing $|N||E|$. Let V be a finitely generated faithful FN-module.

A faithful FE-module W is called *compatible* with V, if the isomorphism $\tau : D_E \to D$ induces an FD-module isomorphism

$$\pi : W_{|D_E} \longrightarrow V_{|D}$$

between the restrictions $W_{|D_E}$ and $V_{|D}$. In particular, $\dim_F W = \dim_F V = n$, say.

Let $\rho : N \to GL_n(F)$ and $\rho_E : E \to GL_n(F)$ be the faithful group representations of N and E afforded by V and W, respectively. Identify N and E with their isomorphic images $\rho(N)$ and $\rho_E(E)$ in $GL_n(F)$, respectively. Then the isomorphism $\tau : D_E \to D$ determines a transformation matrix $T \in GL_n(F)$ such that

$$D = T^{-1}D_E T = N \cap T^{-1}ET \leq GL_n(F).$$

Therefore the canonical n-dimensional F-vector space F^n yields an n-dimensional $F[N *_D E]$-module of the free product $N *_D E$ of N and E with common subgroup D.

With this notation we can restate Thompson's theorem [20, p. 148] determining the number of the isomorphism classes of the n-dimensional $F[N*_D E]$-modules in a special situation.

Proposition 3.2. *Let N, E be a pair of groups intersecting in D. Suppose that F is a finite splitting field with q elements for N, E and their subgroups of characteristic $r > 0$ not dividing $|N||E|$. Let (V, W) be pair of compatible FN- and FE-modules of dimension n. Identify N and E with their images in $GL_n(F)$ of their faithful representations afforded by V and W, respectively.*

Then the set $\mathfrak{S}(V, W)$ of isomorphism classes of n-dimensional $F[N *_D E]$-modules M with faithful restrictions $M_{|N} \cong V$ and $M_{|E} \cong W$ is in one-to-one correspondence with the $C_{GL_n(F)}(N) - C_{GL_n(F)}(E)$ double cosets of the centralizer of $C_{GL_n(F)}(D)$; i.e., if

$$C_{GL_n(F)}(D) = \bigcup_{i=1}^{t} C_{GL_n(F)}(N) T_i C_{GL_n(F)}(E)$$

is a double coset decomposition with representatives $T_i \in GL_n(F)$ for $1 \leq i \leq t$, then there is a unique $T \in \{T_i \mid 1 \leq i \leq t\}$ such that the image $\kappa_M(N *_D E)$ of $N *_D E$ under the representation κ_M afforded by M is given by

$$\kappa_M(N *_D E) = \langle N, T^{-1}ET \rangle \leq GL_n(F), \text{ and } D = N \cap T^{-1}ET.$$

Furthermore, if $V_{|D} \cong W_{|D}$ is multiplicity-free, so are V and W. Let k, l and m be the numbers of composition factors of $V_{|D} \cong W_{|D}$, the FN-module V and the FE-module W, respectively. Suppose also that $|C_{GL_n(F)}(N) \cap C_{GL_n(F)}(E)| = q - 1$. Then all t double coset representatives T_i are diagonal matrices with non-zero coefficients $f_{ij} \in F$, $1 \leq i \leq t$, $1 \leq j \leq n$, and

$$t = (q-1)^{k+1-l-m}$$

Proof. All statements follow from Thompson's theorem [20, p. 148] and Schur's lemma. $\square$

Lemma 3.3. *Let G be a finite group generated by two subgroups M and N intersecting in $D = M \cap N$. Let the finite field F of characteristic $r > 0$ not dividing $|M||N|$ be a splitting field for G and its subgroups.*

Suppose that n is the smallest integer satisfying the following two properties:

(a) *There is a faithful n-dimensional FN-module V with multiplicity-free restriction $V_{|D}$ which is compatible with some faithful FM-module W.*

(b) *Identifying N and M with their isomorphic images in $GL_n(F)$ under their representations afforded by V and W, respectively, and letting*

$$C_{GL_n(F)}(D) = \bigcup_{i=1}^{t} C_{GL_n(F)}(N) T_i C_{GL_n(F)}(M)$$

be a double coset decomposition of the centralizer $C_{GL_n(F)}(D)$ *of* D *in* $GL_n(F)$ *with respect to its subgroups* $C_{GL_n(F)}(N)$ *and* $C_{GL_n(F)}(M)$, *there is at least one double coset representative* $T \in \{T_i \mid 1 \le i \le t\}$ *such that*

$$G \cong \langle N, T^{-1}MT \rangle \le GL_n(F), \text{ and } \quad D = N \cap T^{-1}MT.$$

Then G *has an* n*-dimensional* FG*-module* Y *with multiplicity-free restriction* $Y_{|D} \cong V_{|D}$. *Furthermore, no proper submodule* Y' *of* Y *is a faithful* FG*-module.*

If G *is a simple group, then* Y *is a simple* FG*-module of minimal degree* n.

Proof. From $G = \langle M, N \rangle$ follows that G is an epimorphic image of the free product $N *_D M$ of N and M with amalgamated subgroup D. Hence each n-dimensional FG-module Y is an n-dimensional $F[N *_D M]$-module. By Proposition 3.2 the compatible pair (V, W) of faithful FN- and FM-modules V and W given in (a) and the double coset representative $T \in \{T_i \mid 1 \le i \le t\}$ given in (b) describe a uniquely determined $F[N *_D M]$-module Y such that

$$G \cong \langle N, T^{-1}MT \rangle \le GL_n(F), \text{ and } D = N \cap T^{-1}MT.$$

Thus Y is an n-dimensional FG-module with $Y_{|N} \cong V$. Hence $Y_{|D}$ is multiplicity-free by (a).

Suppose that there is a proper faithful FG-submodule $Y' \neq 0$ of Y. Then the modules $Y'_{|N}$, $Y'_{|M}$ and $Y'_{|D}$ are faithful. Thus $V' = Y'_{|N} < Y_{|N} \cong V$ would be a proper faithful FN-submodule of V which is compatible with the faithful FM-module $W' \cong Y'_{|M}$. Also $Y'_{|D}$ is multiplicity-free, because $r \nmid |N|$ and $V_{|D}$ is multiplicity-free. Let $n_1 = \dim_F Y'$. As Y' is an $F[N *_D M]$-module, Proposition 3.2 asserts that there is a double coset representative T of

$$C_{GL_{n_1}F}(D) = \bigcup_{i=1}^{s} C_{GL_{n_1}(F)}(N) T_i C_{GL_{n_1}(F)}(M)$$

such that $G \cong \langle N, T^{-1}MT \rangle \le GL_{n_1}(F)$, and $D = N \cap T^{-1}MT$, where $T \in \{T_i \mid 1 \le i \le s\}$, and where N, M are identified with their isomorphic images under the faithful representations afforded by their faithful representations V' and W', respectively. Thus the integer $n_1 < n$ satisfies the conditions (a) and (b). This contradiction to the hypothesis on n completes the proof of the assertion. If G is simple, then all non zero FG-submodules Y are faithful. Thus Y is a simple FG-module of minimal degree n. $\square$

In the following the set of all faithful characters of the finite group E is denoted by $\mathrm{fchar}_{\mathbb{C}}(E)$, and $\mathrm{mfchar}_{\mathbb{C}}(E)$ denotes the set of all multiplicity-free faithful characters of E.

Definition 3.4. Let N, M be a pair of finite groups such that $D = N \cap M$. Then

$$\Sigma = \{(\nu, \omega) \in \mathrm{mfchar}_{\mathbb{C}}(N) \times \mathrm{mfchar}_{\mathbb{C}}(M) \mid \nu_{|D} = \omega_{|D}\}$$

is called the set of *compatible pairs* of multiplicity-free faithful characters of N and M.

For each $(\nu, \omega) \in \Sigma$ the integer $n = \nu(1) = \omega(1)$ is called the *degree* of the compatible pair (ν, ω).

