Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 299–304
S 0019-2082

ON SUBGROUPS OF FREE BURNSIDE GROUPS OF LARGE ODD EXPONENT

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ABSTRACT. We prove that every noncyclic subgroup of a free m-generator Burnside group $B(m,n)$ of odd exponent $n \gg 1$ contains a subgroup H isomorphic to a free Burnside group $B(\infty,n)$ of exponent n and countably infinite rank such that, for every normal subgroup K of H, the normal closure $\langle K\rangle^{B(m,n)}$ of K in $B(m,n)$ meets H in K. This implies that every noncyclic subgroup of $B(m,n)$ is SQ-universal in the class of groups of exponent n.

A group G is called SQ-*universal* if every countable group is isomorphic to a subgroup of a quotient of G. One of the classical embedding theorems proved by Higman, B. Neumann and H. Neumann [HNN49] states that every countable group G embeds in a 2-generator group or, equivalently, a free group F_2 of rank 2 is SQ-universal. Recall that the proof of this theorem makes use of the following natural definition. A subgroup H of a group G is called a Q-*subgroup* if for every normal subgroup K of H the normal closure $\langle K\rangle^G$ of K in G meets H in K, i.e., $\langle K\rangle^G \cap H = K$. For example, the factors G_1, G_2 of the free product $G_1 * G_2$ or the direct product $G_1 \times G_2$ are Q-subgroups of $G_1 * G_2$ or $G_1 \times G_2$, respectively. In particular, a free group F_m of rank $m > 1$, where $m = \infty$ means countably infinite rank, contains a Q-subgroup isomorphic to F_k for every $k \le m$. On the other hand, it was proved in [HNN49] that the subgroup $\langle a^{-1}b^{-1}ab^{-i}ab^{-1}a^{-1}b^{i}a^{-1}bab^{-i}aba^{-1}b^{i} \mid i = 1,2,\dots\rangle$ of $F_2 = F_2(a,b)$ is a Q-subgroup of F_2 isomorphic to F_∞ and freely generated by the indicated elements. In [NN59] B. Neumann and H. Neumann found simpler generators and proved that $\langle [b^{-2i+1}ab^{2i-1}, a] \mid i = 1,2,\dots\rangle$, where $[x,y] = xyx^{-1}y^{-1}$ is the commutator of x and y, is a Q-subgroup of F_2 isomorphic to F_∞ and freely generated by the indicated elements. It is obvious that the property of being a Q-subgroup is transitive. Therefore, a group G contains a Q-subgroup isomorphic to F_∞ if and only if G contains a Q-subgroup isomorphic to F_m, where $m \ge 2$.

Received August 30, 2002.
2000 *Mathematics Subject Classification.* Primary 20E07, 20F05, 20F50.
Supported in part by NSF grant DMS 00-99612.

Ol'shanskii [O95] proved that any nonelementary subgroup of a hyperbolic group G (in particular, $G = F_m$) contains a Q-subgroup isomorphic to F_2. In particular, if G is a nonelementary hyperbolic group then G is SQ-universal.

It follows from an embedding theorem of Obraztsov (see Theorem 35.1 of [O89]) that any countable group of odd exponent $n \gg 1$ embeds in a 2-generator group of exponent n and so a free 2-generator Burnside group $B(2,n) = F_2/F_2^n$ of exponent n is SQ-universal in the class of groups of exponent n. Interestingly, the proof of this theorem has nothing to do with free Q-subgroups of the Burnside group $B(2,n)$ and does not imply the existence of such subgroups in $B(2,n)$.

Ol'shanskii and Sapir proved in [OS02] (among many other things) that for odd $n \gg 1$ the group $B(m,n)$ with some $m = m(n)$ does contain Q-subgroups isomorphic to $B(\infty,n) = F_\infty/F_\infty^n$. Sonkin [S02] further refined their arguments to show that for odd $n \gg 1$ the group $B(2,n)$ contains a Q-subgroup isomorphic to $B(\infty,n)$. This also implies that $B(2,n)$ is SQ-universal in the class of groups of exponent n.

Recall that the existence of an embedding $B(\infty,n) \to B(2,n)$ for odd $n \geq 665$, without the Q-subgroup property, was first proved by Shirvanian [Sh76]. Atabekian [A86], [A87] showed that for odd $n \gg 1$ (e.g., $n > 10^{78}$) every noncyclic subgroup of $B(m,n)$ contains a subgroup isomorphic to $B(2,n)$ (and so, by Shirvanian's result, contains a subgroup isomorphic to $B(\infty,n)$). A short proof of this theorem of Atabekian due to the author was incorporated in [O89] (Theorem 39.1). It turns out that the same idea of "fake" letters and using relations of Tarski monsters yields not only embeddings but also embeddings as Q-subgroups and significantly shortens the corresponding arguments in [OS02] and [S02]. The aim of this note is to elaborate on this idea and to strengthen Atabekian's theorem as follows.

