ON THE EXISTENCE OF A COMPLEMENT FOR A FINITE SIMPLE GROUP IN ITS AUTOMORPHISM GROUP

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Abstract. In this paper we determine all finite simple groups $G$ for which the automorphism group $\text{Aut} G$ splits over $G = \text{Inn} G$.

The theory of group extensions, and, in particular, the study of conditions which force the splitting of a given extension or class of extensions, is one of the themes with which the name of Reinhold Baer is associated. The present article gives a concrete, very special instance of this type of study: we examine the automorphism groups of the finite non abelian simple groups to determine those groups $G$ for which $\text{Aut} G$ splits over $G$, where we identify $G$ with the inner automorphism group $\text{Inn} G$. For such groups, the structure of the complement for $\text{Inn} G$ in the automorphism group $\text{Aut} G$ is of course well known: the complement is isomorphic to the outer automorphism group $\text{Out} G$ (see [2]).

The question we are considering is very natural and easily stated; yet, it seems that only very partial results are known (see [6], [7]).

In fact, this is a problem on simple groups of Lie type, since the remaining cases are easily dealt with. Indeed, if $n \geq 5$, $n \neq 6$, $\text{Sym}(n) = \text{Aut}(\text{Alt}(n))$ always splits over $\text{Alt}(n)$, while $\text{Alt}(6) \cong \text{PSL}(2,9)$ has no complement in $\text{Aut}(\text{Alt}(6))$. Similarly, all automorphism groups of the sporadic simple groups split over their socle: if $G$ is a sporadic group, then either $\text{Aut} G = \text{Inn} G$ or $\text{Inn} G$ has index 2 in $\text{Aut} G$, and in each case there exists a conjugacy class of non-inner involutions in $\text{Aut} G$ (see [2]).

On the other hand, the behaviour of groups of Lie type is not so uniform: it depends on the type of the group and on some arithmetical conditions involving the cardinality of the field and the rank of the group. The following theorem collects our results.

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Theorem. Let $G$ be a simple group of Lie type over a finite field with $q = p^m$ elements, $p$ prime, and denote by $d$ the order of the abelian group $H/H$, where $H$ is the group of diagonal automorphisms of $G$ and $H$ is the subgroup of $H$ consisting of those diagonal automorphisms which are inner. (The values of $d$ for untwisted and twisted groups are given in the tables in Sections 3 and 4.) Then $\text{Aut } G$ splits over $G$ if and only if one of the following conditions holds:

1. $G$ is untwisted, not of type $D_l(q)$, and $(\frac{q^l-1}{d}, d, m) = 1$;
2. $G = D_l(q)$ and $(\frac{q^l-1}{d}, d, m) = 1$;
3. $G$ is twisted, not of type $2D_l(q)$, and $(\frac{q+1}{d}, d, m) = 1$;
4. $G = 2D_l(q)$ and either $l$ is odd or $p = 2$.

The paper is divided into four sections. In Sections 1 and 2 we study the groups $A_n(q)$ and $2A_n(q)$, respectively, using their natural projective representations; in Sections 3 and 4 we consider the remaining untwisted (respectively twisted) groups of Lie type.

1. The special linear groups

Let $F = F_q$ be the finite field with $q$ elements, where $q = p^m$ for some prime number $p$. We fix a generator $\lambda$ of the multiplicative group of the field $F_q$. As usual, $GL(n, q)$ (resp. $SL(n, q)$) will denote the general (resp. special) linear group of degree $n$ over the field $F_q$. In the following we will identify $F^*$ with the subgroup of $GL(n, q)$ consisting of scalar matrices, and let $PGL(n, q) = GL(n, q) \langle \phi \rangle = GL(n, q)/(F^*)$, $PSL(n, q) = SL(n, q) F^*/F^*$. For an element $g \in GL(n, q)$ its image in $PGL(n, q)$ will be denoted with $\bar{g}$. Also, as usual, $\det(g)$ will denote the determinant of a matrix $g$.

Throughout this section, we will consider $G = A_{n-1}(q) = PSL(n, q)$, for $n$ and $q$ fixed. Let $\phi$ be the Frobenius automorphism of $F$, defined by $a^\phi = a^p$ (using the exponential notation for automorphisms). Then $\phi$ induces an automorphism of $GL(n, q)$ of order $m$, which will also be denoted by $\phi$, given by $(a_{ij})^\phi = (a_{ij}^p)$ for $i, j = 1, \ldots, n$.

Let $\iota : GL(n, q) \to GL(n, q)$ be the automorphism defined by $g^\iota = (g^T)^{-1}$, where $g^T$ denotes the transpose matrix of $g$.

Both $\phi$ and $\iota$ induce automorphisms $\bar{\phi}$ and $\bar{\iota}$ of $PGL(n, q)$. $\bar{\phi}$ generates the group of field automorphisms, $\bar{\iota}$ is the product of the graph automorphism and an inner automorphism if $n \geq 3$, and it is an inner automorphism if $n = 2$. As $G$ is simple, we may also identify $G$ with $\text{Inn } G \leq \text{Aut } G$.

We have the sequence of normal subgroups

$$\text{SL}(n, q) \leq \text{GL}(n, q) \leq \Gamma L(n, q) = \text{GL}(n, q)/\langle \phi \rangle \leq \Gamma L(n, q)/\langle \iota \rangle.$$ 

Taking quotients modulo the scalar matrices we obtain

$G \leq \text{PGL}(n, q) \leq \text{PGL}(n, q)/\langle \bar{\phi} \rangle \leq \text{Aut } G = \text{PGL}(n, q)/\langle \bar{\iota} \rangle$. 

Also, PGL$(n,q)/G$ is cyclic of order $d = (n,q - 1)$ and $\bar{\phi}$ acts on it as the $p$-th power. We want to prove that $G$ has a complement in Aut $G$ if and only if \(\frac{\alpha}{\beta} \equiv 1 \mod d\). Letting $t$ be the product of all prime factors of $d$ dividing $\frac{\alpha}{\beta}$, counting multiplicities, this is equivalent to proving that $G$ has a complement in Aut $G$ if and only if $(t,m) = 1$.

**Lemma 1.1.**

(i) $\langle \bar{\phi} \rangle$ is a complement for PSL$(n,q)$ in PGL$(n,q)$ if and only if $\det(\bar{\phi}) = \lambda^n$ with $(u,d) = 1$ and $g^d \in F^\ast$.

(ii) Assume that $G$ has a complement $C$ in PTL$(n,q)$. Then it is possible to choose $g \in \text{GL}(n,q)$ and $h \in \text{SL}(n,q)$ such that $C = \langle \bar{\phi} \rangle = \bar{\phi}^1$, $\det(g) = \lambda$, $|\bar{\phi}| = d$ and $\bar{\phi}h = \bar{\phi}^d$.

**Proof.** (i) Suppose that $\det(\bar{\phi}) = \lambda^n$. Then $\bar{\phi}$ generates PGL$(n,q)$ modulo PSL$(n,q)$ if and only if $\lambda^n$ generates $F^\ast$ modulo $(F^n)^n$, that is, and only if $(u,d) = 1$. Therefore $\langle \bar{\phi} \rangle$ is a complement if and only if we have that $\bar{\phi}^d = 1$, that is, $g^d \in F^\ast$.

(ii) Choose $g$ such that $\langle \bar{\phi} \rangle = C \cap G$. As $\bar{\phi}$ generates PGL$(n,q)$ modulo PSL$(n,q)$, we have that $\det(g) = \lambda^n$ with $(u,d) = (u,n,q - 1) = 1$. Let $r,s,v \in \mathbb{Z}$ be such that $ru + sn + v(q - 1) = 1$. Then $\det(\bar{\phi}^r) = \lambda$ and we may replace $g$ by $\bar{\phi}^r$. The remaining statements follow from the fact that the projection $\pi : C \to \langle \bar{\phi} \rangle G/G$ is an isomorphism.

**Lemma 1.2.** Assume that $G$ has a complement in PTL$(n,q)$. Then $(m,t) = 1$.

**Proof.** Let $g,h$ be as in Lemma 1.1 (ii), so that $g^d = \lambda^n \in F^\ast$. Taking the determinant of both sides we have that $\lambda^d = \det(g)^d = (\lambda^n)^d$. So $d \equiv \alpha n \mod q - 1$, that is, $1 \equiv \alpha (n/d) \mod (q - 1)/d$ and thus $(\alpha,\frac{q - 1}{d}) = 1$. It follows that $(\alpha, t) = 1$.

We may view $\phi h$ as a ring automorphism of the ring Mat$(n,q)$ of $n \times n$ matrices with entries in $F$. As $g^dh = g^p$, we have that $g^{\phi h} = (g^d)^p$ for some $z \in F^\ast$, so $\phi h$ normalizes the subring $F[g]$ of Mat$(n,q)$ (where, as usual, $F$ is identified with the ring of scalar matrices). Now the map $\pi : F[g] \to F[g]$, defined by $v^\pi = v^p$ is also a ring automorphism of $F[g]$, and $\phi h^{-1}$ is a ring automorphism which centralizes $F$. So $\lambda^\alpha = g^d = (g^{\phi h})^{-1} = (g^{\phi h^{-1}})^d = (g^d)^3 = \lambda^\alpha z^d$ and $z = 1$. Thus we may assume that $z = \lambda^{\beta(q - 1)/d}$ for some integer $\beta$. It is easy to see that $g^{\phi h^{-1}} = g^{\phi h^{-1}}$ for each natural number $i$. As $(\phi h)^m$ is a scalar matrix, we obtain that $g = g^{\phi h^m} = g^{p^m z^{mp^m}} = g^{q z^m} = g^d z^m$, so $g^{\phi h^{-1}} = z^{-m}$. As $g^{\phi h^{-1}} = g^{d \omega^{-1}} = \lambda^{\alpha \omega^{-1}}$, we have that $\alpha \equiv -\beta m \mod d$, so $(m,t) | (\alpha,t) = 1$, as we wanted to prove.
We now seek a complement for \( G \) in \( \text{PGL}(n,q) \). If \( n = 2 \), we find \( g \in \text{GL}(n,q) \) such that \( \det(g) = \lambda, \ g^d \in F^* \), and \( \langle g \rangle \) is normalized by \( \phi \); if \( n \geq 3 \), we find a matrix \( g \) with the above properties and such that \( \langle g \rangle \) is normalized by \( vu \), for a suitable matrix \( u \in \text{GL}(n,q) \) such that \( \langle vu \rangle^2 = 1 \) and \( vu \) commutes with \( \phi \).

**Lemma 1.3.** Let \( d = tl, \ d_1|d, \ d_1 = t_1l_1 \), where \( t_1 = (d_1, t) \). There exist \( v_1, \ldots, v_{n/t_1} \in F \) and \( u \in Z \) such that \( (u, t_1) = 1, \ v_j^{t_1} = 1 \) for \( j = 1, \ldots, n/t_1 \), and

\[
\prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^u v_j = \lambda^{d/d_1}.
\]

**Proof.** Assume that a prime \( r \) divides \( \frac{d-1}{d} \), so \( r \) divides neither \( l \), as \( \left( \frac{d-1}{d}, l \right) = 1 \), nor \( \frac{d}{d_1} \), as \( \left( \frac{d}{d_1}, \frac{d-1}{d} \right) = 1 \). It follows that \( \left( \frac{d-1}{d}, \frac{n}{d} \right) = \left( \frac{d-1}{d}, \frac{t}{l} \right) = \left( \frac{d-1}{d}, \frac{t}{l} \right) = \frac{d}{d_1} \).

