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p-GROUPS OF MAXIMAL CLASS AS AUTOMORPHISM GROUPS

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Abstract. We classify the (finite) p-groups of maximal class that are isomorphic to the full automorphism group of a (finite or infinite) group. The only such p-groups are the nonabelian groups of order 8 and 3-groups in a certain family, whose structure is fully described. Up to isomorphism there is exactly one such 3-group for each even nilpotency class greater than 2, and none for other classes.

Of several kinds of groups it is known that they cannot be isomorphic to the full automorphism group of any group. The easiest and best-known example probably is that of finite nontrivial cyclic groups of odd order. Other examples include the symmetric group of degree 6 and the alternating groups of degree different from 1, 2 or 8. Among infinite groups the nontrivial free groups [2], the periodic nilpotent groups of infinite exponent [13], and the finite extensions of nontrivial periodic divisible abelian groups [1] also share this property. By contrast, every group is isomorphic to the outer automorphism group $\operatorname{Out} G = \operatorname{Aut} G/\operatorname{Inn} G$ of a suitable group G, as was first proved by Matumoto [10].

In this paper we consider finite p-groups of maximal class. The smallest such groups, those of order 8, are isomorphic to full automorphism groups of groups. Indeed, the dihedral group D_8 is isomorphic to $\operatorname{Aut} G$ if $G \simeq D_8$ or $G \simeq \mathcal{C}_2 \times \mathcal{C}_4$ (and for no other groups), while the quaternion group Q_8 is isomorphic to the automorphism group of a torsion-free abelian group (see [6], p. 272, Example 3). We shall prove that not many other p-groups of maximal class occur as full automorphism groups.

Theorem. *If p is a prime, a p-group A of maximal class is isomorphic to the full automorphism group $\operatorname{Aut} G$ of a group G if and only if one of the following cases occurs:*

1. *$p = 2$ and $A \simeq D_8$, in which case $G \simeq D_8$ or $G \simeq \mathcal{C}_4 \times \mathcal{C}_2$.*
2. *$p = 2$ and $A \simeq Q_8$, in which case G is torsion-free abelian.*

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(3) *$p = 3$ and there exists an integer n greater than 1 such that A is isomorphic to*

$$X_n = \langle x, y, t \mid x^{3^n} = y^{3^n} = 1, [x, y] = t^3 = x^{3^{n-1}}, \\ x^t = x^{-2}y^{-3},\ y^t = xy\rangle.$$

In this case G is infinite of nilpotency class 3.

In each of the three cases the nilpotency class of A is even, and G has cyclic derived subgroup.

As a consequence, this shows that—up to isomorphism—D_8 *is the only p-group of maximal class that is isomorphic to the full automorphism group of a finite group.*

For every integer $n > 1$ the group X_n has order 3^{2n+1} and class $2n$, and is metabelian, like every 3-group of maximal class. Hence *there exists no group G such that $\operatorname{Aut} G$ is a p-group of maximal class c if $p > 3$ or c is odd.* Still with reference to case (3), much more detail on the structure of G and A is given in Theorem 2.11, whose proof takes up the whole of Section 2. One of the features is that $|\operatorname{Out} G| = 3$, regardless of the integer n. An alternative description of the groups X_n as semidirect products is given in the comments following the same Theorem 2.11.

Our results can be compared with those of Fournelle, [4] and [5], who shows—among other things—which dihedral or generalized quaternion groups are isomorphic to the full automorphism group of an infinite group. Contrasting with our results, we also recall that many finite p-groups occur as the full automorphism group of a finite p-group, as shown by U. Martin [9]—see also the discussion on this point in [8].

1. Preliminary results

We start with a rather obvious remark:

Lemma 1.1. *Let L be an abelian nontrivial torsion-free group such that $\operatorname{Aut} L$ is periodic, and let n be a positive integer. Then L/L^n has exponent n.*

Proof. We argue by induction on n. If $n = 1$ there is nothing to prove. Let p be a prime divisor of n and let $m = n/p$. Then $\exp(L/L^m) = m$ by induction. Also, $L \simeq L^m$ and $L^n = (L^m)^p$. If $L^n = L^m$ then the mapping: $x \in L^m \mapsto x^p \in L^m$ would be an automorphism of infinite order, which is impossible since $\operatorname{Aut} L^m \simeq \operatorname{Aut} L$ is periodic. Thus $L^n < L^m$, so that $\exp(L/L^n) = n$. □