Theorem. *Let n be odd, $n \gg 1$ (e.g., $n > 10^{78}$), and $B(m,n)$ be a free m-generator Burnside group of exponent n. Then every noncyclic subgroup of $B(m,n)$ contains a Q-subgroup of $B(m,n)$ isomorphic to $B(\infty,n)$. In particular, every noncyclic subgroup of $B(m,n)$ is* SQ*-universal in the class of groups of exponent n.*

Proof. To be consistent with the notation of [O89], rename the exponent n by n_0. Consider an alphabet $\mathcal{A} = \{a_1, \dots, a_m\}$ with $m \geq 2$. Let $G(\infty)$ be a presentation for the free Burnside group $B(\mathcal{A}, n_0)$ of exponent n_0 in the alphabet $\mathcal{A}$ constructed as in Sect. 18.1 of [O89] and let $\mathcal{H}$ be a noncyclic subgroup of $B(\mathcal{A}, n_0)$. Conjugating if necessary, by Lemma 39.1 of [O89], we can suppose that there are words $F, T \in \mathcal{H}$ such that F is a period of some rank $|F|$ (with respect to the presentation $G(\infty)$ of $B(\mathcal{A}, n_0)$), $|T| < 3|F|$ and $FT \neq TF$ in $B(\mathcal{A}, n_0)$.

Consider a presentation

(1) $$\mathcal{K} = \langle b_1, b_2 \parallel R = 1, R \in \bar{\mathcal{R}}_0 \rangle$$

for a 2-generator group $\mathcal{K}$ of exponent n_0 ($\mathcal{K}$ may be trivial).

Set $\bar{\mathcal{A}} = \mathcal{A} \cup \{b_1, b_2\}$ and define $\bar{G}(0) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \bar{\mathcal{R}}_0 \rangle$. Clearly, $\bar{G}(0)$ is the free product of the free group $G(0) = F(\mathcal{A})$ in $\mathcal{A}$ and $\mathcal{K}$. If W is a word in $\bar{\mathcal{A}}^{\pm 1} = \mathcal{A} \cup \mathcal{A}^{-1}$ then its *length* $|W| = |W|_{\mathcal{A}}$ is defined to be the number of letters $a_k^{\pm 1}$, $a_k \in \mathcal{A}$, in W. In particular, $|b_1| = |b_2| = 0$. Using this new length, we construct groups $\bar{G}(i) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \bar{\mathcal{R}}_i \rangle$ by induction on $i \geq 1$ exactly as in Sect. 39.1 of [O89], that is, the set $\bar{\mathcal{S}}_i = \bar{\mathcal{R}}_i \setminus \bar{\mathcal{R}}_{i-1}$ of defining words of rank i consists of all relators of the first type A^{n_0}, $A \in \bar{\mathcal{X}}_i$, if $i < |F|$. As before, we observe that the set $\bar{\mathcal{X}}_{|F|}$ of periods of rank $i = |F|$ can be chosen so that $F \in \bar{\mathcal{X}}_{|F|}$. For $i = |F|$ the set $\bar{\mathcal{S}}_i = \bar{\mathcal{R}}_i \setminus \bar{\mathcal{R}}_{i-1}$ consists of all relators of the first type A^{n_0}, $A \in \bar{\mathcal{X}}_i$, and two relators of the second type which are words of the form

$$(2) \qquad b_1 F^n T F^{n+2} \dots T F^{n+2h-2}, \quad b_2 F^{n+1} T F^{n+3} \dots T F^{n+2h-1}.$$

For $i > |F|$ the set $\bar{\mathcal{S}}_i = \bar{\mathcal{R}}_i \setminus \bar{\mathcal{R}}_{i-1}$ again consists of all relators A^{n_0} of the first type only, $A \in \bar{\mathcal{X}}_i$. Thus, the groups

$$\bar{G}(i) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \bar{\mathcal{R}}_i \rangle, \qquad \bar{G}(\infty) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \cup_{j=0}^{\infty} \bar{\mathcal{R}}_j \rangle$$

are constructed.

Consider a modification of condition R (see Sect. 25.2 of [O89]), that will be called *condition* R', in which property $R4$ is replaced by the following condition:

$R4'$ The words T_k are not contained in the subgroup $\langle A \rangle$ of the group $\bar{G}(i-1)$, $i \geq 1$, except for the case when $k = 1$, $|T_1| = 0$ and the integers n_1, n_k have the same sign.