Proposition 3.5. *Let N, M be a pair of finite groups such that $D = N \cap M$. Let F be finite field of characteristic $r > 0$ not dividing $|N||M|$ which is assumed to be a splitting field for N, M and D. Then the following assertions hold:*

(a) *The set Σ of all compatible pairs of multiplicity-free faithful characters of M and N is finite.*

(b) *For every integer $n > 1$ there is a bijection between the finite set $\mathfrak{C}_n$ of all compatible pairs (V, W) of faithful multiplicity-free FN-modules V and multiplicity-free faithful FM-modules W with $\dim_F V = \dim_F W = n$ and the set*

$$\Sigma_n = \{(\nu, \omega) \in \mathrm{mfchar}_{\mathbb{C}}(N) \times \mathrm{mfchar}_{\mathbb{C}}(M) \mid \nu_{|D} = \omega_{|D}\}$$

of compatible pairs of faithful characters of N and M with degree $\nu(1) = \omega(1) = n$.

In particular, the finite set $\mathfrak{C}_n$ is computable from the complex character tables of N, M, and D, and the fusion of the conjugacy classes of D in N, and the ones of D in M.

Proof. (a) A finite group E with k conjugacy classes has exactly 2^k inequivalent semisimple multiplicity-free complex characters. Therefore Σ is finite.

(b) By Maschke's theorem the group algebras FN, FM and FD are semisimple. Hence Σ_n is finite, because each semisimple F-algebra has only a finite number of faithful n-dimensional modules. The bijection $\mathfrak{C}_n \leftrightarrow \Sigma_n$ now follows from the bijection between the irreducible group algebra modules and their corresponding irreducible characters. □

Algorithm 3.6. Let G be a finite group which for some prime $p > 1$ has only one conjugacy class of p-central elements of order p.

Let $g \neq 1$ be a central element of order p in the Sylow p-subgroup P of G. Let $L = N_G(\langle g \rangle)$. Suppose that there is a unique non-cyclic elementary abelian normal subgroup A of P containing g such that $D = N_L(A)$ is of maximal order among the normalizers $N_L(B)$ of non cyclic elementary abelian normal subgroups B of P with $g \in B$ and $N_L(B) \neq L$, and that A is also maximal among these normal subgroups B of P.

Suppose that $M = N_G(A) \neq D$. Let F be a finite field of characteristic $r > 0$ not dividing $|L||M|$ which is a splitting field for G and its subgroups. Then perform the following steps:

Step 1:

(a) Compute faithful permutation representations $\pi(L)$ and $\pi(M)$ of each of the groups L and M, respectively.
(b) Compute the character tables of the groups D, L and M.
(c) Compute the automorphism groups $\mathrm{Aut}(D)$ of D, and $L_1 = \{\nu \in \mathrm{Aut}(L) \mid \nu(D) = D\}$ and $L_2 = \{\mu \in \mathrm{Aut}(M) \mid \mu(D) = D\}$. Determine the $L_1 - M_1$ double cosets in $\mathrm{Aut}(D)$. If there are at least two such double cosets, then the algorithm ends. Otherwise, all free products $L *_D M$ with amalgamated subgroup D are isomorphic, and the following steps can be performed.

Step 2: Using Kratzer's algorithm [11] determine the finite set I of integers n such that

$$\Sigma_n = \{(\nu, \omega) \in \mathrm{fchar}_{\mathbb{C}}(L) \times \mathrm{fchar}_{\mathbb{C}}(M) \mid \\ \nu_{|D} = \omega_{|D} \in \mathrm{mfchar}_{\mathbb{C}}(D), \nu(1) = n\} \neq \emptyset.$$

If $I = \emptyset$ for all n, then the algorithms ends. Otherwise the following steps can be performed for each $n \in I$.

Step 3:

(a) For each compatible pair $(\nu, \omega) \in \Sigma_n$ determine the numbers k, l and m of irreducible constituents of ν, ω and $\nu_{|D} = \omega_{|D}$, respectively.
(b) Construct the faithful multiplicity-free semisimple FL-module V corresponding to ν and the faithful multiplicity-free FM-module W corresponding to ω.
(c) Identify L and M with their epimorphic images in $\mathrm{GL}_n(F)$ under the faithful representations afforded by V and W, respectively. Then $C_{\mathrm{GL}_n(F)}(D)$ has

$$t(\nu, \omega) = (q-1)^{k+1-l-m}$$

$C_{\mathrm{GL}_n(F)}(L) - C_{\mathrm{GL}_n(F)}(M)$ double cosets $C_{\mathrm{GL}_n(F)}(L) T_i C_{\mathrm{GL}_n(F)}(M)$, where each $(n \times n)$-matrix T_i is a uniquely determined diagonal matrix of $\mathrm{GL}_n(F)$.
(d) For each $i \in \{1, 2, \ldots, t(\nu, \omega)\}$ let $G_i(\nu, \omega) = \langle L, T_i^{-1} M T_i \rangle \leq \mathrm{GL}_n(F)$. As Σ_n is finite,

$$\mathfrak{K}_n = \{G_i(\nu, \omega) \mid (\nu, \omega \in \Sigma_n, 1 \leq i \leq t(\nu, \omega)\}$$

is also a finite set of finite groups.

Step 4: Determine the subset

$$\mathfrak{S}_n(\nu,\omega) = \{G_i(\nu,\omega) \in \mathfrak{K}_n \mid |G_i(\nu,\omega)| \ \text{ divides } \ |G|\}.$$

If $\mathfrak{S}_n(\nu,\omega) = \emptyset$ or $|\mathfrak{S}_n(\nu,\omega)| > 1$ for all $n \in I$ then the algorithm fails. Otherwise there is a smallest integer $n \in I$ such that the uniquely determined free product $M *_D L$ with amalgamated subgroup D has by Proposition 3.2 and Lemma 3.3 up to algebraic conjugacy or duality exactly one representation $\kappa : M *_D L \to \mathrm{GL}_n(F)$ with faithful restrictions to L and M of minimal degree n over the splitting field F such that $|\kappa(M *_D L)|$ divides $|G|$.

Keep the notation of Algorithm 3.6. Suppose that we have $|\mathfrak{S}_n(\nu,\omega)| = 1$. If the constituents $\nu \in \mathrm{fchar}_{\mathbb{C}}(L)$ and $\omega \in \mathrm{fchar}_{\mathbb{C}}(M)$ of the faithful compatible pair $(\nu,\omega) \in \Sigma_n$ are not both real, then also the conjugate complex pair $(\bar{\nu},\bar{\omega}) \in \Sigma_n$, and $|\mathfrak{S}_n(\bar{\nu},\bar{\omega})| = 1$ by Steps 3 and 4 of Algorithm 3.6. Therefore the free product $L *_D M$ of L and M with amalgamated subgroup D has a unique pair of dual n-dimensional representations $\kappa, \kappa^* : L *_D M \to \mathrm{GL}_n(F)$ provided the characteristic r of F does not divide $|G|$. In particular, $\kappa(L *_D M) \cong \kappa^*(L *_D M)$.

The case of algebraic conjugacy is dealt with similarly.

4. Application: Uniqueness proof for J_1

As an application of the results of the previous sections we give here a short uniqueness proof for Janko's sporadic simple group J_1.

Using the author's algorithm 4.6 of [17] M. Kratzer and the author proved in [12] the following result.

Theorem 4.1. *Let $H = \langle z \rangle \times A_5$, where $z \neq 1$ has order 2. Let A be a fixed Sylow 2-subgroup of H, $D = N_H(A)$, and let $\eta : D \to \mathrm{GL}_3(2)$ be the homomorphism determined by the conjugation of D on A. Then the following assertions hold:*

(a) *Up to conjugacy there exists a unique subgroup $\Phi \cong F_{21}$ of $\mathrm{GL}_3(2)$ containing $\Delta = \eta(D) \cong \langle 3 \rangle$ with odd index and an embedding μ of D into the semidirect product $E_\Phi = A : \Phi$ such that the diagram*

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] & E_\Phi \arrow[r] & \Phi \arrow[r] & 1 \\ 1 \arrow[r] & A \arrow[r] \arrow[u] & D \arrow[r] \arrow[u, "\mu"'] & \Delta \arrow[r] \arrow[u] & 1 \end{tikzcd}$$

commutes.