If a group G has finite automorphism group then $G/Z(G) \simeq \operatorname{Inn} G$ is also finite. Hence G' is finite and the set $\operatorname{tor} G$ of all periodic elements of G is a subgroup of G (containing G'). A theorem due to Nagrebeckiĭ ([11], also see

[12], Theorem 3.1) states that $\operatorname{tor} G$ is finite—still under the hypothesis that $\operatorname{Aut} G$ is finite. If $\operatorname{Aut} G$ is also a p-group then G is clearly nilpotent, because $G/Z(G)$ is, and we are going to show that $\operatorname{tor} G$ is a p-group as well, if G is infinite. We first record as a lemma a very special case of a theorem of Hallett and Hirsch describing the possible structure of the automorphism group of an abelian torsion-free group with finitely many automorphisms.

LEMMA 1.2 (Hallett-Hirsch; see [6], Theorem 116.1). *Let L be an abelian torsion-free group such that $\operatorname{Aut} L$ is finite, and let p be a prime. If Γ is a p-subgroup of $\operatorname{Aut} L$ then either Γ is an elementary abelian 3-group, or Γ embeds in a direct product of copies of Q_8.*

If H and K are normal subgroups of the group G, and $H \leq K$, then the group of the automorphisms of G that act trivially on both G/H and K is isomorphic to the group of derivations $\operatorname{Der}\bigl(G/K, C_H(K)\bigr)$—an isomorphism being obtained by mapping every such automorphism α to the derivation given by $xK \mapsto [x, \alpha]$. We will make frequent use of this well-known fact to produce automorphisms of G, particularly in the special case when $C_H(K)$ is contained in $Z(G)$, in which case $\operatorname{Der}\bigl(G/K, C_H(K)\bigr) = \operatorname{Hom}\bigl(G/K, H \cap Z(G)\bigr)$.

LEMMA 1.3. *Let G be an infinite group such that $\operatorname{Aut} G$ is a finite p-group for some prime p. Then $T := \operatorname{tor} G$ is a finite p-group. Moreover:*

- (i) *The factor $TZ(G)/T$ has a quotient of exponent exactly n, for every $n \in \mathbb{N}$.*
- (ii) *If $T \neq 1$ then G/G^{p^n} is finite, for every $n \in \mathbb{N}$.*
- (iii) *$\operatorname{Aut}(G/T)$ is finite.*
- (iv) *If $p > 3$ then $\operatorname{Aut} G$ acts trivially on G/T.*

Proof. By [12], Corollary 5.4, $Z := Z(G)$ also has finite automorphism group. Let $S = \operatorname{tor} Z = T \cap Z$. Since S is finite by Nagrebeckiĭ's theorem, $Z = S \times L$ for some torsion-free subgroup L. Then $\operatorname{Aut} L$ embeds in $\operatorname{Aut} Z$, hence it is finite as well. Now Lemma 1.1 shows that $\exp(L/L^n) = n$ for every $n \in \mathbb{N}$. As $L \simeq Z/S \simeq TZ/T$ this proves (i). Now, suppose that T has nontrivial q-component T_q for some prime $q \neq p$. Then $T_q \leq Z$, since G/Z is a p-group. From (i) and since G is nilpotent it follows that $\operatorname{Hom}(G/S, T_q)$ is a non-trivial q-group. But this group embeds in $\operatorname{Aut} G$, as $T_q \leq S \leq Z$, a contradiction. Therefore T is a p-group, as required.

Suppose that $T \neq 1$. Then $S = T \cap Z \neq 1$. But $\operatorname{Hom}(G/S, S)$ embeds in $\operatorname{Aut} G$, hence it is finite and so $G/G'G^pS$ is finite. Therefore G/G^p is finite. Part (ii) follows.

To prove (iii) we may assume that $T \neq 1$. Let $B = \operatorname{Aut}(G/T)$, and let $N = N_B(TZ/T)$. Since $TZ \geq G^{p^n}$ for some $n \in \mathbb{N}$, it follows from (ii) that $|B : N|$ is finite. Also, $\operatorname{Aut}\bigl(TZ/T\bigr) \simeq \operatorname{Aut}\bigl(Z/S\bigr)$ is finite, as we have shown in proving (i), so $N/C_N(TZ/T)$ is finite. Clearly $N/C_N(G/TZ)$ is finite too.