We now distinguish two cases. If \( t_1 \) is odd or \( \frac{n}{t_1} \) is even, we take \( u, y \in Z \) such that \( y \frac{d-1}{d} + u \frac{n}{t_1} = \frac{d}{d_1} \). Note that, by dividing both sides by \( \frac{d}{d_1} \), we get

\[
y \frac{d-1}{d} + u \frac{n}{t_1} = 1, \text{ so } (u, t_1) = 1.
\]

If \( t_1 \) is even and \( \frac{n}{t_1} \) is odd, then \( \frac{d}{d_1} \) divides \( \frac{d-1}{d} \), so we may take \( u, y \in Z \) such that \( y \frac{d-1}{d} + u \frac{n}{t_1} = \frac{d}{d_1} \). Again, dividing by \( \frac{d}{d_1} \), we get

\[
y \frac{d-1}{d} + u \frac{n}{t_1} = 1 + \frac{d-1}{d_1}.
\]

In both cases \( u \) has the desired properties, and taking \( \lambda = y \frac{d-1}{d} \), \( v_j = 1 \) for \( j \neq 1 \), we have

\[
\prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^u v_j = (-1)^{n/t_1} \lambda^u \left( y \frac{d-1}{d} + u \frac{n}{t_1} \right) = \lambda^{d/d_1}.
\]

We now describe a construction which will be used in the sequel.

**Lemma 1.4.** Let \( d_1 = t_1l_1 \) be as above. Take \( u \in Z \) and \( v_1, \ldots, v_{n/t_1} \in F \) such that \( v_j^{t_1} = 1 \) for every \( j = 1, \ldots, n/t_1 \), and \( \prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^u v_j = \lambda^{d/d_1} \). Then there exists a matrix \( g \in \text{GL}(n,q) \) such that \( g^{d_1} \in F^* \) and \( \det(g) = \lambda^{d/d_1} \).

**Proof.** Note that Lemma 1.3 ensures the existence of \( u \) and \( v_1, \ldots, v_{n/t_1} \) with the required properties. Let \( j \in \{1, \ldots, n/t_1 \} \), \( c = \lambda^u \) and \( c_j = cv_j \). Consider the commutative ring \( V_j = F[w_j] \), where \( w_j \) has minimal polynomial \( x^{t_1} - c_j \) over \( F \), that is, \( F[w_j] \) is isomorphic to the quotient of the polynomial ring \( F[x] \) over the ideal \( (x^{t_1} - c_j) \). Then \( V_j \) is a vector space of dimension \( t_1 \) over \( F \) and a basis is \( \{1, w_j, w_j^2, \ldots, w_j^{t_1-1} \} \). We have that \( w_j \) acts on \( V_j \) via
right multiplication, and the matrix associated to this endomorphism with respect to the fixed basis is

\[
g_j = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & 1 \\
c_j & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

Note that \(\det(g_j) = (-1)^{t_j-1}\lambda^s v_j\).

Also \(g_j^{d_1} = (g_j^{t_j})^{t_j} = (c_j)^{t_j} = (cv_j)^{t_j} = c^{t_j}\). Let \(V = \oplus_{j=1}^{n/t} V_j\) and let \(g\) be the matrix

\[
g = \begin{pmatrix}
g_1 \\
\ddots \\
g_{n/t_1}
\end{pmatrix};
\]

then \(g^{d_1} = c^{t_1} \in F^*\) and \(\det(g) = \prod_{j=1}^{n/t_1} (-1)^{t_j-1}\lambda^s v_j = \lambda^{d/d_1}\), as required.

\[\square\]

**Proposition 1.5.** \(\text{PSL}(n, q)\) is complemented in \(\text{PGL}(n, q)\).

**Proof.** Take \(v_1, \ldots, v_{n/t} \in F\) and \(u \in \mathbb{Z}\) as in Lemma 1.3, with \(d_1 = d\), and let \(g\) be the matrix constructed in Lemma 1.4. Then \(\langle \bar{g} \rangle\) is the required complement.

We will also need the following observation:

**Observation 1.6.** Consider the polynomial \(x^s - c\), where \(c \in F\) and \(s|q - 1\). If \(c = \lambda^s\), where \((u, s) = 1\), then \(x^s - c\) is irreducible in \(F[x]\).

**Lemma 1.7.** Let \(F[w]\) be a field, where \(w\) has minimal polynomial \(x^s - c\) over \(F\) and \(s|q - 1\). Assume also that \((s, m) = 1\) and let \(k \in \mathbb{N}\) be such that \(mk \equiv -1 \mod s\). Let \(\pi : F[w] \to F[w]\) be the map defined by \(w^\pi = w^p\). Then \(\psi = \pi^{mk+1}\) is an automorphism of \(F[w]\) of order \(m\) such that \(a^\psi = a^p\) for every \(a \in F\) and \(w^\psi = (wz)^p\), where \(z = c^{(q^k-1)/s} \in \langle w \rangle \cap F^*\).

**Proof.** \(F[w]\) is a field of order \(q^s = p^{ms}\). Also, \(\psi = \pi^{mk+1}\) induces \(\phi\) on \(F\), so \(m\) divides the order of \(\psi\). Note that the order of \(\pi\) is \(sm\), so if \(mk + 1 = sh\) we have that \(\psi^m = \pi^{(mk+1)m} = \pi^{shm} = 1\). Hence \(\psi\) has order \(m\). Also, \(w^\psi = w^{\pi^{mk+1}} = (w^{w^{k+1}})^p = (w^{c^{(q^k-1)/s}})^p\), and \(z = c^{(q^k-1)/s} \in \langle w \rangle\).

Next, we recall some well-known facts about symmetric bilinear forms. Let \(K\) be a field and let \(\beta : V \times V \to K\) be a symmetric non-degenerate bilinear form over a \(K\)-vector space \(V\) of dimension \(s\). If \(f \in \text{End}(V)\) is a linear map, then there exists a unique linear map \(f' \in \text{End}(V)\) such that \(\beta(uf, v) = \beta(u, vf')\) for every \(u, v \in V\). The map \(f'\) is called the adjoint map.
of $f$ with respect to $\beta$, and $f$ is said to be self-adjoint if $f' = f$. Take a basis \( \{e_1, e_2, \ldots, e_s\} \) of $V$ and let $A, A'$ and $B$ be the matrices associated to $f, f'$ and $\beta$ with respect to this basis. Then $A' = B^T A^\top (B^\top)^{-1}$. The following lemma is an exercise in [5, p. 367]:

**Lemma 1.8.** Let $V$ be a vector space of dimension $s$ over the field $K$, and let $f \in \text{End}(V)$ be a linear map. Then there exists a symmetric non-degenerate bilinear form $\beta$ with discriminant $\delta \in \{\pm 1(K^*)^2\}$ such that $f$ is self-adjoint with respect to $\beta$.

**Lemma 1.9.** Let $V$ be a vector space of dimension $s$ over the field $K$, and let $\beta$ be a symmetric non-degenerate bilinear form on $V$ with discriminant $\delta$. If $p$ is odd and $\delta = (K^*)^2$ or if $p = 2$ and $s$ is odd, then there exists a basis $E$ of $V$ such that the matrix associated to $\beta$ with respect to $E$ is the identity matrix. If $p$ is odd, $-1$ is not a square in $F$ and $\delta = -(K^*)^2$, then there exists a basis $E$ of $V$ such that the matrix associated to $\beta$ with respect to $E$ is the diagonal matrix $B = \text{diag}(-1, 1, \ldots, 1)$.

**Proof.** See [3, pp. 16,20].

In the sequel, if $R$ is an algebra and $w \in R$, the linear map given by right multiplication by $w$ will be denoted by $r_w$.

**Lemma 1.10.** With the hypotheses and notations of Lemma 1.7, let $V = F[w]$. There exists a basis $E = \{e_1, \ldots, e_s\}$ of $V$ and a matrix $B \in \text{GL}(s, p)$ such that the following hold:

1. $\iota B \in \text{Aut} (\text{SL}(n, q))$ has order 2, and it commutes with $\phi$.
2. The matrix $g$ associated to $r_w$ with respect to $E$ is such that $g^B = g^{-1}$ and $g^\phi = (gz)^p$, where $z = c(q^s - 1)/s \in \langle g \rangle$. Also, $g^* = c$ and $\det(g) = (-1)^{s-1} c$.

**Proof.** We have that $F[w]$ is a field of order $q^s = p^{ms}$. The field $F'$ of fixed points of the automorphism $\psi$ has order $p^s$ and we have $F \cap F' = F_p$, as $(m, s) = 1$.

Let $F' = F_p[v]$ and note that $F[w] = F[v]$ and that every basis of $F'$ over $F_p$ is also a basis of $F[w]$ over $F$. We may view $F'$ as a vector space over $F_p$ and consider the linear map $r_v \in \text{End}_{F_p}(F')$. By Lemma 1.8 there exists a symmetric non-degenerate bilinear form $\beta$ on $F'$ over $F_p$ with discriminant $\delta \in \{\pm 1(F_p)^2\}$ such that $r_v$ is self-adjoint with respect to $\beta$. Note that if $p = 2$, then $s$ is odd. By Lemma 1.9 we may choose a basis $E = \{e_1, \ldots, e_s\}$ of $F'$ such that the matrix $B$ associated to $\beta$ is of the form $B = \text{diag}(s, 1, \ldots, 1)$, where $c \in \{\pm 1\}$. Then the matrix $A$ of $r_v$ with respect to this basis satisfies $A^\top A = A$.

Now consider $V = F[v] = F[w]$. We have that $E$ is a basis for $V$ over $F$. Also, as $w \in F[v]$, $w$ is a linear combination of powers of $v$, so the matrix $g$
associated to \( r_w \) with respect to \( E \) is such that \( g^T B = g \), that is, \( g^B = g^{-1} \), as required. Moreover, \( B \in \text{GL}(s, F_p) \), \( B = B^{-1} = B^{-1} \), so that (i) holds.

Next, let \( x = \lambda_1 + \lambda_2 v + \ldots + \lambda_s v^{s-1} \in V \), with \( \lambda_1, \ldots, \lambda_s \in F \). As \( \psi \) acts trivially on \( E \subseteq F' \), we have \( x^\psi = \lambda_1^p + \lambda_2^p v + \ldots + \lambda_s^p v^{s-1} \), that is, \( \psi \) is the semi-linear map associated to the identity matrix and the automorphism \( \phi \) with respect to the basis \( E \). As \( w^\psi = (zw)^p \), the matrix associated to \( r_{w^p} \) is \( g^\phi = c^{p(q^s-1)/q} g^p \), as we wanted to show.

Note that \( r_{w^p} \) is right multiplication by the scalar \( c \), so \( g^p = c \) and \( x^c - c \) is both the minimal polynomial and the characteristic polynomial of \( g \). It follows that \( \det(g) = (-1)^{s-1}c \).

**Proposition 1.11.** Let \( d_1 | d \), \( d_1 = t_1 l_1 \), where \( t_1 = (d_1, t) \). Assume that \( D \leq \text{PGL}(n, q) \) is such that \( G \leq D \) and \( D/G \) has order \( d_1 \). If \( (m, t_1) = 1 \), then \( G \) has a complement in \( \langle D, \phi, \tilde{\phi} \rangle \).

**Proof.** Take \( v_1, \ldots, v_{n/t_1} \in F \) and \( u \in \mathbb{Z} \) as in Lemma 1.3, and let \( c = \lambda^u \) and \( c_j = cv_j \). Note that \( c_j = \lambda^{u+\alpha_j(q-1)/t_1} \) for some integer \( \alpha_j \), and as \( (u, t_1) = 1 \) we have that \( (u + \alpha_j q^{-1}, t_1) = 1 \), so by Observation 1.6 the polynomials \( x^t - c_j \) are irreducible. Now we may apply Lemma 1.10 and find matrices \( g_j \) and \( B_j \) such that \( B_j \) satisfies (i) of Lemma 1.10, \( g_j^B = g_j^{-1} \), \( g_j^\phi = (cv_j)^{p+q^{s_{1/t_1}}} g_j^p \) and \( g_j^{l_1} = cv_j \) for \( j = 1, \ldots, n/t_1 \). As \( l_1 | q^{s_{1/t_1}} \), it follows that \( v_j^{p+q^{s_{1/t_1}}} = 1 \), so \( g_j^\phi = c^{p(q^s-1)/t_1} g^p \). Also, \( g_j^{d_1} = g_j^{l_1} = (cv_j)^l_1 = c^{l_1} \).