Let $\bar{\Delta}$ be a diagram over the graded presentation $\bar{G}(i)$, $i \geq 0$. According to the new definition of the word length, we define the length $|p|$ of a path p so that $|p| = |\varphi(p)|$. In particular, if e is an edge of $\bar{\Delta}$ with $\varphi(e) = b_k^{\pm 1}$, $k = 1, 2$, then $|e| = 0$. Hence such an edge e is regarded as being a 0-edge of $\bar{\Delta}$ of *type 2*. Recall that if $\varphi(e) = 1$ then e is called in [O89] a 0-edge (we will specify that such an edge e is a 0-edge of *type 1*). All faces labelled by relators of $\bar{G}(0)$ are also called 0-*faces* (or faces of rank 0) of $\bar{\Delta}$. A 0-face Π of $\bar{\Delta}$ has *type 1* if it is a 0-face in the sense of [O89]. Otherwise, i.e., when $\partial \Pi$ has a nontrivial label $R \in \bar{\mathcal{R}}_0^{\pm 1}$, a 0-face Π has *type 2*.

Note that the new definition of length and the existence of 0-edges of type 2 imply a number of straightforward changes in the analogs of definitions and lemmas of Sects. 18 and 25 of [O89] on group presentations with condition R'. (These changes are quite analogous to what was done in similar situations in the papers [I02a] and [I02b].) For example, in the definition of a simple in rank i word A (see Sect. 18.1 of [O89]) it is in addition required that $|A| > 0$. Lemma 25.1 of [O89] now claims that every reduced diagram $\bar{\Delta}$ on a sphere or torus has rank 0. Corollary 25.1 of [O89] is stated for $\bar{\mathcal{R}}_i \setminus \bar{\mathcal{R}}_0$ and Corollary 25.2 of [O89] is now missing. In Lemma 25.2 of [O89] we allow in addition that X be conjugate to a word of length 0. Lemmas 25.12–25.15 of [O89] are left out.

Repeating the proof of Lemma 27.2 of [O89] (and increasing the number of short sections in Lemma 27.1 of [O89] from 2 to 3), we can show that the presentations $\bar{G}(i)$ and $\bar{G}(\infty)$ satisfy condition R'. Furthermore, it is straightforward to check that the proofs of Lemmas 26.1–26.5 of [O89] for a graded presentation with condition R' remain valid (with obvious minor changes in the arguments of proofs of Lemmas 26.1–26.2 of [O89] caused by the possibility that $|T_1| = 0$). Thus, by Lemma 26.5 of [O89], any reduced diagram over $\bar{G}(i)$ (or $\bar{G}(\infty)$) is a B-map.

By definition and by the analogue of Lemma 25.2 of [O89], the group $\bar{G}(\infty)$ has exponent n_0. Suppose U is a word in $\{b_1^{\pm 1}, b_2^{\pm 1}\}$ and $U = 1$ in the group $\bar{G}(\infty)$. Let $\bar{\Delta}$ be a reduced diagram over $\bar{G}(\infty)$ with $\varphi(\partial\bar{\Delta}) \equiv U$. Since $|\partial\bar{\Delta}| = 0$, it follows from Theorem 22.4 of [O89] that $r(\bar{\Delta}) = 0$. Hence $U = 1$ in the group $\mathcal{K}$ given by (1). This means that $\mathcal{K}$ naturally embeds in $\bar{G}(\infty)$.

Let

$$V_1 = (F^n T F^{n+2} \dots T F^{n+2h-2})^{-1}, \quad V_2 = (F^{n+1} T F^{n+3} \dots T F^{n+2h-1})^{-1}.$$

Observe that, in view of the relators (2), the group $\bar{G}(\infty)$ is naturally isomorphic to the quotient

$$B_{\mathcal{K}}(\mathcal{A}, n_0) = \langle B(\mathcal{A}, n_0) \parallel R(V_1, V_2) = 1,\ R(b_1, b_2) \in \bar{\mathcal{R}}_0 \rangle$$

of $B(\mathcal{A}, n_0)$. Hence, the subgroup $\langle V_1, V_2\rangle$ of $B_{\mathcal{K}}(\mathcal{A}, n_0)$ is isomorphic to the group $\mathcal{K}$ given by (1) under the map $V_1 \to b_1$, $V_2 \to b_2$. Since $\mathcal{K}$ is an arbitrary 2-generator group of exponent n_0, it follows that $\langle V_1, V_2\rangle$ is a Q-subgroup of $B(\mathcal{A}, n_0)$ isomorphic to $B(2, n_0)$.