Now $C_N(TZ/T) \cap C_N(G/TZ) \simeq \mathrm{Hom}\big(G/TZ, TZ/T\big)$, and the latter is the trivial group, as G/TZ is finite and TZ/T is torsion-free. Thus B is finite, as required. Also, we may apply Lemma 1.2 to G/T now, and we immediately get (iv). $\square$

It will be crucial for our proofs that the groups that we consider have many characteristic subgroups.

Lemma 1.4. *Let G be group such that $\operatorname{Aut} G$ is a finite p-group for some prime p. Let $F = U/V$ be a characteristic section of G of order p^n for some $n \in \mathbb{N}$. Then there is a composition series between U and V each term of which is characteristic in G. Also, the nilpotency class of $\operatorname{Aut} G/C_{\operatorname{Aut} G}(F)$ is less than n.*

Proof. Let $X = F \rtimes A$. The first statement follows immediately from the fact that X is a finite p-group. The series in the statement has length n and is stabilized by $\operatorname{Aut} G$, hence $\operatorname{Aut} G/C_{\operatorname{Aut} G}(F)$ has class $n-1$ at most (see [7], Satz III.2.9). $\square$

We will also make use of some extension theory.

Lemma 1.5. *Let G be a group and let $G' \leq C \leq Z(G)$. If $\mathrm{Ext}\big((G/C), C\big) = 0$ then G has an automorphism that centralizes C and acts like inversion on G/C.*

Proof. Let $Q = G/C$. Let Δ be the cohomology class of the central extension $C \hookrightarrow G \twoheadrightarrow Q$. If $\alpha \in \operatorname{Aut} C$ and $\beta \in \operatorname{Aut} Q$, then there exists $\gamma \in \operatorname{Aut} G$ inducing α on C and β on Q if and only if $\Delta\alpha_* = \beta^*\Delta$, with reference to the natural actions of $\operatorname{Aut} C$ and $\operatorname{Aut} Q$ on $H^2(Q, C)$ (see [16], Proposition II.4.3); here C is viewed as a trivial Q-module. In the case that we are dealing with, $\alpha = 1$ and $\beta = -1$, so $\Delta\alpha_* = \Delta$ and we have only to check that β^* leaves Δ invariant to prove our statement. The Universal Coefficients Theorem (see [15], 11.4.18) yields a natural exact sequence $\mathrm{Ext}(Q, C) \rightarrowtail H^2(Q, C) \twoheadrightarrow \mathrm{Hom}\big(M(Q), C\big)$, where $M(Q)$ is the Schur multiplier of Q, hence a natural isomorphism $H^2(Q, C) \simeq \mathrm{Hom}\big(M(Q), C\big)$, since $\mathrm{Ext}(Q, C) = 0$ by hypothesis. Furthermore $M(Q) \simeq Q \wedge Q = (Q \otimes Q)/D$, where $D = \langle x \otimes x \mid x \in Q \rangle$ (see [15], 11.4.16). Thus Δ corresponds to a homomorphism in $\mathrm{Hom}(Q \otimes Q, C)$ whose kernel contains D, or, equivalently, to a bilinear map from $Q \times Q$ to C which maps the diagonal subgroup to the identity of C. This map is precisely the commutator map f, defined by $(xC, yC) \mapsto [x, y]$ for all $x, y \in G$ (see [16], Proposition II.5.4, or p. 109). Similarly $\beta^*\Delta$ corresponds to the map $(xC, yC) \in Q \times Q \mapsto \big((xC)^\beta, (yC)^\beta\big) = [x^{-1}, y^{-1}] \in C$. Since G has nilpotency class 2 (at most) then $[x^{-1}, y^{-1}] = [x, y]$ for all $x, y \in G$; thus this latter map is f and $\Delta = \beta^*\Delta$, as we had to prove. $\square$

Finally we state the following lemma for ease of reference. Its proof follows from standard calculations and we omit it.

Lemma 1.6. *Let G be a nilpotent group of class 3, and let p be an odd prime. Suppose that G' is a finite p-group and that $\gamma_3(G)$ has exponent p. Let $q = \exp\bigl(G'/\gamma_3(G)\bigr)$. Then, for all x, $y \in G$:*

(i) $[x^{kp}, y] = [x, y]^{kp}$, *for all* $k \in \mathbb{Z}$.
(ii) $(xy)^{pq} = x^{pq}y^{pq}$.