Now consider the matrices

\[
g = \begin{pmatrix} g_1 & \cdots & g_{n/t_1} \\ & \ddots & \\ g_{n/t_1} & \cdots & g_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \cdots & B_{n/t_1} \\ & \ddots & \\ B_{n/t_1} & \cdots & B_1 \end{pmatrix}.
\]

We have that \( \bar{B} \) has order 2 and commutes with \( \phi \), \( g^B = g^{-1} \) and \( g^\phi = (gz)^p \), where \( z = c^{q^{s_{1/t_1}}} \in F \). Also, \( g^{d_1} = c^{l_1} \) and

\[
\det(g) = \prod_{j=1}^{n/t_1} \det(g_j) = \prod_{j=1}^{n/t_1} (-1)^{l_1} \lambda^u v_j = \lambda^{d_1}.
\]

Then \( \tilde{C} = \langle g, \phi, \bar{B} \rangle \) is the required complement. \( \square \)

Combining Lemma 1.2 with the special case \( d_1 = d \) of Proposition 1.11 we get:

**Theorem 1.12.** \( \text{PSL}(n, q) \) has a complement in \( \text{Aut}(\text{PSL}(n, q)) \) if and only if \( (\frac{q^s-1}{d}, d, m) = 1 \).
2. The unitary groups

In this section, we will consider the group $G = \text{Aut}_G(q) = \text{PSU}(n, q)$, for $n$ and $q$ fixed.

Let $F = F_{q^2}$ be the finite field with $q^2$ elements, where $q = p^m$ for some prime number $p$. We fix a generator $\lambda$ of the multiplicative group of the field $F^*$. Then $U(n, q)$ (resp. $\text{SU}(n, q)$) will denote the general (resp. special) unitary group of degree $n$, that is, $U(n, q) = \{g \in \text{GL}(n, q^2) \mid (g^T)^\sigma = 1\}$, where $\sigma = \phi^m \in \text{Aut}(\text{GL}(n, q^2))$, and $\text{SU}(n, q) = \{g \in U(n, q) \mid \det(g) = 1\}$. All other notations, unless otherwise specified, are as in the previous section, and as usual $F^*$ is identified with the subgroup of $\text{GL}(n, q^2)$ consisting of scalar matrices.

We have the sequence of normal subgroups

$$\text{SU}(n, q) \leq U(n, q) \leq U(n, q)/(\phi),$$

from which, taking images in $U(n, q)/(\phi) F^*/F^*$, we obtain the sequence

$$\text{PSU}(n, q) \leq \text{PU}(n, q) \leq \text{PU}(n, q)/(\phi) F^*/F^* = \text{Aut}(\text{PSU}(n, q)).$$

Also, $\text{PU}(n, q)/G$ is cyclic of order $d = (n, q + 1)$ and $\bar{\phi}$ acts on it as the $p$th power. We want to prove that $G$ has a complement in $\text{Aut} G$ if and only if $(\frac{q+1}{d}, d, m) = 1$. Letting $t$ be the product of all prime factors of $d$ dividing $\frac{q+1}{d}$, counting multiplicities, this is equivalent to proving that $G$ has a complement in $\text{Aut} G$ if and only if $(t, m) = 1$.

**Lemma 2.1.**

(i) If $g \in U(n, q)$, then $\det(g)^{q+1} = 1$.

(ii) $U(n, q) \cap F^* = \{a \in F^* \mid a^{q+1} = 1\}$.

(iii) $\langle \bar{g} \rangle$ is a complement for $\text{PSU}(n, q)$ in $\text{PU}(n, q)$ if and only if $\det(g) = \lambda^{(q-1)u}$, with $(u, d) = 1$, and $g^d \in F^*$.

(iv) Assume that $G$ has a complement $C$ in $\text{Aut} G$. Then it is possible to choose $g \in U(n, q)$ and $h \in \text{SU}(n, q)$ such that $C = \langle \bar{g}, \bar{\phi} \rangle$, $\bar{g}^{\phi h} = \bar{g}^\sigma$, $\det(g) = \lambda^{(q-1)u}$, with $(u, d) = 1$, and $g^d, (\phi h)^{2m} \in F^*$.

**Proof.** (i) and (ii) follow directly from the definition of $U(n, q)$. To obtain (iii) and (iv), we note that, by (i), $\det(g)$ is of the form $\lambda^{(q-1)u}$. The proofs are now analogous to those of Lemma 1.1. \qed

**Lemma 2.2.** Assume that $G$ has a complement in $\text{Aut} G$. Then $(m, t) = 1$.

**Proof.** Let $g, h$ be as in Lemma 2.1 (iv), so that $\det(g) = \lambda^{(q-1)u}$, with $(u, d) = 1$, and $g^d = \lambda^{u(q-1)} \in U(n, q) \cap F^*$ for some natural number $\alpha$ (see Lemma 2.1 (ii)). Taking the determinant on both sides, we obtain $\lambda^{du(q-1)} = \lambda^{\alpha u(q-1)}$, that is, $du(q-1) \equiv du \alpha \equiv \alpha \frac{q+1}{d} \equiv \lambda^{\alpha u(q-1)} \mod(q^2-1)$, and so $u \equiv \alpha \frac{q+1}{d} \mod(q^2-1)$. If $r$ is a prime such that $r \mid t$, then $r \mid \frac{q+1}{d}$ and $r \mid u$, so $r \nmid \alpha$. It follows that $(\alpha, t) = 1$. 

We may view $\phi h$ as a ring automorphism of the ring $\text{Mat}(n, q^2)$. As $\overline{g}h \equiv g\bar{h}$, we have that $g^\phi h = (gz)^\phi$ for some $z \in F^*$, so $\phi h$ normalizes the subring $F[g]$ of $\text{Mat}(n, q^2)$. Now the map $\pi : F[g] \to F[\bar{g}]$, defined by $v^\phi = v^p$, is also a ring automorphism of $F[g]$, and $\phi h \pi^{-1}$ is ring automorphism which centralizes $F$. So $\lambda^{\alpha(q-1)} = g^d = (g^\phi \pi^{-1})^d = (g^\phi \pi^{-1})^d = (gz)^d = \lambda^{\alpha(q-1)}z^d$ and $z^d = 1$. Hence we may assume that $z = \lambda^{\beta(q-1)/d}$ for some integer $\beta$. As $(\phi h)^{2m}$ is a scalar matrix and $(\phi h)^{2m} = g^{q_0}z^{ip'}$ for each natural number $i$, we obtain that $g = g^{\phi h} = g^{q_0}z^{2m}$, so $g^q = z^{-2m}$. Moreover, $g^{-1} = g^{(q-1)/d}$, so we have $\alpha(q-1)g^{q-1} = (2m\beta) \frac{q^2-1}{d}$ mod $q^2 - 1$. It follows that $\alpha(q-1) \equiv -2\beta m \mod d$.

Let $r$ be a prime which divides $t$. If $r = 2$, then $p = 2$. Both $\frac{q+1}{2}$ and $d$ are even, so $q + 1 = p^m + 1 \equiv 0 \mod 4$ and $m$ is odd. If $r \neq 2$, then $r|d, r|q + 1, r \not\mid q - 1$, and $r \not\mid \alpha$ (by what we have just proved), so $r \not\mid m$. It follows that $(m, t) = 1$, as we wanted to prove.

We now seek a complement for $G$ in $\text{Aut} G$. We find $g, h \in U(n, q)$ such that $\det(g) = \lambda^d, g^d \in F^*$, $(\phi h)^{2m} \in F^*$ and $\langle g \rangle$ is normalized by $\phi h$.

**Lemma 2.3.** Assume that $d = tl$, $d_1 \mid d$, $d_1 = t_1l_1$, where $t_1 = (d_1, t)$. Then there exist $v_1, \ldots, v_{n/t_1} \in (F^*)^{q-1}$ and $u \in Z$ such that $v_j^{t_1} = 1$ for $j = 1, \ldots, n/t_1$ and

$$\prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^{u(q-1)v_j} = \lambda^{q-1)d/d_1}.$$

**Proof.** The proof is analogous to that of Lemma 1.3.

**Lemma 2.4.** Let $d_1 = t_1l_1$ as above. Take $u \in Z$ and $v_1, \ldots, v_{n/t_1} \in F$ such that $v_j^{t_1} = 1$ for every $j = 1, \ldots, n/t_1$, and $\prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^{u(q-1)v_j} = \lambda^{(q-1)d/d_1}$. Then there exists a matrix $g \in U(n, q)$ such that $g^{d_1} \in F^*$ and $\det(g) = \lambda^{(q-1)d/d_1}$.

**Proof.** Note that Lemma 2.3 ensures the existence of $u$ and $v_1, \ldots, v_{n/t_1}$ with the required properties. Then construct the matrix $g$ as in Lemma 1.4, using $c = \lambda^{u(q-1)}$ in place of $c = \lambda^u$. It is easy to see that $g_j(g_j^\phi)^\phi = \text{diag}(1, \ldots, 1, c^{q+1}) = 1$, as $c^{q+1} = (\lambda^{u(q-1)}v_j)^{q+1} = 1$, because $l_1|q + 1$. It follows that $g(g_j^\phi)^\phi = 1$, so $g \in U(n, q)$.

**Proposition 2.5.** $\text{PSU}(n, q)$ is complemented in $\text{PU}(n, q)$.

**Proof.** Take $v_1, \ldots, v_{n/t} \in F$ and $u \in Z$ in Lemma 2.3, with $d_1 = d$ and let $g$ be the matrix constructed in Lemma 2.4. Then $\langle g \rangle$ is the required complement.
LEMMA 2.6. Let $F[w]$ be a commutative ring, where $w$ has minimal polynomial $x^{t_1} - c$ over $F$ (where $t_1$ is as in Lemma 2.3), that is, $F[w]$ is isomorphic to the quotient of the polynomial ring $F[x]$ over the ideal $(x^{t_1} - c)$. Let $c = \lambda^u(q - 1)$ and assume also that $(t_1, u) = (t_1, m) = 1$. Then $F[w]$ has a ring automorphism $\psi$ of order $2m$ such that $\psi^a = a^p$ for every $a \in F$ and $w^{\psi} = (w^2)^p$, with $z \in \langle c \rangle$. More specifically, we have:

(i) If $t_1$ is odd, let $k \in \mathbb{N}$ be such that $2mk \equiv -1 \mod t_1$. Then $z = c(q^{x_1-1}/t_1) \in \langle w \rangle$.

(ii) If $t_1$ is even let $k \in \mathbb{N}$ be such that $k$ is odd and $mk \equiv -1 \mod t_1/2$.

Then $z = c(q^{x_1-2}/(2t_1)) \in \langle w \rangle$.

Proof. (i) In this case $(t_1, 2m) = 1$. Note that, as $t_1|q + 1$, we have that $(t_1, q - 1) = 1$, so by Observation 1.6 the polynomial $x^{t_1} - \lambda^u(q - 1)$ is irreducible. Then the map $\psi = \pi z^{\epsilon}$ constructed in Lemma 1.7 with $s = t_1$ and $2m$ in place of $m$ has the required properties.

(ii) As $(m, \frac{t_1}{2}) = 1$, there exist an odd $k \in \mathbb{N}$ and $s \in \mathbb{Z}$ such that $mk + \frac{s}{2}t_1 + 1 = 0$.

Let $\epsilon = c(q^{x_1-1}/(2t_1))$. As $q^2 \equiv 1 \mod t$, it follows that $1 + q^2 + \cdots + q^{2(k-1)} \equiv k \mod t$. Also, it is clear that $(\frac{x_1}{2}, t_1) = 1$, so if $\alpha = u \frac{2k-1}{t_1}(1 + q^2 + \cdots + q^{2(k-1)})$ we have that $(\alpha, t_1) = 1$. It follows that $\epsilon = \lambda^u(q - 1)/2^{(q^{x_1-1}/(2t_1) - 1)} \lambda^u(q - 1)/t_1$ has order $t_1$, so $\epsilon^{t_1/2} = -1$.