(b) *The free product $H *_D E_\Phi$ of H and E_Φ with amalgamated subgroup D has a unique 7-dimensional irreducible representation $\kappa : H *_D E_\Phi \to$*

$\mathrm{GL}_7(11)$ *over the prime field* $F = \mathrm{GF}(11)$ *such that the group*

$$J = \langle \kappa(H), \kappa(E_\Phi)\rangle = \langle x, y, s\rangle \le \mathrm{GL}_7(11)$$

generated by the matrices

$$x = \begin{pmatrix} 8 & 5 & 5 & 5 & & & \\ 9 & 8 & 9 & 8 & & & \\ 9 & 8 & 8 & 9 & & & \\ 6 & 8 & 8 & 8 & & & \\ & & & & 7 & 5 & 9 \\ & & & & 9 & 7 & 5 \\ & & & & 1 & 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 3 & 6 & 6 & 6 & & & \\ 2 & 2 & 3 & 3 & & & \\ 5 & 3 & 3 & 3 & & & \\ 2 & 3 & 3 & 2 & & & \\ & & & & 5 & 9 & 7 \\ & & & & 1 & 1 & 1 \\ & & & & 9 & 7 & 5 \end{pmatrix},$$

$$s = \begin{pmatrix} 0 & 8 & 0 & 8 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 & 2 & 1 & 1 \\ 0 & 6 & 6 & 0 & 0 & 1 & 10 \\ 0 & 5 & 5 & 0 & 9 & 10 & 10 \\ 10 & 0 & 3 & 3 & 0 & 5 & 5 \\ 1 & 0 & 3 & 3 & 0 & 6 & 6 \\ 1 & 0 & 8 & 8 & 0 & 5 & 5 \end{pmatrix},$$

is a finite simple group of order $J = 175560$ *having an involution* $z = y^3$ *with centralizer* $C_J(z) = \langle x, y\rangle \cong H$.

In particular, the finite simple group $J = \langle x, y, s\rangle$ of H-type $2 \times A_5$ satisfies the conditions (1), (2), (3) and (5) of Theorem 2.1 for the primes $p = 2$, $r = 11$, and the minimal dimension $n = 7$. The remaining condition (4) follows from the following subsidiary result, because $\mathrm{Aut}(D) \cong 2 \times S_4$, where $D \cong 2 \times A_4$. Its proof is due to M. Kratzer.

LEMMA 4.2. *Let* $H = \langle z\rangle \times A_5$, $z^2 = 1 \ne z$. *Then:*

(a) *H can be generated by two elements x and y satisfying the following relations:*

$$x^5 = y^6 = (xy)^{10} = (xy^2)^2 = [x, y^3] = 1, \text{ and } (xy)^5 = y^3.$$

(b) *$A = \langle z = y^3,\ a_1 = xy^2,\ a_2 = (x^2, y)^2 x\rangle$ is a Sylow 2-subgroup of H, and $D = N_H(A) = \langle A, d = y^2x^2\rangle \cong \langle z\rangle \times A_4$.*

(c) *The automorphism group* $\mathrm{Aut}(H) \cong \langle z\rangle \times S_5$ *contains an outer automorphism* $\beta \in \mathrm{Aut}(D)$ *of order* 2.

Proof. (a) and (b) follow from Lemma 2.2 of [12, p. 40].

(c) Using MAGMA it can be shown that there is an automorphism β of H defined by $\beta(x) = x^3yx^2yx$, $\beta(y) = y^{-1}$.

Another application of MAGMA yields that $\beta^2 = 1$, and $\beta(a_1) = a_1a_2$, $\beta(a_2) = a_2$, and $\beta(d) = a_1d^2$.

Since the Sylow 2-subgroup of H is elementary abelian of order 8, and $\{a_1, a_2, \beta\}$ is a dihedral group of order 8, β is not an inner automorphism of H. □

REMARK 4.3. The free product $H *_D E_\Phi$ of H and $N_J(A) \cong E_\Phi$ with amalgamated subgroup D described in Theorem 4.1 (b) is uniquely determined by H up to isomorphism. This follows from Lemma 4.2 and statement (2.7) of [6, p. 381].

DEFINITION 4.4. A finite group G having a 2-central involution z with centralizer $H = C_G(z) \cong \langle z \rangle \times A_5$ such that G has no subgroup of index 2 is said to be of *J_1-type*.

We now verify conditions (a), (b), (c) and (d) of Theorem 2.1 for any finite group G of J_1-type.

LEMMA 4.5 ([10, p. 160]). *Let G be a finite group of J_1-type. Then the following assertions hold:*

(a) *A Sylow 2-subgroup A of G is elementary abelian of order 8.*
(b) *$N_G(A)$ is a split extension of A by a Frobenius group F_{21} of order 21.*
(c) *All involutions of G are conjugate in G.*

Proof. (a) follows immediately from Definition 4.1.

(b) Let z be an involution in the Sylow 2-subgroup A of G. Then $H = C_G(z) \cong \langle x \rangle \times A_4$ by (a) and Definition 4.1. In particular, $C_G(A) = C_H(A) = A$, and $D = N_H(A) \cong \langle z \rangle \times A_4$. Since G does not have a proper subgroup of index 2, Theorem 7.1.6 of [14, p. 151] implies that $M = N_G(A)$ contains D properly. Now $1 \neq D/A < M/A < \mathrm{GL}_3(2)$, and $|M/A : D/A|$ is odd. As $|D/A| = 3$, Sylow's theorem implies that up to conjugacy $M/A \cong F_{21}$, a Frobenius group of order 21. Now the Schur-Zassenhaus theorem completes the proof of (b).

(c) As the normal Sylow 7-subgroup of F_{21} does not fix the involution $z \in A$, it follows that all involutions of A are conjugate in $M = A : F_{21}$. Thus (c) holds by Sylow's theorem. □

In [2, p. 228] Brauer proved a general group order formula, a special case of which is stated in the following proposition.

PROPOSITION 4.6. *Let G be a finite group of order $|G| = pq$, where p is a prime and $(p, q) = 1$. Let $D = \langle t \rangle$ a Sylow p-subgroup with normalizer $Y = N_G(D)$. Let B be a p-block of G with defect group $\delta(B) =_G D$ and Brauer correspondent b in Y. Then*

$$\frac{|G|}{|C_G(z)|^2} \sum_{\chi} \frac{\chi(z)^2 \chi(t)}{\chi(1)} = |Y| \sum_i \sum_j \sum_{\psi} \frac{\psi(z_i)\psi(z_j)\psi(t)}{|C_Y(z_i)||C_Y(z_j)|\psi(1)},$$

where the sum on the left ranges over all irreducible characters in B and on the right, the elements z_i and z_j range independently over a set of representatives for the Y-conjugacy classes comprising $z^G \cap Y$, while ψ ranges over all irreducible characters of Y which belong to the p-block b of Y.

LEMMA 4.7. *Let G be a finite group of J_1-type with $H = C_G(z) \cong \langle z\rangle \times A_5$. Let U be a subgroup in H of prime order $p \neq 2$. Then the following statements hold:*

(a) *$C_G(U) = A\langle z\rangle$, where the normal subgroup A of $C_G(U)$ is cyclic of order* 15.

(b) *$p \in \{3, 5\}$, and U is a Sylow p-subgroup of G.*

Proof. Assertion (b) holds by statement 4.1 of Bender [1, p. 225].

(a) Let S be a Sylow 3-subgroup of G generated by an element $t \in H$ of order 3. Suppose that $|C_G(S)| \neq 30$. Then $C_G(S) = S \times \langle z\rangle$, and $C^*_G(t) = \{x \in G\} \mid t^x = t^{\pm 1}\} \leq N_G(S) \leq H$ by Lemma 3.1 of Janko [10, p. 152]. We now give the character table of H, using the notation of Lemma 4.2 and setting $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $*\alpha = \frac{1}{2}(1-\sqrt{5})$.

Class	1	2_1	2_2	2_3	3	5_1	5_2	6	10_1	10_2
Length	1	1	15	15	20	12	12	20	12	12
Repr.	z^2	z	a_1	za_1	d	x	a_2x	zd	zx	za_2x
ψ_1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	-1	1	-1	1	1	1	-1	-1	-1
ψ_3	3	-3	-1	1	0	α	$*\alpha$	0	$-\alpha$	$-*\alpha$
ψ_4	3	-3	-1	1	0	$*\alpha$	α	0	$-*\alpha$	$-\alpha$
ψ_5	3	3	-1	-1	0	α	$*\alpha$	0	α	$*\alpha$
ψ_6	3	3	-1	-1	0	$*\alpha$	α	0	$*\alpha$	α
ψ_7	4	-4	0	0	1	-1	-1	-1	1	1
ψ_8	4	4	0	0	1	-1	-1	1	-1	-1
ψ_9	5	-5	1	-1	-1	0	0	1	0	0
ψ_{10}	5	5	1	1	-1	0	0	-1	0	0

Let B_0 and b_0 be the principal 3-blocks of G and H, respectively. Then $\mathrm{Irr}(b_0) = \{\psi_1, \psi_8, \psi_{10}\}$ by the character table of H. Theorem 4 of R. Brauer [3, p. 417] asserts that B_0 has three irreducible characters $\chi_1 = 1_G$, χ_2 and χ_3 which can be ordered such that

(1) $\chi_1(t) = \psi_1(t)$, $\chi_2(t) = \psi_8(t)$ and $\chi_3(t) = \psi_{10}(t)$ for $1 \neq t \in S$, and
(2) $\chi_1(y) + \chi_2(y) - \chi_3(y) = 0$ for all 3-regular elements y of G.