2. Necessity

The aim of this section is to prove that the only p-groups of maximal class that can occur as full automorphism groups of groups are those listed in the Theorem in the introduction—that they actually occur will be shown in the next section.

If the automorphism group of a group G is non-abelian of order 8 then it is easy to check that G must be as required in cases (1) and (2) of the Theorem—for instance this follows from [3], [4] and [5]. Thus we shall not need to consider this case any further.

For the sake of brevity, we fix some notation and hypotheses that will be in effect throughout the whole section.

Thus we let G be a group and p be a prime, and assume that $A := \operatorname{Aut} G$ is a p-group of maximal class c. We shall further assume that $|A| > 8$. Then we shall prove that $p = 3$ and A is isomorphic to one of the groups X_n defined in the introduction, for some integer $n > 1$, hence $c = 2n$. We shall also gain information on the structure of G. Coming back to notation, we set $Z := Z(G)$, $T = \operatorname{tor} G$ and $S = \operatorname{tor} Z = Z \cap T$, which is consistent with the usage in Lemma 1.3.

To start with, observe that Lemma 1.2 shows that G cannot be torsion-free abelian, for otherwise $A \simeq Q_8$. Hence $T \neq 1$ and so $S \neq 1$. Next, we see that if G is finite then we may always assume that G is p-group (as usual, G_π denotes the π-component of G):

Lemma 2.1. *If G is finite then $|G_{p'}| \leq 2$ and $A \simeq \operatorname{Aut} G_p$.*

Proof. Let q be a prime divisor of $|G|$ different from p. Then $\operatorname{Aut} G_q$ is isomorphic to a direct factor of A. Hence it is a p-group. It follows easily that either $p \neq 2$ and $|G_q| = 2$ or $p = 2$ and $|G_q| = q$. In this latter case $\operatorname{Aut} G_q$ is abelian and nontrivial. On the other hand, A has no nontrivial abelian direct factor, since it has maximal class. The lemma follows. □

We shall often use the fact that the normal structure of groups of maximal class is very restricted to obtain information on characteristic subgroups of G. A first instance is the next lemma.

Lemma 2.2. *There are two subgroups N and M of G such that:*

(i) *M is the only characteristic subgroup of index p in G.*
(ii) *N is the only characteristic subgroup of order p in G.*
(iii) *$Z(A) = C_A(M) \cap C_A(G/N)$.*

For every $n \in \mathbb{N}$ the quotient $G\big/[G,A]G^{p^n}$ is cyclic.

Proof. Lemma 1.4 and the fact that $S \neq 1$ ensure the existence of a characteristic subgroup N of order p in G; the existence of M is proved similarly, because G/G^p is finite and non-trivial (by Lemma 1.3 if G is infinite, and by Lemma 2.1 if G is finite). If $N \not\leq M$ then $G = N \times M$, which is impossible since M is characteristic in G and $\mathrm{Hom}(M,N) \neq 0$. Hence $N \leq M$. Now, $\mathit{\Gamma} := C_A(M) \cap C_A(G/N) \lhd A$, and $\mathit{\Gamma} \simeq \mathrm{Hom}(G/M,N) \simeq \mathcal{C}_p$, because $N \leq Z(G)$. Hence $\mathit{\Gamma} = Z(A)$. Thus $Z(A)$ centralizes every characteristic subgroup of index p in G. If there were another such subgroup, say M^*, then $Z(A)$ would centralize $MM^* = G$, which is impossible. This proves the uniqueness of M. Similarly, $[G, Z(A)]$ is contained in every characteristic subgroup of order p of G, and this proves the uniqueness of N. Parts (i)–(iii) are proved.

Finally, all subgroups of G containing $[G,A]$ are characteristic, so only one of them (namely M) has index p. From this we deduce the remaining claim. □

From now on, by N and M we will always mean the subgroups introduced in the previous lemma.