Let $b = \lambda^u(q - 1)/2$, so that $b^2 = c$. Then $x_1^t - c = (x^{t_1/2} - b)(x^{t_1/2} + b)$. Consider the ring $K[w_1]$, where $w_1$ has minimal polynomial $x^{t_1/2} - b$. Note that, as $(u + \frac{1}{2}, \frac{t_1}{2}) = 1$ and $\frac{t_1}{2} \frac{q^{x_1-1}}{2}$, the polynomials $x^{t_1/2} - b$ and $x^{t_1/2} + b$ are irreducible.

We have that $(w_1 \epsilon)^{t_1/2} = -b$ and we may assume that $F[w] = F[w_1] \times F[w_1 \epsilon] = F[w_1] \times F[w_1]$ as $\epsilon \in F$. Moreover, we may assume that $w = (w_1, w_1 \epsilon)$ and that $F \leq F[w]$ is identified with the subfield $\tilde{F} = \{(a, a) \mid a \in F\}$ of the direct product.

Define $\psi : F[w] \to F[w]$ by $(a_1, a_2)^{\psi} = (a_1^p, a_2^{p2m+1})$. For every $a \in F$ we have that $(a, a)^{\psi} = (a^p, a^{2m+1}) = (a^p, a^p) = (a, a)^p$, so that $\psi$ acts on $\tilde{F}$ as the $p$-th power $\pi$. In particular, the order of $\psi$ is at least $2m$.

We also have that $(a_1, a_2)^{\psi^2} = (a_1^{p2m+2}, a_2^{p2m+2})$, so $\psi^2$ stabilizes $F[w_1]$. Moreover, $\psi^{2m} = \pi^{(2m+1)/2} \pi^{t_1/2} = \pi^{-2mt_1/2}$. But $\pi^{2mt_1/2}$ acts trivially on $F[w_1]$, so $\psi$ has order $2m$. Note that

$$w^{\psi w^{-p}} = (w_1, w_1 \epsilon)^{\psi}(w_1, w_1 \epsilon)^{-p} = (\epsilon^p, w_1^{2m+1-p} \epsilon^{-p}) = (\epsilon, w_1^{2m+1-p} \epsilon^{-1})^p$$

and

$$w_1^{p2m+1} = w_1^{q^{x_1-1}} = w_1^{t_1(q^{x_1-1}/t_1)} = c(q^{x_1-1}/t_1) = c^2.$$
Therefore $w^\psi w^{-p} = (e, e)^p$, that is,
\[
w^\psi = (gz)^p, \quad z = (c(q_{2k-1})/(2t_1), c(q_{2k-1})/(2t_1)) \in \tilde{\Gamma} \cap \langle w \rangle.
\]

**Lemma 2.7.** With the hypotheses and notation of Lemma 2.6, there exist two matrices $g, h \in U(t_1, q)$ such that $g^{t_1} = c$, $\det(g) = (-1)^{t_1-1}c$, $g^{\phi h} = (gz)^p$, where $z = c(q_{2k-1})/(2t_1) \in \langle g \rangle$ if $t_1$ is odd, and $z = c(q_{2k-1})/(2t_1) \in \langle g \rangle$ if $t_1$ is even. Also, $(\phi h)^{2m} = 1$.

**Proof.** We have that $E = \{1, w, w^2, \ldots, w^{d_1-1}\}$ is a basis of $V = F[w]$ as a vector space over $F$. The matrix $g$ associated to $r_w$ with respect to $E$ is
\[
g = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & 0 & 1 \\
c & 0 & 0 & 0
\end{pmatrix}.
\]
We have that $g \in U(t_1, q)$, by the same argument as in Lemma 2.4. Also, $g^{t_1} = c$ and $\det(g) = (-1)^{t_1-1}c$. Note that $\psi$ is a semilinear map associated with the automorphism $\phi$ of $F$. We have that $(w^i)^\psi = c^{\alpha_i(w)\alpha_i(w+1)}$, where $\sigma \in \text{Sym}(t_1)$. So $\psi$ permutes the subspaces $F w^i$. Let $h$ be the matrix associated to the linear map which acts in the same way as $\psi$ on the given basis. Then $h$ is monomial. Also, $h(h^{-1})^\psi$ is a diagonal matrix with all non-zero entries of the form $c^{\alpha_i(q+1)} = \lambda^{\alpha_i(u(q-1)/(q+1))} = 1$, so $h \in U(t_1, q)$.

Next, note that the group $\Gamma L(V)$ of semilinear maps is isomorphic to $\Gamma L(n, q)$ and, with respect to the chosen basis $E$, we have that $\psi$ corresponds to $\phi h$, so $(\phi h)^{2m} = 1$. Also, $(\phi h) \cap F^* = 1$.

Finally, right multiplication by $w^\psi$ is right multiplication by $(wz)^p$, so $g^{\phi h} = (gz)^p$. \qed

**Proposition 2.8.** Let $d_1|d$, $d_1 = t_1t_1$, where $t_1 = (d_1, t)$. Let $D \leq \text{PU}(n, q)$ be such that $G \leq D$ and $D/G$ has order $d_1$. If $(m, t_1) = 1$, then $G$ has a complement in $(D, \phi)$.

**Proof.** Take $v_1, \ldots, v_n/t_1 \in F$ and $u \in Z$ as in Lemma 2.3, and let $c = \lambda^{u(q-1)}$ and $c_j = cv_j$. Note that $c_j = \lambda^{u(q-1)+\alpha_j(q^2-1)/t_1}$ for some integer $\alpha_j$, and as $(u, t_1) = 1$ and $t_1|q^{2k-1}$, we have that $(u + \alpha_j(q^{2k-1}/t_1), t_1) = 1$, so the hypotheses of Lemma 2.6 are satisfied. Now we may apply Lemma 2.7 and find matrices $g_j, h_j \in U(t_1, q)$ such that $(\phi h_j)^{2m} = 1$, $g_j^{\phi h_j} = (g_j z_j)^p$, with $z_j \in \langle g_j \rangle$. If $t_1$ is odd, we have
\[
z_j = c_j^{(q_{2k-1})/(2t_1)} = (cv_j)^{(q_{2k-1})/(2t_1)} = c^{(q_{2k-1})/(2t_1)}.
\]
for every $j = 1, \ldots, n/t_1$, as $l_1 \mid q^2 - 1$. If $t_1$ is even, we have

$$z_j = c_j^{(q^{2k} - 1)/(2t_1)} = (cv_j)^{(q^{2k} - 1)/(2t_1)} = c^{(q^{2k} - 1)/(2t_1)}$$

for every $j = 1, \ldots, n/t_1$, as $l_1 \mid q^2 - 1$ (where $l_1$ is odd). Also, $g_j^{d_1} = g_j^{t_1} = (cv_j)^{t_1} = c^{t_1}$.

Now consider the matrices

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ g_{n/t_1} & & \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & & \\ & \ddots & \\ h_{n/t_1} & & \end{pmatrix}.$$

We have that $g, h \in U(n, q)$, $(\phi h)^{2m} = 1$ and $g^{\phi h} = (gz)^{p}$, where $z \in F \cap \langle g \rangle$. Also, $g^{d_1} = c^{l_1}$ and

$$\det(g) = \prod_{j=1}^{n/t_1} \det(g_j) = \prod_{j=1}^{n/t_1} (-1)^{t_1-1} \lambda^{a(q-1)v_j} = \lambda^{(q-1)d/d_1}.$$

Then $\overline{C} = \langle \overline{g}, \overline{\phi h} \rangle$ is the required complement. □

Combining Lemma 2.2 with the special case $d_1 = d$ of Proposition 2.8 we get:

**Theorem 2.9.** PSU$(n, q)$ has a complement in Aut(PSU$(n, q)$) if and only if $(q^{2^{m-1}}, d, m) = 1$.

3. Untwisted groups of Lie type

In the following, we denote by $F_q$ the finite field of order $q = p^m$, with $p$ a prime and $m$ a positive integer. Moreover, we denote by $\lambda$ a generator of the multiplicative group of $F_q$. Let $\Phi$ be a root system corresponding to a simple Lie algebra $L$ over the complex field $\mathbb{C}$, and let us consider a fundamental system $\Pi = \{a_1, \ldots, a_l\}$ in $\Phi$. For any choice of $\Pi$ and for any finite field $F_q$, we let $L(q)$ denote the corresponding finite group (where $L$ denotes the type of the group; i.e., $L = A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$).

We assume that for the various possible root systems the elements of $\Pi$ are labelled in such a way that $(a, a) = 2$ and $(a, b) = 0$ for each pair of roots in
Π, with the following exceptions:

\[ A_1 : (a_1, a_{i+1}) = -1 \text{ for } 1 \leq i \leq l - 1; \]
\[ B_1 : (a_1, a_1) = 1, (a_i, a_{i+1}) = -1 \text{ for } 1 \leq i \leq l - 1; \]
\[ C_l : (a_i, a_i) = 1, (a_i, a_{i+1}) = -1/2 \text{ for } 1 \leq i \leq l - 2, \]
\[ (a_{l-1}, a_{l-1}) = -(a_1, a_l) = 1; \]
\[ D_l : (a_1, a_3) = (a_i, a_{i+1}) = -1 \text{ for } 2 \leq i \leq l - 1; \]
\[ E_l : (a_i, a_{i+1}) = (a_{l-3}, a_l) = -1 \text{ for } 1 \leq i \leq l - 2; \]
\[ F_4 : (a_1, a_1) = (a_2, a_2) = 1, (a_1, a_2) = -1/2, \]
\[ (a_2, a_3) = (a_3, a_4) = -1; \]
\[ G_2 : (a_1, a_1) = 2/3, (a_1, a_2) = -1. \]

The Chevalley group \( L(q) \), viewed as a group of automorphisms of a Lie algebra \( L_K \) over the field \( K = F_q \), obtained from a simple Lie algebra \( L \) over the complex field \( C \), is the group generated by certain automorphisms \( x_r(t) \), where \( t \) runs over \( F_q \) and \( r \) runs over the root system \( \Phi \) associated to \( L \). For each \( r \in \Phi \), \( X_r = \{ x_r(t) \mid t \in F_q \} \) is a subgroup of \( L(q) \) isomorphic to the additive group of the field. \( X_r \) is called a root subgroup, and the group \( L(q) \) is generated by the root-subgroups \( X_r, \pm r \in \Pi \). In the following we will use the notations and the terminology introduced in [1].

Let us recall some facts about the automorphism group of \( L(q) \).

Any automorphism \( \sigma \) of the field \( F_q \) induces a field automorphism (also denoted by \( \sigma \)) of \( L(q) \), defined by

\[ (x_r(t))^\sigma = x_r(t^\sigma). \]

The set of the field automorphisms of \( L(q) \) is a cyclic group of order \( m \) generated by the Frobenius automorphism \( \phi \).

We recall that a symmetry of the Dynkin diagram of \( L(q) \) is a permutation \( \rho \) of the nodes of the diagram, such that the number of bonds joining nodes \( i \), \( j \) is the same as the number of bonds joining nodes \( i\rho, j\rho \), for any \( i \neq j \). A non trivial symmetry \( \rho \) of the Dynkin diagram can be extended to a map of the space \( (\Phi) \) into itself, which we also denote by \( \rho \). This map yields an outer automorphism \( \epsilon \) of \( L(q) \); \( \epsilon \) is said to be a graph automorphism of \( L(q) \).

If \( L(q) \) is \( A_l(q), l \geq 2, D_l(q) \) or \( E_6(q) \), then \( (x_r(t))^\epsilon = x_{r\rho}(\gamma_r t) \), where \( r \in \Phi, t \in F_q, \gamma_r \in Z \). Moreover, the \( \gamma_r \) can be chosen so that \( \gamma_r = 1 \) if \( r \in \Pi \), and \( \gamma_r = -1 \) if \( -r \in \Pi \).