Now we will show that $B(\infty, n_0)$ embeds in $B(2, n_0)$ as a Q-subgroup. To do this we repeat the above arguments with some changes. We let $\mathcal{A} = \{a_1, a_2\}$ (so that $m = 2$), and $\mathcal{B} = \{b_1, b_2, \dots\}$ be a countably infinite alphabet. Let

$$\mathcal{K} = \langle \mathcal{B} \parallel R = 1, R \in \bar{\mathcal{R}}_0 \rangle \tag{3}$$

be a presentation of a finite or countable group of exponent n_0, $\bar{\mathcal{A}} = \mathcal{A} \cup \mathcal{B}$, and $\bar{G}(0) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \bar{\mathcal{R}}_0\rangle$.

As before, constructing groups $\bar{G}(i)$ by induction on $i \geq 1$, we first define the set $\bar{\mathcal{X}}_i$ of periods of rank $i \geq 1$. It is easy to show that each $\bar{\mathcal{X}}_i$, $i \geq 1$, contains a word A_i in the alphabet $\{a_1, a_2\}$ such that A_i is not in the cyclic subgroup $\langle a_1 \rangle$ of $\bar{G}(i-1)$. Then for every $F \in \bar{\mathcal{X}}_i$ we define the relator F^{n_0} and for the distinguished period $A_i \in \bar{\mathcal{X}}_i$ we introduce the second relator

$$b_i A_i^n a_1 A_i^{n+2} \dots a_1 A_i^{n+2h-2}.$$

These relators over all $F \in \bar{\mathcal{X}}_i$ form the set $\bar{\mathcal{S}}_i = \bar{\mathcal{R}}_i \setminus \bar{\mathcal{R}}_{i-1}$. As above, we set

$$\bar{G}(i) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \bar{\mathcal{R}}_i\rangle, \qquad \bar{G}(\infty) = \langle \bar{\mathcal{A}} \parallel R = 1, R \in \cup_{j=0}^{\infty} \bar{\mathcal{R}}_j\rangle$$

and show that these presentations satisfy condition R'. Similarly, we establish analogues of corresponding claims of Sects. 18 and 25–27 of [O89].

Suppose $U = U(\mathcal{B})$ is a word in $\mathcal{B}^{\pm 1}$ and $U = 1$ in the group $\bar{G}(\infty)$. Let $\bar{\Delta}$ be a reduced disk diagram over $\bar{G}(\infty)$ with $\varphi(\partial\bar{\Delta}) \equiv U$. It follows from Lemma 26.5 of [O89], Theorem 22.4 of [O89] and the equality $|\partial\bar{\Delta}| = 0$ that $r(\bar{\Delta}) = 0$. Hence $U = 1$ in the group $\mathcal{K}$ given by (3) and so $\mathcal{K}$ naturally embeds in $\bar{G}(\infty)$. As above, by definition and by Lemma 25.2 of [O89], the group $\bar{G}(\infty)$ has exponent n_0 and we can see that $\bar{G}(\infty)$ is naturally isomorphic to the quotient

$$B_{\mathcal{K}}(\mathcal{A}, n_0) = \langle B(\mathcal{A}, n_0) \parallel R(V_1, V_2, \dots) = 1,\ R(b_1, b_2, \dots) \in \bar{\mathcal{R}}_0 \rangle$$

of $B(\mathcal{A}, n_0)$, where $V_i = (A_i^n a_1 A_i^{n+2} \dots a_1 A_i^{n+2h-2})^{-1}$, $i = 1, 2, \dots$. Hence, the subgroup $\langle V_1, V_2, \dots \rangle$ of $B_{\mathcal{K}}(\mathcal{A}, n_0)$ is isomorphic to the group $\mathcal{K}$ under the map $V_i \to b_i$, $i = 1, 2, \dots$. Since $\mathcal{K}$ is an arbitrary finite or countable group of exponent n_0, it follows that $\langle V_1, V_2, \dots \rangle$ is a Q-subgroup of $B(\mathcal{A}, n_0) = B(2, n_0)$ isomorphic to $B(\infty, n_0)$.

The explicit estimate $n = n_0 > 10^{78}$ of the Theorem can be obtained by using the lemmas and explicit estimates of articles [O82] and [AI87] (see also [O85]) instead of those of [O89]. The proof of the Theorem is complete. $\square$

In conclusion, we remark that it is not difficult to show that $B(\infty, n)$ embeds in $B(2, n)$ for $n = 2^k \gg 1$ (see [IO97], [I94]) but it is not clear how to embed $B(\infty, n)$ in $B(2, n)$ as a Q-subgroup and it would be interesting to do so. It would also be of interest to find out whether $B(\infty, n)$ embeds (as a Q-subgroup) in every nonlocally finite subgroup of $B(m, n)$ for $n = 2^k \gg 1$.

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