Let $\chi_2(1) = x$. Then $\chi_3(1) = x + 1$ by (2). Since $\chi_2, \chi_3 \in \mathrm{Irr}(B_0)$, both numbers x and $x + 1$ are coprime to 3 by Theorem 1.16 of Feit [5, p. 278].

By Lemma 4.5 any group G of J_1-type has only one conjugacy class z^G of involutions. Lemma 2.15 of Isaacs [9, p. 20] asserts that $\chi_2(z) = a$ is an integer. Hence $\chi_3(z) = a + 1$ by (2). Let $c_i = |C_H(z_i)|$ for $i = 1, 2, 3$. Then $c_1 = 120$, $c_2 = c_3 = 8$ by the character table of H.

Inserting all these data into Brauer's group order formula stated in Proposition 4.6, we get:

$$\begin{aligned}
\frac{|G|}{|H|^2}&\left(1 + \frac{a^2}{x} - \frac{(a+1)^2}{x+1}\right) \\
&= |H|\Bigg[\sum_{i=1}^{3}\frac{1}{c_i^2} + \frac{4}{c_1c_2} + \frac{2}{c_2c_3} + \frac{\psi_2(2_1)^2\psi_2(t)}{c_1^2\psi_2(1)} - \sum_{i=1}^{3}\frac{\psi_3(2_i)^2}{c_i^2\psi_3(1)} \\
&\qquad\qquad - \frac{4\psi_3(2_1)\psi_3(2_2)}{c_1c_2\psi_3(1)} - \frac{2\psi_3(2_2)\psi_3(2_3)}{c_2c_3\psi_3(1)}\Bigg] \\
&= 120\left[\frac{5}{120^2} + \frac{4}{120\cdot 8} + \frac{4}{64} - \frac{1}{5}\left(\frac{25}{120^2} + \frac{4\cdot 5}{120\cdot 8} + \frac{4}{64}\right)\right] = 6,
\end{aligned}$$

and

$$u(x-a)^2 = 2^4 3^2 \cdot 5x(x+1). \tag{$*$}$$

As one of the numbers x and $x + 1$ is even, this equation implies that the precise power of 2 dividing x or $x + 1$ is 4, because both numbers divide $|G|$ by Theorem 3.12 of Isaacs [9, p. 38], and the order of a Sylow 2-subgroup of G is 8.

Let $\chi \in \{\chi_2, \chi_3\}$ be the character of G with $4 \mid \chi(1)$. Since a Sylow 2-subgroup D of G is self centralizing by the above character table, the principal 2-block $B_0^{(2)}$ is the only 2-block of G with full defect by Brauer's first main theorem on blocks and Corollary 3.11 of Feit [5, p. 200]. As $|D| = 8$, Theorem 3.8 of Landrock [15, p. 438] asserts that all irreducible characters $B_0^{(2)}$ have odd degree. Hence χ belongs to a 2-block $B_1^{(2)}$ with defect group D_1 of order smaller than 4. Now Theorem 2 of Brauer and Feit [4, p. 116] asserts that χ belongs to a 2-block $B_1^{(2)}$ of G with defect group D_1 of order 2. Since all involutions of G are conjugate $D_1 =_G \langle z\rangle$, it follows from Theorem 4 of R. Brauer [3, p. 417] and the character table of H that $a = \chi_2(z) = \psi_8(z) = 4$ or $a + 1 = \chi_3(z) = \psi_{10}(z) = 5$ by (2). In any case $a = 4$.

Since $\gcd(x(x+1), 3) = 1$, and 5 is the highest power of 5 dividing $x(x+1)$, it follows from $(*)$ that $x(x + 1) = 4 \cdot 5 \cdot w$, where w is a divisor of u. As u is coprime to $2, 3$ and 5, equation $(*)$ implies that

$$(x-4)^2 = 2^6 \cdot 3^2 \cdot 5^2.$$

Hence $x = 124$, and $x+1 = 125 = 5^3$, a contradiction to the fact that a Sylow 5-subgroup of G has order 5. Thus $|C_G(S)| = 30$. $\square$

LEMMA 4.8. *Let G be a finite group of J_1-type. Let T_3 and T_5 be a Sylow 3-subgroup and a Sylow 5-subgroup of G, respectively. Then:*

(a) *$N_G(T_3) = N_G(T_5) \cong D_{10} \times S_3$, where D_{10} denotes the dihedral group of order 10, and S_3 the symmetric group of degree 3.*
(b) *The character table of $N_G(T_3)$ is given as follows, where $\alpha = -\frac{1}{2}(1+\sqrt{5})$, and $\beta = -\frac{1}{2}(1-\sqrt{5})$.*

Class	1	2	3	4	5	6	7	8	9	10	11	12
Size	1	3	5	15	2	2	2	10	6	6	4	4
Order	1	2_1	2_2	2_3	3	5_1	5_2	6	10_1	10_2	15_1	15_2
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	-1	-1	1	1	1	1	-1	-1	-1	1	1
ψ_3	1	-1	1	-1	1	1	1	1	-1	-1	1	1
ψ_4	1	1	-1	-1	1	1	1	-1	1	1	1	1
ψ_5	2	0	2	0	-1	2	2	-1	0	0	-1	-1
ψ_6	2	0	-2	0	-1	2	2	1	0	0	-1	-1
ψ_7	2	2	0	0	2	α	β	0	α	β	β	α
ψ_8	2	-2	0	0	2	α	β	0	$-\alpha$	$-\beta$	β	α
ψ_9	2	2	0	0	2	β	α	0	β	α	α	β
ψ_{10}	2	-2	0	0	2	β	α	0	$-\beta$	$-\alpha$	α	β
ψ_{11}	4	0	0	0	-2	2α	2β	0	0	0	$-\beta$	$-\alpha$
ψ_{12}	4	0	0	0	-2	2β	2α	0	0	0	$-\alpha$	$-\beta$

Proof. (a) follows at once from Lemma 4.5 and Lemma 3.1 of [10, p. 152]. (b) is an easy consequence of (a) by GAP [19]. $\square$

THEOREM 4.9. *All finite groups G of J_1-type have the same group order*

$$|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19.$$

Proof. By Lemma 4.7 each group G of J_1-type has a cyclic Sylow 3-subgroup S generated by an element t of order 3. Let $N = N_G(S)$ be its normalizer in G. The character table of N is given in Lemma 4.8. It follows that the principal 3-block b_0 of N consists of the irreducible characters $\psi_1 = 1_N$, ψ_3 and ψ_5. As in the proof of Lemma 4.7 it follows from Theorem 4 of [3, p. 417] that the principal 3-block B_0 of G has three ordinary characters $\chi_1 = 1_G, \chi_2$ and χ_3 satisfying:

(1) $\chi_2(s) = \psi_3(s)$ for all 3-singular elements $1 \neq s$ of G.

(2) $\chi_1(y) + \chi_2(y) - \chi_3(y) = 0$ for all 3-regular elements $y \in G$.

Let $x = \chi_2(1)$. Then $\chi_3(1) = x + 1$, and both numbers x and $x + 1$ are coprime to 3. By Lemma 4.5 G has only one conjugacy class z^G of involutions. If $\chi_2(z) = a$, then $\chi_3(z) = a + 1$ by (2). By Lemma 2.15 of [9, p. 20] a is an integer.

Let $c_i = |C_N(2_i)|$ for $i = 1, 2, 3$. Then $c_1 = 20$, $c_2 = 12$, and $c_3 = 4$ by Lemma 4.8.