Let \( P = Z\Phi \) be the additive group generated by the roots in \( \Phi \); a homomorphism from \( P \) into the multiplicative group \( F_q^* \) will be called an \( F_q \)-character of \( P \). From each \( F_q \)-character \( \chi \) of \( P \) arises a diagonal automorphism \( h(\chi) \) of \( L(q) \) which maps \( x_r(t) \) to \( x_r(\chi(r)t) \). The automorphisms of the form \( h(\chi) \) form an abelian subgroup \( H \) of the full automorphism group of \( L(q) \). Now consider the additive group \( Q \) generated by the fundamental weights \( \lambda_1, \ldots, \lambda_l \),
Any element of \( P \) is an integral combination of \( \lambda_1, \ldots, \lambda_l \). (More precisely, \( a_i = \sum_{1 \leq j \leq l} A_{ij} \lambda_j \), where \((A_{ij})\) is the Cartan matrix of \( L \).) Thus \( P \) is a subgroup of \( Q \). Every \( F_q \)-character of \( Q \) gives rise to an \( F_q \)-character of \( P \) by restriction. However, an \( F_q \)-character of \( P \) need not be the restriction of some \( F_q \)-character of \( Q \). More precisely, if an \( F_q \)-character of \( P \), say \( \chi \), can be extended to an \( F_q \)-character of \( Q \), then the automorphism \( h(\chi) \) is inner, and vice versa. In the following we will often apply the above criterion to decide whether a diagonal automorphism \( h(\chi) \) is inner; this will be done using the information coming from the Cartan matrix. Namely, if \( \chi(a_i) = \lambda^{a_i} \), \( 1 \leq i \leq l \), then \( \chi \) can be extended to a \( F_q \)-character of \( Q \) by setting \( \chi(\lambda) = \lambda^{\beta_i} \) for \( 1 \leq i \leq l \) if and only the integers \( \beta_1, \ldots, \beta_l \) satisfy the conditions \( \alpha_i \equiv \sum_{1 \leq j \leq l} A_{ij} \beta_j \mod q - 1 \) for \( 1 \leq i \leq l \).

We denote by \( H \) the group of the diagonal automorphisms that are inner and by \( d \) the order of the abelian group \( \hat{H}/H \). The value of \( d \) is given by the following table.

<table>
<thead>
<tr>
<th>( L(q) )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_l(q) )</td>
<td>( (l + 1, q - 1) )</td>
</tr>
<tr>
<td>( B_l(q) )</td>
<td>( (2, q - 1) )</td>
</tr>
<tr>
<td>( C_l(q) )</td>
<td>( (2, q - 1) )</td>
</tr>
<tr>
<td>( D_l(q) )</td>
<td>( (4, q^2 - 1) )</td>
</tr>
<tr>
<td>( E_6(q) )</td>
<td>( (3, q - 1) )</td>
</tr>
<tr>
<td>( E_7(q) )</td>
<td>( (2, q - 1) )</td>
</tr>
<tr>
<td>( E_8(q) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( G_2(q) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( F_4(q) )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

The main result about the automorphism group of \( L(q) \) is as follows:

For each automorphism \( \theta \in \text{Aut} \ L(q) \) there exist an inner automorphism \( i \), a diagonal automorphism \( h \), a field automorphism \( f \) and a graph automorphism \( \epsilon \), such that \( \theta = ihfe \); moreover,

\[
L(q) \leq \langle L(q), \hat{H} \rangle \leq \langle L(q), \hat{H}, \phi \rangle \leq \text{Aut} \ L(q).
\]

We will prove the following result:

**Theorem 3.1.** Suppose that \( q = p^m \) and let \( L(q) \) be an untwisted group of Lie type. Define \( \tilde{q} = q^l \) if \( L = D_l \), and \( \tilde{q} = q \) otherwise. Then \( L(q) \) has a complement in \( \text{Aut} \ L(q) \) if and only if the following condition is satisfied:

\[
(\tilde{q} - 1, d, m) = 1.
\]
We have already proved that this is true for $A_l(q) \cong \text{PSL}(l + 1, q)$. In this section we discuss the remaining cases.

The subgroup $\langle L(q), H \rangle$ of inner-diagonal automorphisms is always complemented in $\text{Aut} L(q)$, so we only have to deal with the cases when $d \neq 1$.

We first prove that the condition $(*)$ is necessary in order for $L(q)$ to have a complement.

As had already been noticed by Pandya [6, Lemma 3.5], Lang’s Theorem implies the following result.

**Lemma 3.2.** Suppose that $L(q)$ has a complement $X$ in $\text{Aut} L(q)$. Then there exists $g \in L(q)$ such that the Frobenius automorphism $\phi$ belongs to $X^g$.

Thus, if $L(q)$ has a complement $X$ in $\text{Aut} L(q)$, we may assume without loss of generality that $\phi \in X$. In particular, $Y = \langle L(q), H \rangle \cap X$ is a subgroup of $X$ isomorphic to $H/H$ and normalized by $\phi$. We will show that this is possible only if $(q - 1/d, d, m) = 1$. To this end we use the Bruhat Decomposition. As is well known, if $N$ is the normalizer of $H$ in $L(q)$, then there exists a homomorphism from $N$ onto the Weyl group $W$ of $L$, with kernel $H$. For each $w \in W$ we fix an element $n_w \in N$ which maps to $w$ under this homomorphism and such that $[n_w, \phi] = 1$. Let $U = \langle X_r \mid r \in \Pi \rangle$ and let $U_w$ be the subgroup generated by those root subgroups $X_r$ for which $r$ is positive and $rw$ is negative. Each element $x$ of $\langle L(q), H \rangle$ has a unique representation in the form $x = u_1 h(\chi)n_w u$, where $u_1 \in U$, $h(\chi) \in H$, $w \in W$, $u \in U_w$.

**Lemma 3.3.** Suppose that $L(q) = B_l(q), C_l(q)$, or $E_7(q)$ and that there exists a complement $Y$ of $L(q)$ in $\langle L(q), H \rangle$ normalized by the Frobenius automorphism $\phi$. Then $(*)$ is satisfied.

**Proof.** We may assume $d \neq 1$. Hence $d = (q - 1, 2) = 2$ and $q = p^m$ with $p$ an odd prime. In this case $Y = \langle x \rangle$, with $|x| = 2$. Using the Bruhat Decomposition we may write $x$ in the form $x = u_1 h(\chi)n_w u$, where $u_1 \in U$ and $u \in U_w$. Then

$$x = x^\phi = u_1^\phi h(\chi)^\phi n_w^\phi u^\phi = u_1^\phi h(\chi)^\phi n_w u^\phi.$$ 

Note that $u_1^\phi \in U$ and $u^\phi \in U_w$, so by the uniqueness of the representation of $x$ we deduce $h(\chi)^\phi = h(\chi)$, and this implies $\chi^\phi = \chi$. Since $x \notin L(q)$, we have $h(\chi) \in \hat{H} \setminus H$, which implies that there exists $1 \leq i \leq l$ with $\chi(a_i) = \lambda^s$ for an odd integer $s$. Therefore $sp \equiv s \mod q - 1$. Hence $(q - 1)_{12} \leq (p - 1)_{12}$, and this is possible only if $m$ is odd. To conclude the proof, it is enough to notice that if $d = 2$, then $(q - 1/d, d, m) = 1$ if and only if $m$ is odd. \qed

**Lemma 3.4.** Suppose that $L(q) = D_l(q)$ with $l$ even and that there exists a complement $Y$ of $L(q)$ in $\langle L(q), H \rangle$ normalized by the Frobenius automorphism $\phi$. Then $(*)$ is satisfied.
Proof. Again we may assume \( d \neq 1 \). In this case \( d = 4 \), \( H/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \phi \) centralizes \( H/H \). In particular, \( Y \) contains an element \( x \) of order 2 centralized by \( \phi \). Arguing as in Lemma 3.3, we deduce that \( m \) is odd, and this is equivalent to the condition that \((q^4 - 1)/4, 4, m) = 1\). \(\square\)

**Lemma 3.5.** Suppose that \( L(q) = D_1(q) \) with \( l \) odd and that there exists a complement \( Y \) of \( L(q) \) in \( (L(q), H) \) normalized by the Frobenius automorphism \( \phi \). Then \((*)\) is satisfied.

**Proof.** Again it is enough to prove that either \( d = 1 \) or \( m \) is odd. Assume \( d \neq 1 \). Then \( H/H \) is cyclic of order \( d \in \{2, 4\} \). Let \( x \) be a generator of \( Y \). If \([\phi, x] = 1\) we may repeat the argument of Lemma 3.3 to deduce that \( m \) is odd. So assume that \( \phi \) does not centralize \( x \). This occurs only if \( d = 4 \), \( p \equiv 3 \) mod 4, and \( m \) is even. In this case we take an element \( y \in Y \) of order 2 and write \( y \) in the form \( y = u_1 h(x) n_w u \) with \( u_1 \in U \) and \( u \in U_w \). As \( \phi \) centralizes \( y \), using the uniqueness of this representation, we deduce \( \chi^p = \chi \). Since \( y \notin L(q) \), we have \( h(x) \in H \setminus H \), which implies that there exists \( 1 \leq i \leq l \) with \( \chi(a_i) = \lambda^i \) for some integer \( s \) not divisible by 4. Therefore \( sp \equiv r \) mod \( q - 1 \), hence \((q - 1) / 2 \leq (s(p - 1)) / 2 \leq 4 \), but this is impossible, since if \( p \equiv 3 \) mod 4 and \( m \) is even, then \( q \equiv 1 \) mod 8. \(\square\)

**Lemma 3.6.** Suppose that \( L(q) = E_6(q) \) and that there exists a complement \( Y \) of \( L(q) \) in \( (L(q), H) \) normalized by the Frobenius automorphism \( \phi \). Then \((*)\) is satisfied.

**Proof.** In this case \( d = (3, q - 1) \) and \((*)\) is equivalent to the condition that either \( d = 1 \) or \((3, m) = 1 \). Suppose that \( d \neq 1 \). \( H/H \) is cyclic of order 3. Let \( x \) be a generator of \( Y \) and write \( x \) in the form \( x = u h(x) n_w u \) with \( u_1 \in U \) and \( u \in U_w \). Since \( \phi^2 \) centralizes \( x \), arguing as in the proof of Lemma 3.3 we deduce \( h(x)^{\phi^2} = h(x) \), and this implies \( \chi^{p^2} = \chi \). Since \( x \notin L(q) \), we have \( h(x) \in H \setminus H \), which implies that there exists \( 1 \leq i \leq l \) with \( \chi(a_i) = \lambda^i \) for an integer \( s \) not divisible by 3. Therefore \( sp \equiv s \) mod \( q - 1 \). Hence \((q - 1) / 3 \leq (p^2 - 1) / 3 \), which implies \((3, m) = 1\). \(\square\)

It remains to prove that if \((*)\) is satisfied, then \( L(q) \) has a complement in \( \text{Aut} L(q) \). As we have already noticed, \( (L(q), H) \) is always complemented in \( \text{Aut} L(q) \), so we only have to consider the case when \( d \neq 1 \).

We first recall the following useful result (see [1, Theorem 7.2.2]):

**Lemma 3.7.** If \( n \in N \) and \( n \) maps to \( w \) under the natural homomorphism from \( N \) onto \( W \), then \( h(x)^n = h(x^w) \), where \( \chi^w(r) = \chi(rw) \) for each \( r \in \Phi \).

For any \( r \in \Phi \) let \( w_r \) be the reflection in the hyperplane orthogonal to \( r \) and let \( n_r = x_r(1)x_r(-1)x_r(1) \). Then \( n_r \in N \) and \( n_r \) maps to \( w_r \) under the
natural homomorphism from $N$ onto $W$. In the following we write $w_i, n_i$ in place of $w_{a_i}, n_{a_i}$, for any $a_i \in \Pi$.