Lemma 2.3. *G is not abelian.*

Proof. If G is abelian then the inverting automorphism $g \mapsto g^{-1}$ belongs to $Z(A)$ and so centralizes M and G/N. Therefore $\exp M = \exp(G/N) = 2$. Hence either $G \simeq \mathcal{C}_4 \times E$ or $G \simeq E$, where E is elementary abelian. Since $\mathrm{Aut}\, E$ embeds in A then $\mathrm{Aut}\, E$ is a p-group, hence $|E| \leq 2$. Thus $G \simeq \mathcal{C}_4 \times \mathcal{C}_2$ and $A \simeq D_8$, a contradiction. □

Lemma 2.4. $\mathrm{Inn}\, G < A$. *Moreover:*

(i) *The characteristic subgroups H of G containing Z form a chain and a composition series between Z and G.*
(ii) *$M = Z[G,A]$.*

Proof. We have $|\mathrm{Inn}\, G| \geq p^2$ by Lemma 2.3. If $\mathrm{Inn}\, G = A$ then the group $\mathrm{Aut}_c\, G = C_A(\mathrm{Inn}\, G)$ of all central automorphisms of G would be $Z(A)$, which has order p. However, $\mathrm{Hom}(G/Z, S)$ can be embedded in $\mathrm{Aut}_c\, G$, and since $S \neq 1$ we also have $|\mathrm{Aut}_c\, G| \geq p^2$. Thus $\mathrm{Inn}\, G < A$. Now, let φ be the natural isomorphism from G/Z to $\mathrm{Inn}\, G$. If H is a subgroup of G containing Z, then H is characteristic in G if and only if $(H/Z)^\varphi \lhd A$. The subgroups of $\mathrm{Inn}\, G$

that are normal in A form a chain, because A has maximal class. This and Lemma 1.4 prove (i). Two characteristic subgroups lying between Z and G are $K = Z[G, A]$ and $F = ZG'G^p$. Since G/Z is not cyclic by Lemma 2.3 the factor G/F is also not cyclic, while G/K is cyclic by Lemma 2.2. Thus $K \nleq F$ and so $F < K$, which implies that K has index p, because G/F has exponent p. Lemma 2.2 (i) yields $K = M$, that is (ii). □

Lemma 2.5. *Assume that the class c of A is greater than 2. Then $C_A\big(Z_2(A)\big)$ is a maximal subgroup of A. Moreover:*

- (i) *Both $\operatorname{Inn} G$ and $\operatorname{Aut}_c G$ contain $Z_2(A)$ and are contained in $C_A\big(Z_2(A)\big)$.*
- (ii) *If A has an abelian maximal subgroup B then $B = C_A\big(Z_2(A)\big)$ and G has nilpotency class 2.*

Proof. Since $c > 2$ and A has maximal class, $|Z_2(A)| = p^2$ and $C_A\big(Z_2(A)\big) \lessdot A$. (Here, as elsewhere, the symbol '$\lessdot$' means 'is a maximal subgroup of'.) As in the proof for Lemma 2.4 both $\operatorname{Inn} G$ and $\operatorname{Aut}_c G$ have order at least p^2, hence $Z_2(A) \leq \operatorname{Inn} G$ and $Z_2(A) \leq \operatorname{Aut}_c G$. Also, $[\operatorname{Inn} G, \operatorname{Aut}_c G] = 1$, so $\operatorname{Inn} G \cap \operatorname{Aut}_c G \leq Z(\operatorname{Inn} G) \cap Z(\operatorname{Aut}_c G)$ and (i) holds. If B is as in (ii) then $Z_2(A) \leq B$ because $B \lessdot A$. Since B is abelian, $B = C_A\big(Z_2(A)\big)$. Thus $\operatorname{Inn} G \leq B$ by (i), so that G/Z is abelian (but G is not by Lemma 2.3). □

Lemma 2.6. $p > 2$.

Proof. Assume that $p = 2$. As $|A| > 8$ we have $c > 2$. Every 2-group of maximal class has a cyclic maximal subgroup ([7], Satz III.11.9), hence Lemma 2.5 shows that $\operatorname{Inn} G$ is cyclic. This is impossible by Lemma 2.3. □

An obvious consequence is that no automorphism of G induces the inverting automorphism on any section of G of exponent greater than 2.