Computing the right hand side of Brauer's group order formula stated in Proposition 4.3 one gets

$$|N| \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{\psi \in b_0} \frac{\psi(2_i)\psi(2j)\psi(t)}{c_i \cdot c_j \cdot \psi(1)} = \frac{54}{5}.$$

Another application of Lemma 4.7 yields that $|G| = H \cdot u$, where u is coprime to 2, 3, and 5. Inserting these values into Brauer's group order formula of Proposition 4.6 one gets:

$$\frac{|G|}{|H|^2} \left(1 + \frac{a^2}{x} - \frac{(a+1)^2}{x+1} \right) = \frac{54}{5},$$

$$(*) \qquad u(x-a)^2 = 2^4 \cdot 3^4 x(x+1).$$

Since one of the numbers x and $x+1$ is even, both divide $|G| = 2^3 \cdot 3 \cdot 5 \cdot u$, and both are coprime to 3, equation $(*)$ implies that 4 is the highest power of 2 dividing either x or $x+1$. Another application of Theorem 3.8 of Landrock [15, p. 438] and Theorem 2 of Brauer and Feit [4, p. 116] asserts that either χ_2 or χ_3 belongs to 2-block B_2 with cyclic defect group $D_2 =_G \langle z \rangle$.

Hence either $a = \chi_2(z) \in \{4, -4\}$ or $a + 1 = \chi_3(z) \in \{4, -4\}$ by the character table of H and the Brauer correspondence between the 2-blocks $\tilde{B}$ of G and the 2-blocks $\tilde{b}$ of H with defect group $\tilde{D} =_G \langle z \rangle$. Thus $a \in \{4, -4, 3, -5\}$. Since $|z^G| = |G : H| \not\equiv 0 \mod (3)$ by Lemma 4.7 (b) and χ_2 belongs to the principal 3-block B_0 of G,

$$\frac{|z^G|\chi_2(z)}{\chi_2(1)} = |G : H| \frac{a}{x} \equiv |G : H| 1 \mod (3).$$

As $\chi_2(1) = x \equiv \psi_3(1) = 1 \mod (3)$, it follows that $a \in \{4, -5\}$.

Suppose that $a = -5$. Then 4 is the precise power of 2 dividing $x + 1$. As x and $x + 1$ are coprime, Theorem 3.12 of Isaacs [9, p. 38] implies that $x(x + 1)$ divides $|G| = 2^3 \cdot 3 \cdot 5u$. By Lemma 4.7 and a previous argument $x(x + 1) = 4 \cdot 5^\sigma v$ for some $\sigma \in \{0, 1\}$ and some odd number v not divisible by 3 and 5. Now $(*)$ implies that

$$u(x + 5)^2 = 2^6 \cdot 3^4 \cdot 5^\sigma v = 2^4 \cdot 3^4 x(x + 1).$$

As 5 is coprime to u, and 5^2 does not divide $x(x + 1)$, we get $\sigma = 0$. Hence u divides v. Since $4v$ divides $|G| = 2^3 \cdot 3 \cdot 5u$, also v divides u. Thus $u = v$, and $(x+5)^2 = 2^6 \cdot 3^4$. Hence $x+5 = 72$ which by $(*)$ implies that $g = 2^3 \cdot 3 \cdot 5 \cdot 17 \cdot 67$. By Lemma 4.8 G has $g/60 = 2278$ Sylow 5-subgroups, a contradiction to Sylow's theorem.

Therefore $a = 4$, and $(x - 4)^2 = 2^6 \cdot 3^4$ by equation $(*)$. Thus $\chi_2(1) = x = 76$, $\chi_3(1) = x + 1 = 77$, and $u = 7 \cdot 11 \cdot 19$, which completes the proof. □

Corollary 4.10. *Let G be a finite group of J_1-type. Then:*

(a) *A Sylow* 7*-normalizer is a Frobenius group of order* 42.
(b) *A Sylow* 11*-normalizer is a Frobenius group of order* 110.
(c) *A Sylow* 19*-normalizer is a Frobenius group of order* 114.

Proof. (a) follows from Lemma 4.5, Theorem 4.9 and Sylow's theorem.

(b) follows from Theorem 4.9, the structure of $C_G(z) = H$, assertion (a), and Sylow's theorem.

(c) is proved in the same way as (b). □

Corollary 4.11. *Let G be a finite group of J_1-type. Then G has* 15 *conjugacy classes $(g_j)^G$ whose representatives g_j and their centralizers $C_G(g_j)$ have the following orders, respectively:*

g_j	1	2	3	5_1	5_2	6	7	10_1	10_2	11	15_1	15_2	19_1	19_2	19_3
$\lvert C_G(g_j)\rvert$	$\lvert G\rvert$	120	30	30	30	6	7	10	10	11	15	15	19	19	19

Furthermore, all conjugacy classes of G are real.

Proof. All assertions follow from Theorem 4.9, Lemma 4.5, Lemma 4.8 and Corollary 4.10. □

Proposition 4.12. *All finite groups G of J_1-type are simple and have the same character table. With the notation of the representatives of the conjugacy classes as in Corollary 4.11 the character values are given in the following table, where $\alpha = -\frac{1}{2}(1+\sqrt{5})$, $\beta = -\frac{1}{2}(1-\sqrt{5})$, and η is a primitive* 19*th root of unity.*

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
1	1	76	77	76	77	77	133	209	56	133	56	133	120	120	120
2	1	4	5	-4	-3	-3	5	1	0	-3	0	-3	0	0	0
3	1	1	-1	1	2	2	1	-1	2	-2	-2	-2	0	0	0
5_1	1	1	2	1	α	β	-2	-1	-2β	$-\alpha$	-2α	$-\beta$	0	0	0
5_2	1	1	2	1	β	α	-2	-1	-2α	$-\beta$	-2β	$-\alpha$	0	0	0
6	1	1	-1	-1	0	0	-1	1	0	0	0	0	0	0	0
7	1	-1	0	-1	0	0	0	-1	0	0	0	0	1	1	1
10_1	1	-1	0	-1	α	β	0	1	0	α	0	β	0	0	0
10_2	1	-1	0	-1	β	α	0	1	0	β	0	α	0	0	0
11	1	-1	0	-1	0	0	1	0	1	1	1	1	-1	-1	-1
15_1	1	1	-1	1	β	α	1	-1	β	$-\alpha$	α	$-\beta$	0	0	0
15_2	1	1	-1	1	α	β	1	-1	α	$-\beta$	β	$-\alpha$	0	0	0
19_1	1	0	1	0	1	1	0	0	-1	0	-1	0	η	η^2	η^4
19_2	1	0	1	0	1	1	0	0	-1	0	-1	0	η^4	η	η^2
19_3	1	0	1	0	1	1	0	0	-1	0	-1	0	η^2	η^4	η

Proof. Let S_p be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. By Theorem 4.9 all these subgroups S_p of G are cyclic of order $|S_p| = p$. Let $N_p = N_G(S_p)$. Then by Brauer's first main theorem on p-blocks G and N_p have the same number of p-blocks of defect one. In particular, each $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ whose degree is divisible by p vanishes on all p-singular elements g of G by Theorem 8.17 of Isaacs [9, p. 133].

Thus G has four 3-blocks, $B_0^{(3)} = \{\chi_1 = 1_G, \chi_2, \chi_3\}$, $B_1^{(3)}$, $B_2^{(3)}$ and $B_3^{(3)}$, of defect one by Lemma 4.8 (b), where χ_1, χ_2, χ_3 are the characters of degrees 1, 76 and 77, respectively, determined in the proof of Theorem 4.9. Using the character table of $N_3 = N$ given in Lemma 4.8, and Theorem 4 of [3, p. 417], one obtains all the values of χ_1, χ_2, and χ_3, because 19 divides $\chi_2(1) = 76$, and 7 and 11 divide $\chi_3(1) = 77$.

The principal 5-block $b_0^{(5)}$ of $N_5 = N$ has four irreducible characters by Lemma 4.8, i.e., $\mathrm{Irr}(b_0^{(5)}) = \{\psi_1, \psi_4, \psi_7, \psi_9\}$. Therefore the principal 5-block $B_0^{(5)}$ of G has 2 non exceptional characters $\lambda_1 = 1_G$, λ_2 and two exceptional characters λ_3 and λ_4. Another application of Theorem 4 of [3, p. 417] yields:

(1) $\lambda_2(s) = \psi_4(s)$ for all 5-singular elements $1 \neq s \in G$.
(2) $\lambda_1(g) + \lambda_2(g) - \lambda_3(g) = 0$ for all 5-regular elements $g \in G$.