**Lemma 3.8.** If $L(q) = B_3(q)$ and $(\ast)$ is satisfied, then there is a complement $X$ of $L(q)$ in $\text{Aut} \ L(q)$.

**Proof.** We may assume $d = 2$ (in which case $L(q)$ has no graph automorphism). Let $\mu$ be a generator of the 2-Sylow subgroup of $F'_q$ and define $\chi$ by $\chi(a_1) = \mu, \chi(a_2) = \mu^{-1}$, and $\chi(a_i) = 1$ for $i > 2$. Consider the element $x = h(\chi)n_1$. We have $n_1^2 = h_1(-1) = 1$ (see [4, p. 20]), $\chi^{w_1}(a_1) = \chi(a_1w_1) = \chi(-a_1) = \mu^{-1}$ and $\chi^{w_1}(a_2) = \chi(a_1w_1) = \chi(2a_1 + a_2) = \mu$. Hence $x^2 = h(\chi)h(\chi)^{n_1} = h(\chi)h(\chi^w) = 1$. Moreover, since $(\ast)$ is satisfied, we have $(q - 1)/2 = (p - 1)/2$, so $\mu^p = \mu$ and $[x, \phi] = 1$. We claim that $x \notin L(q)$. Indeed, if $x \in L(q)$, then $h(\chi) \in H$, and $\chi$ could be extended to an $F'_q$-character of $Q$: as $2\lambda_1 = la_1 + (l - 1)a_2 + \cdots + a_i$, we would then have $\chi(\lambda_1)^2 = \chi(la_1 + (l - 1)a_2) = \mu \in F'_q$, a contradiction. But then $X = \langle x, \phi \rangle$ is a complement for $L(q)$ in $\text{Aut} \ L(q)$. \hfill \Box

**Lemma 3.9.** If $L(q) = C_3(q)$ and $(\ast)$ is satisfied, then there is a complement $X$ of $L(q)$ in $\text{Aut} \ L(q)$.

**Proof.** We may assume $d = 2$. Let $\mu$ be a generator of the 2-Sylow subgroup of $F'_q$ and define $\chi$ by $\chi(a_i) = \mu$ if $i = 1 \mod 4, \chi(a_i) = \mu^{-1}$ if $i = 3 \mod 4, \chi(a_i) = 1$ if $i$ is even and $i \neq l$, and $\chi(a_i) = \chi(a_{i-1})^{-1}$ if $l$ is even. Let $n = n_1n_3 \cdots n_k$ with $k = 2\left\lfloor \frac{-1}{2} \right\rfloor + 1$ and consider the element $x = h(\chi)n$. Let $w = w_1w_3 \cdots w_k$. Then $\chi(w)(a_i) = \chi(a_iw) = \chi(-a_i)$ if $i$ is odd, $\chi(w)(a_i) = \chi(a_1w) = \chi(a_{i-1} + a_i + a_{i+1}) = 1$ if $i$ is even and $i \neq l$, and $\chi(w)(a_i) = \chi(a_iw) = \chi(2a_{i-1} + a_i) = \chi(a_i)^{-1}$ if $l$ is even. Since $n^2 = h_1(-1)h_3(-1) \cdots h_k(-1) = 1$ (see [4, p. 20]), we have $x^2 = h(\chi)h(\chi)^n = h(\chi)h(\chi^w) = 1$. Moreover, since $(\ast)$ is satisfied, $(q - 1)/2 = (p - 1)/2$, so $\mu^p = \mu$ and $[x, \phi] = 1$. We claim that $x \notin L(q)$. Indeed, if $x \in L(q)$, then $h(\chi) \in H$, and $\chi$ could be extended to an $F'_q$-character of $Q$: as $2\lambda_1 - a_i \in \langle 2a_1, 2a_2, \ldots, 2a_{i-1} \rangle$ we would then have $\chi(a_i) = \chi(\lambda_1)^2 \mod F'_q$, and hence $\chi(a_i) \in F'_q$, a contradiction. But then $X = \langle x, \phi \rangle$ is a complement for $L(q)$ in $\text{Aut} \ L(q)$. \hfill \Box

**Lemma 3.10.** If $L(q) = E_7(q)$ and $(\ast)$ is satisfied, then there is a complement $X$ of $L(q)$ in $\text{Aut} \ L(q)$.

**Proof.** We may assume $d = 2$. Let $\mu$ be a generator of the 2-Sylow subgroup of $F'_q$ and define $\chi$ by $\chi(a_1) = \chi(a_7) = \mu, \chi(a_3) = \mu^{-1}$, and $\chi(a_i) = 1$ otherwise. Let $n = n_1n_3n_7$ and consider the element $x = h(\chi)n$. Let $w = w_1w_3w_7$. Then $\chi(w)(a_i) = \chi(a_iw) = \chi(-a_i)$ if $i \in \{1, 3, 7\}$, $\chi(w)(a_i) = \chi(a_iw) = \chi(a_i) = 1$ if $i \in \{5, 6\}$, $\chi(w)(a_2) = \chi(a_2w) = \chi(a_1 + a_2 + a_3) = 1$, and $\chi(w)(a_4) = \chi(a_4w) = \chi(a_3 + a_4 + a_7) = 1$. Since $n^2 = h_1(-1)h_3(-1)h_7(-1) = 1$ (see
we have $x^2 = h(\chi)h(\chi)^n = h(\chi)h(\chi^m) = 1$. Moreover, since $(\ast)$ is satisfied, $(q - 1)_2 = (p - 1)_2$, so $\mu^0 = \mu$ and $[x, \phi] = 1$. We claim that $x \notin L(q)$. Indeed, if $x \in L(q)$, then $h(\chi) \in H$, and $\chi$ could be extended to an $F_q$-character of $Q$: as $2\lambda_1 = 3a_1 + 4a_2 + 5a_3 + 6a_4 + 4a_5 + 2a_6 + 3a_7$, we would then have $\chi(\lambda_1)^2 = \chi(3a_1 + 5a_3 + 3a_7) = \mu \in F_q^3$, a contradiction. But then $X = \langle x, \phi \rangle$ is a complement for $L(q)$ in $\operatorname{Aut} L(q)$. \hfill \Box

**Lemma 3.11.** If $L(q) = E_6(q)$ and $(\ast)$ is satisfied, then there is a complement $X$ of $L(q)$ in $\operatorname{Aut} L(q)$.

**Proof.** We may assume $d = 3$. Consider the subgroup $S = \langle X_{a_1}, X_{-a_1} \mid 1 \leq i \leq 5 \rangle$ of $E_6(q)$ and let $T$ be the subgroup of $\operatorname{Aut} E_6(q)$ consisting of the elements of the form $sh(\chi)$ with $s \in S$ and $\chi(a_1) = 1$. Let $Z = Z(S)$. Then $Z$ is cyclic of order 2, generated by $z = h_{a_1}(-1)h_{a_2}(-1)h_{a_3}(-1)$. Moreover, $S \cong \operatorname{SL}(6, q)/\langle \omega \rangle$ with $\omega$ a primitive 3rd root of unity in $F_q$, $S/Z \cong A_5(q) \cong \operatorname{PSL}(6, q)$, $T$ normalizes $S$ and acts by conjugation on $S/Z \cong A_5(q)$ as the group of the inner-diagonal automorphism of $A_5(q)$. We have proved in Proposition 1.11 that if $(\ast)$ is satisfied, then there exist $g_1 \in \operatorname{GL}(6, q) \setminus \operatorname{SL}(6, q)$ and $g_2 \in \operatorname{SL}(6, q)$ such that $(g_2)^2 = 1$, $[g_2, \phi] = 1$, $\phi$ and $g_2$ normalize $\langle g_1 \rangle$ and $g_2^3 \in Z(\operatorname{SL}(6, q))$. Thus there exist an element $y \in S$, centralized by $\phi$ and such that $ye$ has order 2 (where $e$ is the graph automorphism of $L(q)$), and an element $x = sh(\chi) \in T$ such that $x \notin S$, $x^3 \in Z$, and $\langle x \rangle$ is normalized by $\phi$ and by $ye$. We claim that $X = \langle x^2, \phi, ye \rangle$ is a complement for $L(q)$ in $\operatorname{Aut} L(q)$. We only have to prove that $x^2 \notin L(q)$. Since $x \notin S$, we have $\chi(a_1)\chi(a_2)^{-1}\chi(a_4)^{-1} \notin F_q^3$. If $x^2 \in L(q)$, then $h(\chi^2) \in H$, and $\chi^2$ could be extended to a $F_q$-character $\tilde{\chi}$ of $Q$: as $3\lambda_1 = 4a_1 + 5a_2 + 6a_3 + 4a_4 + 2a_5 + 3a_6$, we would then have $(\chi(a_1))^2 \chi(a_2)^{-1} \chi(a_4)^{-1} \chi(a_5)^{-1} = \tilde{\chi}(\lambda_1)^3$ mod $F_q^3$, a contradiction. \hfill \Box

**Lemma 3.12.** If $L(q) = D_l(q)$ with $l$ even and $(\ast)$ is satisfied, then there is a complement $X$ of $L(q)$ in $\langle L(q), H \rangle$, which is normalized by the Frobenius and the graph automorphisms.

**Proof.** We may assume $d \neq 1$. In this case $H/H \cong Z_2 \times Z_2$. Moreover, if $\chi$ is an $F_q$-character of $P$ with $\chi(a_i) = 1$ for $i > 4$ then $h(\chi) \in H$ only if $\chi(a_i)\chi(a_j) \in F_q^3$ for each $(i, j) \in \{(1, 2, 4)\}^2$. Let $\mu$ be a generator of the Sylow 2-subgroup of the multiplicative group of the field $F_q$. For $i \in \{1, 2, 4\}$ let $\chi_i$ be the $F_q$-character of $P$ defined by $\chi_i(a_3) = \mu^{-1}$, $\chi_i(a_i) = 1$, and $\chi_i(a_j) = \mu$ if $j \notin \{i, 3\}$. Consider the elements $x_1 = h(\chi_1)n_2n_4$, $x_2 = h(\chi_2)n_1n_4$, and $x_4 = h(\chi_4)n_1n_2$. It can be easily verified that $x_1, x_2, x_4$ generate a complement $X$ of $L(q)$ in $\langle L(q), H \rangle$. Since $(q - 1)/2 = 1$, $(q - 1)/(p - 1)$ is odd and $\mu^\phi = \mu$. This implies that $X$ is centralized by the field automorphisms. Any graph automorphism $\epsilon$ of $D_l(q)$ arises from a permutation of the roots $a_1, a_2$ when $l \neq 4$, and from a permutation of the roots $a_1, a_2, a_4$ when $l = 4$. This
automorphism $\epsilon$ permutes in the same way the three generators $x_1, x_2, x_4$ of $X$, so $X$ is normalized by the graph automorphisms.

**Lemma 3.13.** If $L(q) = D_l(q)$ with $l$ odd and (*) is satisfied, then there is a complement $X$ of $L(q)$ in Aut $L(q)$.