Lemma 2.7. *If G is infinite, then $TC_G(T) \leq M < G$. In particular, $Z < TZ < G$.*

Proof. Suppose that $T \leq Z$. As $G' \leq T$ we may apply Lemma 1.5 to produce an automorphism of G inducing the inversion on G/T, a contradiction. Hence $T \nleq Z$ and $C_G(T) < G$. Lemma 2.4 (i) yields $C_G(T) \leq M$. Parts (i) and (ii) of Lemma 1.3 show that T is contained in a proper characteristic subgroup K of G such that G/K is a finite p-group. Thus $T \leq K \leq M$ by Lemmas 1.4 and 2.2 (i). Hence $TC_G(T) \leq M$. □

Lemma 2.8. *Let L be an abelian normal subgroup with a complement in G. Then $|L| \leq 2$.*

Proof. We have $G = L \rtimes K$ for some $K \leq G$. Then G has an automorphism α that centralizes K and induces the inversion on L. As $p > 2$, we have $\alpha = 1$, so that $\exp L \leq 2$. But $p > 2$, hence $|L| \leq 2$ by Lemmas 1.3 and 2.1. $\square$

Lemma 2.9. *$\left|G/Z\right| > p^2$ and $c > 2$.*

Proof. Suppose that $\left|G/Z\right| = p^2$. Then G has class 2. If G is finite we may assume that it is a p-group by Lemma 2.1. Then the mapping α given by $x \mapsto x^{p+1}$ is an automorphism of G lying in $Z(A)$, so that $[M, \alpha] = 1$ by Lemma 2.2. Hence Z has exponent p. A theorem by Faudree [3] shows that $|G| \leq |A| = p^{c+1}$; hence $|G| = p^{c+1}$ by Lemma 1.4. Therefore $|Z| = p^{c-1}$ and $\left|\mathrm{Hom}(G/Z, Z)\right| = p^{2(c-1)}$. Now, $\mathrm{Hom}(G/Z, Z)$ embeds in $\mathrm{Aut}_c\, G$, thus $2(c-1) \leq c$ and $c \leq 2$. So $|G| = p^3$, whence G has a complemented normal subgroup of order p^2, contradicting Lemma 2.8. If G is infinite Lemma 2.7 shows that $G/Z = \big(TZ/Z\big) \times \big(V/Z\big)$ for some $V \leq G$, and Z is maximal in both TZ and V. Hence T and V are abelian. Since $S = \mathrm{tor}\, V$ is finite, $V = S \times K$ and so $G = T \rtimes K$ for a suitable $K \leq V$, but this is excluded by Lemma 2.8 again. Thus $\left|G/Z\right| > p^2$. Finally, if $c = 2$ then $|A| = p^3$ and Lemma 2.4 yields $\left|G/Z\right| \leq p^2$, a contradiction. $\square$

Lemma 2.10.

(i) *$Z(A) = C_A(M)$.*
(ii) *$Z(M) = C_G(M)$ and $\left|Z(M)/Z\right| = p$.*
(iii) *M is not abelian, neither is $[G, A]$.*
(iv) *$\left|G/Z\right| > p^3$.*

Proof. We know from Lemma 2.2 that $Z(A) = C_A(M) \cap C_A(G/N)$. Since A has maximal class, if a normal subgroup of A different from A' is the intersection of two normal subgroups of A then it must be one of the two. We have $Z(A) \neq A'$ because $c > 2$, hence $Z(A)$ is one of $C_A(M)$ and $C_A(G/N)$. Now, $\varGamma := C_A(Z) \cap C_A(G/N) \lhd A$, and $\varGamma \simeq \mathrm{Hom}(G/Z, N)$. This group has order greater than p, as G/Z is not cyclic; hence $Z(A) < \varGamma \leq C_A(G/N)$. Therefore $Z(A) = C_A(M)$, i.e., (i) holds. Since $Z(A) \leq \mathrm{Inn}\, G$ we also have that $Z(A) = C_{\mathrm{Inn}\, G}(M) \simeq C_G(M)/Z$. On the other hand, $C_G(M)$ is characteristic in G, hence Lemma 2.4 (i) shows that $C_G(M) \leq M$, so $C_G(M) = Z(M)$. Thus (ii) is also proved. Next, if M were abelian then $\left|G/Z\right| = p^2$ by (ii), in contradiction to Lemma 2.9. As $M = Z[G, A]$ by Lemma 2.4, neither is $[G, A]$ abelian, and we have (iii). Finally, $|G/Z| = \left|G/M\right|\left|M/Z(M)\right|\left|Z(M)/Z\right| \geq pp^2p = p^4$. $\square$

We are now in position to describe the structure of A and, to a large extent, that of G.