Let $y = \lambda_2(1)$. Then $\lambda_3(1) = \lambda_4(1) = y+1$. By Lemma 4.5 G has only one conjugacy class z^G of involutions $z \neq 1$. Let $\lambda_2(z) = b$. Then b is an integer by Lemma 2.15 of [9, p. 20].

Let $c_i = |C_N(2_i)|$ for $i = 1, 2, 3$. Then $c_1 = 20$, $c_2 = 12$ and $c_3 = 4$ by Lemma 4.8 (b). Computing the right hand side of Brauer's group order formula stated in Proposition 4.6 one obtains the equation

$$(*) \qquad \frac{|G|}{|H|^2}\left(1 + \frac{b^2}{y} - \frac{(b+1)^2}{y+1}\right) = \frac{40}{3}.$$

From Theorem 4.8 follows that

$$(**) \qquad 7 \cdot 11 \cdot 19(y-b)^2 = 2^6 \cdot 5^2 y(y+1).$$

Using then an argument similar to that at the end of the proof of Theorem 4.9 and the facts that $y(y+1)$ is even, y and $(y+1)$ are coprime to each other, and both are coprime to 5, it follows from $(**)$ that $(y-b)^2 = 2^8 5^2$ and $b \in \{-4, 3\}$. As 83 does not divide the order of G, we have $b = -4$.

In particular, $\lambda_2 = \chi_4 \neq \chi_2$ is the remaining irreducible character of the unique 2-block $B_1^{(2)}$ of 2-defect 1 of G. Using the character table of H and all the values of χ_2 it is easy to obtain all the values of χ_4.

Furthermore, $\lambda_3 = \chi_5$, $\lambda_4 = \chi_6$ are two new irreducible characters of G with common degree $\chi_5(1) = \chi_6(1) = 77$. Using the character table of $N_5 = N$ stated in Lemma 4.8 and the equations (1) and (2) it is now easy to obtain all the values of χ_5 and χ_6 on all conjugacy classes of G, except on the classes 19_1, 19_2 and 19_3.

From the partial character table of G calculated so far it follows that χ_4, χ_5 and χ_6 belong to the three non principal 3-blocks $B_1^{(3)}$, $B_2^{(3)}$ and $B_3^{(3)}$, respectively. Using the character table of $N_3 = N$ stated in Lemma 4.8 it can be checked that their Brauer correspondents $b_1^{(3)}$, $b_2^{(3)}$ and $b_3^{(2)}$ in N consist of the irreducible characters $\mathrm{Irr}(b_1^{(3)}) = \{\psi_2, \psi_4, \psi_6\}$, $\mathrm{Irr}(b_2^{(3)}) = \{\psi_7, \psi_8, \psi_{11}\}$ and $\mathrm{Irr}(b_3^{(3)}) = \{\psi_9, \psi_{10}, \psi_{12}\}$, respectively.

Let $\mathrm{Irr}(B_1^{(3)}) = \{\chi_4, \mu_1, \mu_2\}$, $x = \mu_1(1)$ and $\mu_1(z) = c$. Then Proposition 4.6 and the character table of N given in Lemma 4.8 yield the equation

$$(***) \qquad \frac{|G|}{|H|^2}\left(\frac{4^2}{76} + \frac{c^2}{x} - \frac{(c-4)^2}{x+76}\right) = \frac{24}{5}.$$

From Theorem 4.9 follows that $7 \cdot 11(x + 19c)^2 = 2^4 \cdot 3^2 x(x+76)$. Hence $x = 133$ and $c = 5$. Thus $\mu_2(1) = 133 + 76 = 209$, and $\mu_2(z) = -4 + \mu_1(z) = 1$. Therefore μ_1 and μ_2 are new irreducible characters χ_7 and χ_8 of G, respectively. Furthermore, their character values are completely determined on all conjugacy classes of G, except on 5_1, 5_2, 10_1 and 10_2.

The remaining 3-blocks $B_2^{(3)} = \{\chi_5, \chi_9, \chi_{10}\}$ and $B_3^{(3)} = \{\chi_6, \chi_{11}, \chi_{12}\}$ of G are algebraic conjugate because their Brauer correspondents $b_2^{(3)}$ and $b_3^{(3)}$

are algebraic conjugate 3-blocks of $N_3 = N$ by Lemma 4.8 (b). Thus it suffices to determine the character values of $\chi \in \mathrm{Irr}_{\mathbb{C}}(B_2^{(3)})$.

Let $\chi_9(1) = x$ and $\chi_9(z) = e$. Then by Proposition 4.6 and Lemma 4.8 (b) we get

$$\frac{|G|}{|H|^2}\left(\frac{2\cdot 9}{77} - \frac{2e^2}{x} - \frac{2(e-3)^2}{x+77}\right) = \frac{6}{5}.$$

Now Theorem 4.9 implies that

$$19(3x + 77e)^2 = 2^3 3^2 x(x + 77).$$

If e were not zero, then $e = 3e_1$ for some integer $e_1 > 0$. Hence

$$19(x + 77e_1)^2 = 8x(x + 77),$$

a contradiction. Therefore $e = 0$, and $x = 56$. Thus $\chi_{10} = 133$, and $\chi_{10}(z) = \chi_5(z) = -3$. Using Lemma 4.8 all other character values of these two irreducible characters of G can be computed on all other conjugacy classes of G which are different from 5_i and 10_i for $i = 1, 2$.

The missing values of the six irreducible character χ_j with $7 \le j \le 12$ on the G-classes 5_i, 10_i for $i = 1, 2$ follow easily from Lemma 4.8 and the 5-block distribution of these characters of G into the 5-blocks:

$$B_1^{(5)} = \{\chi_3, \chi_7, \chi_9, \chi_{11}\} \text{ and } B_2^{(5)} = \{\chi_2, \chi_8, \chi_{10}, \chi_{12}\}.$$

By Corollary 4.10 the principal 19-block $B_0^{(19)}$ of G has six non-exceptional characters and three exceptional characters. From the character table of G determined so far follows that

$$\mathrm{Irr}_{\mathbb{C}}(B_0^{(19)}) = \{\chi_1, \chi_3, \chi_5, \chi_6, \chi_9, \chi_{11}, \chi_{13}, \chi_{14}, \chi_{15}\},$$

where χ_{13}, χ_{14} and χ_{15} are the exceptional characters. All other $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ are of 19-defect zero. Therefore Theorem 4.9 and Maschke's theorem imply that $\chi_i(1) = 120$ for $i = 13, 14, 15$. In particular, these three characters vanish on all 2-, 3-, and 5-singular elements. Their other values can be calculated from the character table of the Frobenius group $N_{19} = 19 : 6$ and Theorem 4 of Brauer [3, p. 416–417]. $\square$

Proposition 4.13. *Each finite group G of J_1-type has an absolutely irreducible 11-modular representation of degree 7.*

Proof. By Proposition 4.12 all groups G of J_1-type have the same character table. It coincides with the character table of Janko's sporadic simple group J_1 stated in [10, p. 148–149]. Using only the data contained in this character table, G. Hiss and K. Lux have proved in [8, p. 74] that the principal 11-block $B_0^{(11)}$ of J_1 and therefore of G has a uniquely determined Brauer tree:

$$\underset{\chi_2}{\bullet}\overset{1}{\text{———}}\underset{\chi_{13}}{\bullet}\overset{119}{\text{———}}\underset{\chi_{10}}{\bullet}\overset{14}{\text{———}}\underset{\chi_{14}}{\bullet}\overset{106}{\text{———}}\underset{\chi_7}{\bullet}\overset{27}{\text{———}}\underset{\chi_2}{\bullet}\overset{49}{\text{———}}\underset{\chi_{11}}{\bullet}\overset{7}{\text{———}}\underset{\chi_4}{\bullet}\overset{69}{\text{———}}\underset{\chi_{12}}{\bullet}\overset{64}{\text{———}}\underset{\chi_{15}}{\bullet}\overset{56}{\text{———}}\underset{\chi_9}{\bullet}$$

In particular, each finite group G of J_1-type has an absolutely irreducible 11-modular character φ of degree $\varphi(1) = 7$. $\square$

THEOREM 4.14. *Up to isomorphism there exists only one finite group G of J_1-type with involution centralizer*

$$C_G(z) \cong H = \langle z \rangle \times A_5.$$

It is isomorphic to the finite simple group J of order $|J| = 175560$ described in Theorem 4.1.