**Proof.** We may assume $d = (4, q - 1) \neq 1$. We first deal with the case $d = 2$. Consider the subgroup $S = \langle X_{a_i}, X_{-a_i} \mid 1 \leq i \leq 3 \rangle$ of $D_l(q)$ and let $T$ be the subgroup of Aut $D_l(q)$ consisting of the elements of the form $sh(\chi)$ with $s \in S$ and $\chi(a_i) = 1$ for $i \geq 4$. Then $S \cong A_3(q) \cong \text{PSL}(4, q)$ and $T$ acts by conjugation on $S$ as the group of the inner-diagonal automorphism of $S$. We have proved in Theorem 1.12 that if (*) is satisfied, then there exists a complement $(x)$ of $\text{PSL}(4, q)$ in $\text{PGL}(4, q)$, normalized by $\phi$ and $\iota$. When we identify $\text{PSL}(4, q)$ with $A_3(q)$, the automorphism $\iota$ can be written as the product of an inner automorphism centralized by $\phi$ with the graph automorphism. Note that the graph automorphism $\epsilon$ of $D_l(q)$ centralizes the root subgroup $X_{a_i}, 3 \leq i \leq l$, and acts on $T$ as the graph automorphism of $A_3(q)$. Thus there exist an element $y \in S$, centralized by $\phi$ and such that $ye$ has order 2, and an element $x = sh(\chi) \in T$ of order $d$ modulo $S$, which generate a subgroup normalized by $ye$ and $\phi$. We claim that $X = \langle x, \phi, ye \rangle$ is a complement for $L(q)$ in Aut $L(q)$. We only have to prove that $x \notin L(q)$. Since $x \notin S$, we have $\chi(a_1) \chi(a_2) \notin F_q^2$. If $x \in L(q)$, then $\chi$ could be extended to a $F_q$-character $\tilde{\chi}$ of $Q$; as $4\lambda_1 \in a_1 + a_2 + 2(a_1, a_2, a_3, a_4, \ldots, a_l)$, we would then have $\chi(a_1) \chi(a_2) \equiv \tilde{\chi}(\lambda_1)^4 \mod F_q^2$, which implies $\chi(a_1) \chi(a_2) \in F_q^2$, a contradiction.

Now assume $d = (q - 1, 4) = 4$. Let $\mu$ be a generator of the 2-Sylow subgroup of $F_q^2$ and define $\chi$ by $\chi(a_2) = \mu$, $\chi(a_1) = \chi(a_3) = 1$, $\chi(a_i) = 1$ if $i$ is even and $i \neq 2$, $\chi(a_4) = -\mu^{-1}$ if $i$ is odd, $i > 3$ and $i \equiv 1 \mod 4$, and $\chi(a_i) = -\mu$ if $i$ is odd, $i > 3$ and $i \equiv 3 \mod 4$. Let $n = n_1n_3n_2n_5n_7 \ldots n_l$ and consider the element $x = h(\chi)n$. Since $n^4 = 1$, we have $x^4 = (h(\chi)n)^4 = h(\chi)h(\chi)^n h(\chi)^n h(\chi)^{n^3} = h(\chi)w^\chi w^\chi w^w$, where $w = w_1w_3w_2w_5w_7\ldots w_1$. But $a_1(1 + w + w^2 + w^3) = 0$ if $i$ is odd or $i = 2$, $a_2(1 + w + w^2 + w^3) = 2(a_1 + a_2 + a_3 + a_4 + a_5)$ and $a_1(1 + w + w^2 + w^3) = 2(a_i + 1 + 2a_i + a_i + 1)$ if $i$ is even and $i > 3$. Hence $\chi w^\chi w^w = 1$ and $x^4 = 1$. Moreover, $x^2x = h(\chi)^n h(\chi)n = h(\chi)^c h_5(-1)h_7(-1)\ldots h_l(-1)h(\chi)^n = h(\tilde{\chi}w\tilde{\chi}w)$, where $\tilde{\chi}(a_1) = \chi(a_2)$, $\tilde{\chi}(a_2) = \chi(a_1)$, and $\tilde{\chi}(a_i) = \chi(a_i)$ otherwise, $\psi(a_4) = -1$, and $\psi(a_i) = 1$ otherwise. Now,

\[
\begin{align*}
\tilde{\chi}\psi(\chi(a_1)) = \chi(a_2)\chi(a_1w) = \chi(a_2)(-a_1 - a_2 - a_3) = 1,
\tilde{\chi}\psi(\chi(a_2)) = \chi(a_1)\chi(a_2w) = \chi(a_3) = 1,
\tilde{\chi}\psi(\chi(a_3)) = \chi(a_3)\chi(a_3w) = \chi(a_1) = 1,
\tilde{\chi}\psi(\chi(a_4)) = \chi(a_4)\chi(a_4w) = -\chi(a_2 + a_3 + a_4 + a_5) = 1,
\tilde{\chi}\psi(\chi(a_i)) = \chi(a_i)\chi(a_iw) = \chi(a_i)(-a_i) = 1 \text{ if } i \text{ is odd, } i \geq 5.
\end{align*}
\]
\[ \chi\psi\chi(a_i) = \chi(a_i)\chi(a_iw) = \chi(a_i)\chi(a_i - 1 + a_i + a_{i+1}) = 1 \]

if \( i \) is even, \( i \geq 6 \).

Hence we conclude \( x^* = x^{-1} \). Moreover, since \((*)\) is satisfied, we have \((q - 1)2 = (p - 1)2\), so \( \mu^p = \mu \) and \([x, \phi] = 1 \). We claim that \( x^2 \notin L(q) \). Since \( x^2 \notin S \), we have \( \chi(a_1)\chi(a_2) \notin F_q^2 \). If \( x^2 \in L(q) \), then \( \chi^2 \) could be extended to a \( F_q \)-character \( \tilde{\chi} \) of \( Q \); as \( 4\lambda_1 \in a_1 + a_2 + 2(a_1, a_2, a_3, \ldots, a_l) \), we would then have \( \mu^2 = \chi(a_1)\chi(a_2)^2 \equiv \tilde{\chi}(\lambda_1)^4 \mod F_q^4 \), a contradiction. But then \( X = \langle x, \phi, \epsilon \rangle \) is a complement for \( L(q) \) in \( \text{Aut} L(q) \). \( \square \)

4. Twisted groups of Lie type

We begin with a short description of the twisted groups. Let \( G = L(q^*) \) be a group of Lie type whose Dynkin diagram has a non trivial symmetry \( \rho \) of order \( \rho \). If \( \epsilon \) is the graph automorphism corresponding to \( \rho \), let us suppose that \( L(q^*) \) admits a non trivial field automorphism \( \alpha \) such that the automorphism \( \sigma = \epsilon \alpha \) satisfies \( \sigma^* = 1 \). If such an automorphism \( \sigma \) does exist, the twisted group \( ^sL(q) \) is defined as the subgroup of the group \( L(q^*) \) which is fixed elementwise by \( \sigma \). The structure of \( ^sL(q) \) is similar to that of a Chevalley group: if \( \Phi \) is the root-system fixed in \( L(q^*) \), the automorphism \( \sigma \) determines a partition of \( \Phi = \cup S_i \). If \( R \) is an element of the partition, we denote by \( X_R \) the subgroup \( \langle X_r \mid r \in R \rangle \) of \( L(q^*) \), and by \( X_R^k \) the subgroup \( \{ x \in X_R \mid x^k = x \} \) of \( ^sL(q) \). The group \( ^sL(q) \) is generated by the groups \( X_{S_i}^1, \Phi = \cup S_i \); in fact, the subgroups \( X_{S_i}^1 \) play the role of the root-subgroups.

An element \( R \) of the partition which contains a simple root is said to be a simple set. We have \( \text{Aut}(^sL(q)) = \langle L(q), \hat{H}^1, \phi \rangle, \) where \( \phi \) is the Frobenius automorphism and \( \hat{H}^1 = N_{\hat{H}}(^sL(q)) \). Note that \( h(\chi) \in \hat{H}^1 \) if and only if \( \chi(rp) = \chi(r)^a \) for any \( s \in \Phi \). Moreover, a diagonal automorphism \( h \in \hat{H}^1 \) is inner if and only if \( h \in H^1 = H \cap \hat{L}(q) \). Let \( d \) be the order of \( H^1/H^1 \). Then \( d = 1 \) except in the following cases:

<table>
<thead>
<tr>
<th>(^sL(q))</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^sA_1(q))</td>
<td>((l + 1, q + 1))</td>
</tr>
<tr>
<td>(^sD_1(q))</td>
<td>((q', q + 1))</td>
</tr>
<tr>
<td>(^sE_6(q))</td>
<td>((3, q + 1))</td>
</tr>
</tbody>
</table>

We will prove the following result:

**Theorem 4.1.** Suppose that \( q = p^m \) and let \( ^sL(q) \) a twisted group of Lie type.

1. If \( ^sL(q) \neq ^sD_1(q) \), then \( ^sL(q) \) has a complement in \( \text{Aut} ^sL(q) \) if and only if \( (q', d, m) = 1 \).
2. If \( l \) is odd, then \( ^sD_1(q) \) has a complement in \( \text{Aut} ^sD_1(q) \) for any choice of \( q \).
(3) If \( l \) is even, then \( 2D_l(q) \) has a complement in \( \text{Aut}^2 D_l(q) \) if and only if \( d = 1 \).

We have already shown that this is true for \( 2A_l(q) \cong \text{PSU}(l + 1, q) \). When \( d = 1 \), \( \langle \phi \rangle \) is a complement for \( 2^l L(q) \) in \( \text{Aut}^+ L(q) \), so we only have to deal with the cases \( 2D_l(q) \) and \( 2E_6(q) \).

**Lemma 4.2.** If \( l \) is odd, there exists a complement \( X \) of \( 2D_l(q) \) in \( \text{Aut}^2 D_l(q) \).

**Proof.** We may assume \( d \neq 1 \). First suppose \( d = (q^l + 1, 4) = 2 \) and note that this implies \( \left( \frac{q^l + 1}{2}, 2, m \right) = 1 \). Consider the simple sets \( R_1 = \{ a_1, a_2 \} \), \( R_2 = \{ -a_1, -a_2 \} \), \( R_3 = \{ a_3 \} \), \( R_4 = \{ -a_3 \} \). Let \( S = \langle X_{R_1}, X_{R_2}, X_{R_3}, X_{R_4} \rangle \leq 2D_l(q) \) and let \( T \) be the subgroup of \( \text{Aut}^2 D_l(q) \) consisting of the elements of the form \( sh(\chi) \) with \( s \in S, h(\chi) \in H^1 \) and \( \chi(a_i) = 1 \) for \( i \geq 4 \). Then \( S \cong 2A_4(q) \cong \text{PSU}(4, q) \) and \( T \) acts by conjugation on \( S \) as the group of the inner-diagonal automorphism of \( S \). Since \( \left( \frac{q^l + 1}{2}, 2, m \right) = 1 \), by Theorem 2.9 \( \text{PSU}(4, q) \) has a complement in \( \text{Aut}(\text{PSU}(4, q)) \). Therefore there exist \( t = s_1 h(\chi) \in T \) and \( s_2 \in S \) such that \( \langle t \rangle \) is a complement for \( S \) in \( T \) normalized by \( s_2 \). We claim that \( X = \langle t, s_2 \rangle \) is a complement for \( 2D_l(q) \) in \( \text{Aut}^2 D_l(q) \). We only have to prove that \( t \notin 2D_l(q) \). Since \( t \notin S \), we have \( \chi(a_1) \notin (F_q)^2 \). If \( t \notin 2D_l(q) \), then \( \chi \) could be extended to an \( F^+ \)-character \( \chi \) of \( Q \) satisfying \( \hat{\chi}(\lambda_2) = \chi(\lambda_1)^q \). As \( 2(\lambda_1 - \lambda_2) = a_1 - a_2 \), we would then have \( \chi(\lambda_1) = \chi(\lambda_1)^{q-1} \), which implies \( \chi(a_1) \notin (F_q)^2 \), a contradiction.

Now assume \( d = (q^l + 1, 4) = 4 \). Let \( \mu \) be a generator of the 2-Sylow subgroup of \( F_q^* \) and define \( \chi \) by \( \chi(a_1) = \mu, \chi(a_2) = \mu^q, \chi(a_3) = -1 \), and \( \chi(a_i) = 1 \) otherwise. Let \( n = n_3 n_1 n_2 n_5 n_7 \ldots n_i \) and consider the element \( x = h(\chi)n \). Since \( [n, \phi] = [n, \epsilon] = 1 \), we have \( x \in 2D_l(q) \). Arguing as in the proof of Lemma 3.13 it can be shown that \( x^4 = 1 \). Now let \( y = n_1 n_2 \phi \). We claim that \( x^y = x^{-1} \). Indeed, \( x^y x = (h(\chi)n)^y h(\chi)n = h(\chi)^{n_1 n_2 \phi n_1 n_2 \phi n_1 n_2 \phi n_1 n_2 \phi} h(\chi)n = h(\chi^\phi) n_1 n_2 \phi (-1)^{h(-1)^h(\chi)n} = h(\chi^\phi)^{n_1 n_2 \phi} (-1)^h(-1)^h(\chi)n \), where \( \psi(a_1) = \psi(a_2) = -1 \), and \( \psi(a_i) = 1 \) otherwise, and \( w = w_3 w_1 w_2 w_5 w_7 \ldots w_l \). Let \( \bar{\chi} = (\chi^\phi)^{w_1 w_2 \psi} \chi^n \). Then

\[
\bar{\chi}(a_1) = -\chi(a_1 w_1 w_2)^p \chi(a_1 w) = -\chi(-a_1)^p \chi(a_2 + a_3) = \mu^{q-p} = 1,
\]

\[
\bar{\chi}(a_2) = -\chi(a_2 w_1 w_2)^p \chi(a_2 w) = -\chi(-a_2)^p \chi(a_1 + a_3) = \mu^{q-p} = 1,
\]

\[
\bar{\chi}(a_3) = \chi(a_3 w_1 w_2)^p \chi(a_3 w) = \chi(a_1 + a_2 + a_3)^p \chi(-a_1 - a_2 - a_3) = \mu^{q+1(p-1)} = 1,
\]

\[
\bar{\chi}(a_4) = \chi(a_4 w_1 w_2)^p \chi(a_4 w) = \chi(a_4)^p \chi(a_1 + a_2 + a_3 + a_4 + a_5) = -\mu^q = 1,
\]

\[
\bar{\chi}(a_i) = \chi(a_i w_1 w_2)^p \chi(a_i w) = \chi(a_i)^p \chi(-a_i) = 1 \text{ if } i \text{ is odd, } i \geq 5,
\]
\[ \bar{\chi}(a_i) = \chi(a_iw_1w_2)^p\chi(a_iw) = \chi(a_i)^p\chi(a_{i-1} + a_i + a_{i+1}) = 1 \]

if \( i \) is even, \( i > 4 \).

We claim that \( \langle x, n_1n_2\phi \rangle \) is a complement for \( 2D_l(q) \) in \( \text{Aut}^2 D_l(q) \). We only have to prove that \( x^2 \notin 2D_l(q) \). If \( x^2 \in 2D_l(q) \), then \( \chi^2 \) could be extended to a \( F_q^* \)-character of \( Q \) satisfying \( \bar{\chi}(\lambda_2) = \chi(\lambda_1)^q \). As \( 2(\lambda_1 - \lambda_2) = a_1 - a_2 \), we have \( \bar{\chi}(\lambda_1)^{2(q-1)} = \mu^{2(q-1)} \). Moreover, from \( \lambda_1 + \lambda_2 = \frac{q-1}{2}(a_1 + a_2) + (l - 2)a_3 + \cdots + a_l \) we deduce \( \bar{\chi}(\lambda_1)^{q+1} \in \langle F_q^* \rangle^{2(q+1)} \), so

\[ \mu^{q^2-1} = \mu^{2(q-1)2^{q+1}} = \bar{\chi}(\lambda_1)^{2(1-q)2^{q+1}} = 1, \]

which is again a contradiction. \( \square \)

**Lemma 4.3.** If \( l \) is even and \( q \) is odd, then \( 2D_l(q) \) has no complement in \( \text{Aut}^2 D_l(q) \).

**Proof.** Assume that \( X \) is a complement of \( 2D_l(q) \) in \( \text{Aut}^2 D_l(q) \). We may assume \( X = \langle x, \phi y \rangle \), where \( y \in 2D_l(q) \) and \( x \) is an inner-diagonal automorphism of \( 2D_l(q) \) of order 2, centralized by \( \phi y \). We may write \( x = h(\chi)z \), with \( z \in 2D_l(q) \), \( \chi(a_1) = \lambda, \chi(a_2) = \lambda^q \), and \( \chi(a_i) = 1 \) for \( i \geq 3 \) (where \( \lambda \) is a generator of \( F_q^* \)). The inner diagonal automorphism group \( \langle 2D_l(q), \hat{H} \rangle \) can be viewed as a subgroup of \( \langle D_l(q^2), \hat{H} \rangle \). We claim that \( h(\chi) \notin H \). Indeed, if \( h(\chi) \in H \), then \( \chi \) could be extended to an \( F_q^* \)-character of \( Q \); as \( 2(\lambda_1 - \lambda_2) = a_1 + a_2, a_3, \ldots, a_l \) we would then have \( \lambda = \chi(a_1) \in \langle F_q^* \rangle^2 \), a contradiction. This implies that \( x \notin D_l(q^2) \). By Lang’s Theorem there exists \( g \in D_l(q^2) \) with \( (\phi y)^g = \phi \). In particular, \( x^g \in \langle D_l(q^2), \hat{H} \rangle \backslash D_l(q^2) \) and is centralized by \( \phi \). Using the Bruhat Decomposition in \( D_l(q^2) \) we may write \( x^g \) in the form \( x^g = u_1 h(\chi_1) n_{w} u \) with \( u_1 \in U \) and \( w \in U_w \). Then

\[ x^g = (x^g)^\phi = u_1^\phi h(\chi_1)^{\phi} n_{w}^{\phi} u^\phi = u_1^\phi h(\chi_1)^{\phi} n_{w}^{\phi} u^\phi. \]

Note that \( u_1^\phi \in U \) and \( u^\phi \in U_w \), so, by the uniqueness of the representation of \( x^g \), we deduce \( h(\chi_1)^{\phi} = h(\chi_1) \), and this implies \( \chi_1^\phi = \chi_1 \). Since \( x^g \notin D_l(q^2) \), we have \( h(\chi_1) \in \hat{H} \backslash \hat{H} \), which implies that there exists \( 1 \leq i \leq l \) with \( \chi(a_i) = \lambda^s \), for an odd integer \( s \). Therefore \( sp \equiv s \mod q^2 - 1 \). Hence \( (q^2 - 1)2 \leq (p - 1)2 \), but this is impossible. \( \square \)

**Lemma 4.4.** If \( 2E_6(q) \) has a complement in \( \text{Aut}^2 E_6(q) \), then \( \left\langle \frac{q+1}{q-1}, d, m \right\rangle = 1 \).

**Proof.** In this case \( d = (3, q + 1) \) and \( \left\langle \frac{q+1}{q-1}, d, m \right\rangle = 1 \) is equivalent to the condition that either \( d = 1 \) or \( (3, m) = 1 \). Suppose that \( d \neq 1 \). Assume that \( X \) is a complement of \( 2E_6(q) \) in \( \text{Aut}^2 E_6(q) \). We may assume \( X = \langle x, \phi y \rangle \), where \( y \in 2E_6(q) \) and \( x \) is an inner-diagonal automorphism of \( 2E_6(q) \) of order 3, centralized by \( (\phi y)^2 \). We may write \( x = \chi(h)z \), with \( z \in 2E_6(q) \), \( \chi(a_1) = \lambda \),
\(\chi(a_5) = \lambda^q\), and \(\chi(a_i) = 1\) otherwise (where \(\lambda\) is a generator of \(F_q^\times\)). The inner diagonal automorphism group \((^2E_6(q), H)\) can be viewed as a subgroup of \(\langle E_6(q^2), H \rangle\). We claim that \(h(\chi) \notin H\). Indeed, if \(h(\chi) \in H\), then \(\chi\) could be extended to an \(F_q^2\)-character of \(Q\); as \(3\lambda_1 = 4a_1 + 5a_2 + 6a_3 + 4a_4 + 2a_5 + 3a_6\), we would then have \(\chi(\lambda_1)^3 = \lambda^{3+2a}\), which implies \(\lambda \in (F_q^2)^3\), a contradiction. This implies that \(x \notin E_6(q^2)\). By Lang's Theorem there exists \(g \in E_6(q^2)\) with \((\phi g) = \phi\). In particular, \(x^g \in \langle E_6(q^2), H \rangle \setminus E_6(q^2)\) and is centralized by \(\phi^2\). Using the Bruhat Decomposition in \(E_6(q^2)\) we may write \(x^g\) in the form \(x^g = u_1 h(\chi_1) n_w u_2\) with \(u_1 \in U\) and \(u_2 \in U_w\). Arguing as in the previous lemma we deduce that \(h(\chi_1)^{\phi^2} = h(\chi_1)\), and this implies \(\chi_1^q = \chi_1\). Since \(x^g \notin E_6(q^2)\), we have \(h(\chi_1) \in H \setminus H\), which implies that there exists \(1 \leq i \leq 6\) with \(\chi(a_i) = \lambda^a\) for some integer \(a\) not divisible by 3. Therefore \(sp^2 \equiv s \mod q^2 - 1\).

Hence \((q^2 - 1)_3 \leq (p^2 - 1)_3\), which implies \((m, 3) = 1\) for otherwise \((q^2 - 1)_3 = (q + 1)_3 = (p + 1)_3 = (p^2 - 1)_3\).

\[\square\]

**Lemma 4.5.** If \((2^i, d, m) = 1\), then there is a complement of \(^2E_6(q)\) in \(\text{Aut}^2 E_6(q)\).

**Proof.** We may assume \(d = 3\). Consider the simple sets \(R_1 = \{a_1, a_5\}, R_2 = \{-a_1, -a_5\}, R_3 = \{a_2, a_4\}, R_4 = \{-a_2, -a_4\}, R_5 = \{a_3\}, R_6 = \{-a_3\}\). Let \(S = \langle X^R_i \mid 1 \leq i \leq 6 \rangle \leq ^2E_6(q)\) and let \(T = \text{Aut}^2 E_6(q)\) consisting of the elements of form \(sh(\chi)\) with \(s \in S, h(\chi) \in H\) and \(\chi(a_6) = 1\). Let \(Z = Z(S)\). Then \(Z\) is cyclic of order 2, generated by \(z = h_{a_1}(-1)h_{a_3}(-1)h_{a_5}(-1)\). Moreover, \(S \cong SU(6, q)/\langle \omega \rangle\) with \(\omega\) a primitive 3rd root of unity in \(F_q^2\), \(S/Z \cong \Delta A_5(q) \cong \text{PSU}(6, q)\), \(T\) normalizes \(S\) and acts by conjugation on \(S/Z \cong A_5(q)\) as the group of the inner-diagonal automorphism of \(A_5(q)\). We have proved in Proposition 2.8 that if \((2^i, d, m) = 1\), then there exist \(g_1 \in U(6, q) \setminus SU(6, q)\) and \(g_2 \in SU(6, q)\) such that \(|\phi g_2| = |\phi| = 2m\), \(\phi g_2\) normalizes \((g_1)\) and \(g_1^3 \in Z(\text{SL}(6, q))\). Thus there exist \(x \in S\) and \(x^3 \in Z\) (\(x\) is normalized by \(\phi y\) and \(|\phi y| = |\phi| = 2m\). We claim that \(X = \langle x^2, \phi y \rangle\) is a complement for \(^2E_6(q)\) in \(\text{Aut}^2 E_6(q)\). We only have to prove that \(x^2 \notin L(q)\). Since \(x \notin S\), we have \(\chi(a_1)\chi(a_2) \neq F_3\). If \(x^2 \in L(q)\), then \(h(\chi^2) \in H\), and \(\chi^2\) could be extended to an \(F_q^2\)-character \(\chi^g\) of \(Q\); as \(3\lambda_1 = 4a_1 + 5a_2 + 6a_3 + 4a_4 + 2a_5 + 3a_6\), we would then have \((\chi(a_1)\chi(a_2)^{-1}) \equiv \chi(\lambda_1) mod F_3\), a contradiction. \[\square\]

**References**


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