Proof. This is done by verifying the conditions of the uniqueness criterion given in Theorem 2.1. By Theorem 4.1 there exists a finite simple group J having an involution $\mathcal{Z}$ with $C_J(\mathcal{Z}) \cong H \cong \langle z \rangle \times A_5$ such that $|J| = 175560$, and J satisfies all the conditions (1) and (2), (3) and (5) of Theorem 2.1 for the primes $p = 2$, $r = 11$ and the minimal dimension $n = 7$. Furthermore, by Remark 4.3 condition (4) also holds.

Theorem 4.9 asserts that all finite groups G of J_1-type have order $|G| = 175560 = |J|$. By Proposition 4.13 each such group G has an absolutely irreducible 11-modular representation of degree 7. Furthermore, Lemma 4.2 asserts that $N_G(A) \cong N_J(\mathfrak{A})$, where A and $\mathfrak{A}$ are Sylow 2-subgroups of G and J, respectively. Applying now Theorem 2.1 for the primes $p = 2$ and $r = 11$ we get $G \cong J$. $\square$

5. Uniform uniqueness proofs for other sporadic simple groups

Since the literature on the classification of the finite simple groups with dihedral and semihedral Sylow 2-subgroups is comprehensive, the case of the smallest Mathieu group M_{11} has not been considered in [17], nor is it considered here. As mentioned in the introduction, the 2-central involution centralizers H of the Mathieu groups M_{12}, M_{22}, M_{23} and M_{24} do not satisfy condition (4) of Theorem 2.1.

On the other hand, the common 2-central involution centralizer H of Janko's simple groups J_2 and J_3 does not satisfy the (theoretical) conditions (a) and (b) of Theorem 2.1. Finally, O'Nan's simple group ON fails to have suitable local subgroups which satisfy conditions (1), (3) and (5) of Theorem 2.1.

However, Theorem 2.1 does not only allow a uniqueness proof for Janko's simple group J_1, but also *uniform* uniqueness proofs for the ten other sporadic simple groups: J_4, HS, McL, Suz, Ru, Ly, Co_1, Co_3, Fi_{22} and Fi_{23}.

In [7, p. 86 and p. 173] Gorenstein states that all these groups have a characterization by means of the structure of a 2-central involution centralizer H. In the papers cited there it is mentioned that for a given H all simple groups G of H-type have the same character table and the same group order. Furthermore, in each of these 10 isomorphism classes of groups it follows that such a simple group G has an irreducible ordinary character χ of minimal degree $n = \dim_F(W)$ given in Table 5.1 below.

Hence the theoretical conditions (a), (b) and (c) of Theorem 2.1 hold for all these groups. Furthermore, conditions (1), (2), and (d) can easily be verified from the character table and the given group structure of H in all cases except Ly and Fi_{23}. For Lyons' simple group a complete uniqueness proof is given by Weller in [21]. It builds on the article [13] by M. Kratzer, W. Lempken, K. Waki and the author, which gives a new uniqueness proof for McLaughlin's group. In [11] and a subsequent paper, M. Kratzer has verified all conditions of Theorem 2.1 for the 2-central involution centralizer H of Rudvalis type. The same has been done in [18] for the 2-central involution centralizer H of J_4-type by Weller, R. Staszewski and the author.

In Table 5.1 below, each of the 10 sporadic simple groups J_4, HS, McL, Suz, Ru, Ly, Co_1, Co_3, Fi_{22} and Fi_{23} is defined by the structure of a given centralizer $H = C_G(t)$ of a 2-central involution. For each of them the explicit group structure of H is given in the table. The other data given there are obtained as follows.

For each of these 10 groups H Algorithm 4.6 of [17] has been applied to determine:

(a) A Sylow 2-subgroup S of H.
(b) A non-cyclic elementary abelian normal subgroup A of order $|A| = 2^d$ of S such that $D = N_H(A)$ is maximal in the sense of condition (2) of Theorem 2.1.
(c) An extension $\hat{\eta} : M \to \mathrm{GL}_d(2)$ with kernel $\ker(\hat{\eta}) = C = C_H(A)$ of the natural homomorphism $\eta : D \to \mathrm{GL}_d(2)$ induced by conjugation of D on A such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] & M \arrow[r, "\hat{\eta}"] & \hat{\eta}(M) \arrow[r] & 1 \\
1 \arrow[r] & C \arrow[r] \arrow[u, "id"'] & D \arrow[r, "\eta"] \arrow[u, "\mu"'] & \eta(D) \arrow[r] \arrow[u, "id"'] & 1
\end{tikzcd}$$

is commutative for some embedding $\mu : D \to M$ with the following properties:
(c1) $\eta(D) = \{\varphi \in \hat{\eta}(M) \mid t\varphi = t \in A\} < \hat{\eta}(M)$,
(c2) $|\hat{\eta}(M) : \eta(D)|$ is odd.
For each of the eight cases different from Ly and Fi_{23} the structure of the groups D and M is given in Table 5.1 below.
(d) It has also been shown that the groups M are uniquely determined (up to isomorphism) by H being the centralizer of a 2-central involution in any finite simple group G of H-type.

Using then Kratzer's algorithms [11] it has been shown:

(e) Condition (4) of Theorem 2.1 holds, i.e., there is up to equivalence only one embedding $\mu : D \to M$.
(f) There is a uniquely determined faithful compatible pair

$$(\nu, \omega) \in \mathrm{mfchar}_{\mathbb{C}}(H) \times \mathrm{mfchar}_{\mathbb{C}}(M)$$

of minimal degree n in the sense of Algorithm 3.6, Step 4.

In the five cases leading to the simple sporadic groups HS, McL, Co_1, Co_3 and Fi_{22} the compatible pair determines uniquely an irreducible r-modular representation $\kappa : H *_D M \to \mathrm{GL}_n(\mathrm{GF}(r))$ of the free product $H *_D M$ of H and M with amalgamated subgroup D, because the number t of double cosets described in Proposition 3.2 equals

$$t = (r-1)^{k+1-l-m} = (r-1)^0 = 1.$$

In each case the irreducible constituents of the restrictions κ_H, κ_D and κ_M of κ to H, D and M are given in Table 5.1 by the natural numbers describing their dimensions over the Galois field $\mathrm{GF}(r)$ with r elements. The invariant t is also 1 in the case of Fischer's sporadic group Fi_{23} if the 2-centralizer H is replaced by the normalizer $L = 3^{1+8} \cdot 2^{1+6} \cdot 3^{1+2} \cdot 2S_4$ of a 3-central element; see Table 5.1 below.

In the cases J_4, Suz, Ru also Step 4 of Algorithm 3.6 has been performed by Kratzer and Weller. It turned out that in each case $H *_D M$ has a uniquely determined irreducible r-modular representation of degree n given in Table 5.1.

Much more difficult is the remaining case leading to Lyons' simple group Ly. In [21] Weller constructs from the given group $H = 2A_8$ a 5-central element y and shows that in each finite simple group G of H-type its normalizer satisfies $L = N_G(\langle y \rangle) \cong 5^{1+4} : 4S_6$. Then he determines a unique maximal elementary abelian normal subgroup A of order $|A| = 5^3$ in a Sylow 5-subgroup S of L. From there he constructs $D = N_L(A) \cong 5^3(5 : \mathrm{GL}_2(5))$. He shows that D can be embedded uniquely into $M \cong 5^3 \cdot L_3(5)$ by a result analogous to Theorem 2.3 of [17]. Applying Kratzer's algorithm [11] he gets a uniquely determined compatible pair $(\nu, \omega) \in \mathrm{mfchar}_{\mathbb{C}} \times \mathrm{mfchar}_{\mathbb{C}} M$ of minimal degree $n = 2480$. For $r = 11$ the corresponding semi-simple multiplicity-free 11-modular representations of H, D, and M are constructed. Weller applied Step 4 of Algorithm 3.8 and checked that the free product $L *_D M$ with amalgamated subgroup D has precisely one irreducible 11-modular representation of degree 2480. The dimensions of the irreducible constituents of the 3 semi-simple restrictions are given in Table 5.1 below. Using the theoretical results of Lyons' original paper Weller has verified all remaining conditions of the uniqueness criterion stated in Theorem 2.1 in his paper [21].

In the table below the degrees of the faithful irreducible constituents of an r-modular semi-simple module M are printed in boldface.

G	$H = C_G(t)$	p	r	$\dim_F(W)$	$W_{\vert L} = W_1 \oplus \mathbf{W_2}$
J_4	$2^{1+12}.3M_{22}:2$	2			
	$L = H$		13	1333	$693 \oplus \mathbf{640}$
	$D = 2^{11}:(2^6:3S_6)$				$(45 \oplus 288 \oplus 360) \oplus \mathbf{640}$
	$M = 2^{11}:M_{24}$				$45 \oplus \mathbf{1288}$
HS	$(4 * 2^4):S_5$	2			
	$L = H$		7	22	$(4 \oplus 10) \oplus \mathbf{8}$
	$D = 2^4.(S_4 \times 2)$				$(1 \oplus 3 \oplus 4 \oplus 6) \oplus \mathbf{8}$
	$M = 2^4S_6$				$(1 \oplus 7) \oplus \mathbf{14}$
McL	$2A_8$	2			
	$L = H$		11	22	$14 \oplus \mathbf{8}$
	$D = 2^4:L_3(2)$				$(1 \oplus 6 \oplus 7) \oplus \mathbf{8}$
	$M = 2^4:A_7$				$(1 \oplus 6) \oplus \mathbf{15}$
Suz	$2^{1+6}.U_4(2)$	2			
	$L = H$		11	143	$(1 \oplus 15 \oplus 27 \oplus 36) \oplus (\mathbf{32_1} \oplus \mathbf{32_2})$
	$D = (2^{4+6}:3):S_4$				$(1_1 \oplus 1_2 \oplus 3 \oplus 6 \oplus 8 \oplus 12 \oplus 24_1 \oplus 24_2)$ $\oplus(\mathbf{32_1} \oplus \mathbf{32_2})$
	$M = 2^{4+6}:3A_6$				$(5 \oplus 18) \oplus \mathbf{120}$
Ru	$2.2^{4+6}:S_5$	2			
	$L = H$		11	378	$(6 \oplus 60 \oplus 120) \oplus \mathbf{192}$
	$D = 2^{3+8}:S_4$				$(6 \oplus 12 \oplus 24 \oplus 48) \oplus (\mathbf{64} \oplus \mathbf{96} \oplus \mathbf{128})$
	$M = 2^{3+8}:L_3(2)$				$42 \oplus \mathbf{336}$
Co_1	$2^{1+8}.O_8^+(2)$	2			
	$L = H$		13	276	$(28 \oplus 120) \oplus \mathbf{128}$
	$D = 2^{11}:(2^4:A_8)$				$(28 \oplus 120) \oplus \mathbf{128}$
	$M = 2^{11}:M_{24}$				$\mathbf{276}$
Co_3	$2Sp_6(2)$	2			
	$L = H$		11	23	$15 \oplus \mathbf{8}$
	$D = 2^4(2^3:L_3(2))$				$(1 \oplus 7_1 \oplus 7_1) \oplus \mathbf{8}$
	$M = 2^4A_8$				$(1 \oplus 7) \oplus \mathbf{15}$

TABLE 5.1. Sporadic simple groups satisfying conditions (4), (5) of Theorem 2.1

G	$H = C_G(t)$	p	r	$\dim_F(W)$	$W_{\mid L} = W_1 \oplus \mathbf{W_2}$
Ly	$2A_{11}$	5			
	$L = 5^{1+4} : 4S_6$		11	2480	$480 \oplus (\mathbf{400} \oplus \mathbf{800_1} \oplus \mathbf{800_2})$
	$D = 5^3.(5^2 : GL_2(5))$				$(240_1 \oplus 240_2) \oplus (\mathbf{200} \oplus \mathbf{300} \oplus \mathbf{400}$ $\oplus \mathbf{500} \oplus \mathbf{600})$
	$M = 5^3.L_3(5)$				$\mathbf{2480}$
Fi_{22}	$(2 \times [2^{1+8} : U_4(2)]) : 2$	2			
	$L = H$		13	78	$(6 \oplus 40) \oplus \mathbf{32}$
	$D = 2^{10} : (2^4 : S_5)$				$(1 \oplus 5 \oplus 40) \oplus \mathbf{32}$
	$M = 2^{10} : M_{22}$				$1 \oplus \mathbf{77}$
Fi_{23}	$2Fi_{22}$	3			
	$L = 3^{1+8}.2^{1+6}.3^{1+2}.2S_4$		13	782	$(8 \oplus 288) \oplus \mathbf{486}$
	$D = 3^{3+7}.[2 \times (3^2 : 2S_4)]$				$(8 \oplus 18 \oplus 54 \oplus 216) \oplus \mathbf{486}$
	$M = 3^{3+7}.(2 \times L_3(2))$				$(26 \oplus 54) \oplus \mathbf{702}$

TABLE 5.1. Sporadic simple groups satisfying conditions (4), (5) of Theorem 2.1 (continued)

Acknowledgements. The author gratefully acknowledges mathematical advice and computational assistance from his collaborators M. Kratzer, W. Lempken and M. Weller.

REFERENCES

[1] H. Bender, *Finite groups with large subgroups*, Illinois J. Math. **18** (1974), 223–228.

[2] R. Brauer, *Some applications of the theory of blocks of characters of finite groups III*, J. Algebra **3** (1966), 225–255.

[3] ———, *On groups whose order contain a prime number to the first power I*, Amer. J. Math. **64** (1942), 401–420.

[4] R. Brauer and W. Feit, *On the number of irreducible characters of finite groups in a given block*, Proc. National Acad. Sciences **45** (1959), 361–365.

[5] W. Feit, *The representation theory of finite groups*, North–Holland, Amsterdam, 1982.

[6] D.M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. **111** (1980), 377–406.

[7] D. Gorenstein, *Finite simple groups. An introduction to their classification*, Plenum Press, New York, 1982.

[8] G. Hiss and K. Lux, *Brauer trees of sporadic groups*, Clarendon, Oxford, 1989.

[9] I.M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

[10] Z. Janko, *A new finite simple group with abelian Sylow 2–subgroups and its characterization*, J. Algebra **3** (1966), 147–186.

[11] M. Kratzer, *Konkrete Charaktertafeln und kompatible Charaktere*, Dissertation, Universität Essen, 2001.

[12] M. Kratzer and G.O. Michler, *A natural existence proof for Janko's sporadic group J_1*, Representation theory of finite groups and related topics (Kyoto, 1998), Surikaisekikenkyusho Kokyuroku No. 1149, Research Inst. Math. Sciences, Kyoto University, Kyoto, 2000, pp. 39–53.

[13] M. Kratzer, W. Lempken, G.O. Michler, and K. Waki, *Another existence and uniqueness proof for McLaughlin's simple group*, J. Group Theory, to appear.

[14] H. Kurzweil and B. Stellmacher, *Theorie der endlichen Gruppen*, Springer–Verlag, Berlin, 1998.

[15] P. Landrock, *On the number of irreducible characters in a 2–block*, J. Algebra **68** (1981), 426–442.

[16] W. Lempken, *2–local amalgams for the simple groups* $\mathrm{GL}(5,2)$*, M_{24} and He*, Illinois J. Math. **47** (2003), 361–393.

[17] G.O. Michler, *On the construction of the finite simple groups with a given centralizer of a 2–central involution*, J. Algebra **234** (2000), 668–693.

[18] G.O. Michler, R. Staszewski, and M. Weller, *A new uniqueness proof for Janko's simple group J_4*, in preparation.

[19] M. Schönert et al., *GAP 3.4 Manual (Groups, Algorithms and Programming)*, RWTH Aachen, Aachen, Germany, 1994.

[20] J.G. Thompson, *Finite–dimensional representations of free products with an amalgamated subgroup*, J. Algebra **69** (1981), 146–149.

[21] M. Weller, *A new uniqueness proof for Lyons' simple group*, in preparation.

Institut für Experimentelle Mathematik, Universität GH Essen, Ellernstrasse 29, D-45326 Essen, Germany

E-mail address: `michler@exp-math.uni-essen.de`