Theorem 2.11. *Let p be a prime and G be a group such that $A = \operatorname{Aut} G$ is a finite p-group of maximal class and $|A| > 8$. Let $T = \operatorname{tor} G$ and let M be as in Lemma 2.2. Then G is a central product HL, where:*

- (g.i) *$H = \bigl(\langle c\rangle \times \langle u\rangle\bigr) \rtimes \langle v\rangle$, where u and v have infinite order, c has order 3^n for some integer $n > 1$, and the following relations hold: $c = [u, v]$ and $[c, v] = c^{3^{n-1}}$.*
- (g.ii) *$L = G^{3^n} \simeq G_{\mathrm{ab}}$ is a torsion-free abelian group with finite automorphism group. Thus L is characteristic in G.*
- (g.iii) *$T = G' = H' = \langle c\rangle$ and $\bigl|\gamma_3(G)\bigr| = 3$, so that G has nilpotency class 3. Also, $M = C_G(G')$, and G/H is 3-divisible.*

Moreover, $p = 3$ and A has the following structure: $\bigl|A/\operatorname{Inn} G\bigr| = 3$ and:

- (a.i) *$\operatorname{Inn} G = \langle \tilde u\rangle \rtimes \langle \tilde v\rangle$, where $\tilde u$ and $\tilde v$, the inner automorphisms of G induced by u and v, respectively, both have order 3^n, and $[\tilde u, \tilde v] = \tilde u^{3^{n-1}}$.*
- (a.ii) *$A = (\operatorname{Inn} G)\langle\varphi\rangle$, where φ is an automorphism that centralizes G' and such that $v^\varphi = uv$ and $u^\varphi = u^{-2}v^{-3}z$ for some $z \in Z$ such that $z^\varphi = z$, and $\varphi^3 = [\tilde u, \tilde v]$, the inner automorphisms of G induced by c.*
- (a.ii′) *$A = (\operatorname{Inn} G) \rtimes \langle\psi\rangle$, where $\psi = \varphi\tilde v$ has order 3 and its action on $\operatorname{Inn} G$ is defined by $\tilde u^\psi = \tilde u^{-2+3^{n-1}}\tilde v^{-3}$ and $\tilde v^\psi = \tilde u^{1+3^{n-1}}\tilde v$.*
- (a.iii) *A is metabelian of nilpotency class $2n$.*

Proof. We shall first prove first that T is cyclic. Suppose that this is false. By Lemma 2.1 we may assume that T is a p-group if G is finite. Then, since $T \rtimes A$ is a p-group and $p \neq 2$, there exists an A-invariant subgroup P of T which is isomorphic to $\mathcal{C}_p \times \mathcal{C}_p$ (see, for instance, [7], Hilfssatz III.7.5). Since $\bigl|PZ/Z\bigr| \le p = \bigl|Z(M)/Z\bigr|$ (see Lemma 2.10) and by Lemma 2.4 (i) we have that $P \le Z(M)$. By Lemma 2.10 again, $Z(A) = C_A(M) \cap C_A(G/M) \simeq \operatorname{Der}\bigl(G/M, Z(M)\bigr)$. Now, the (elementary) description of derivations of cyclic groups, which is also an easy consequence of the description of the cohomology of cyclic groups, yields $\operatorname{Der}\bigl(G/M, Z(M)\bigr) \simeq K := \ker(1+\alpha+\alpha^2+\cdots+\alpha^{p-1})$, where α is the automorphism of $Z(M)$ induced by conjugation by an element of $G \setminus M$. It is straightforward to check that $P \le K$, since α induces on P an automorphism of order p at most. This is a contradiction, because $|K| = |Z(A)| = p$. Hence T is cyclic, and G is therefore infinite. Also, $[G, A] \nleq T$ by Lemma 2.10 (iii), hence Lemma 1.3 (iv) and Lemma 2.6 show that $p = 3$. Another consequence of the fact that T is cyclic is that $A/C_A(T)$ is cyclic, as it embeds in $\operatorname{Aut} T$. Since A has maximal class, $\bigl|A/C_A(T)\bigr| \le 3$. Now, $T \nleq Z$, by Lemma 2.7, hence $\operatorname{Inn} G \nleq C_A(T)$. We may now employ Lemma 2.4 and deduce that $\operatorname{Inn} G$ and $C_A(T)$ are different maximal subgroups of A. Thus $A' = \operatorname{Inn} G \cap C_A(T) \lessdot \operatorname{Inn} G$ and hence $C_G(T) \lessdot G$. By Lemma 2.2 (i), $C_G(T) = M$. Moreover, since $\bigl|A/C_A(T)\bigr| = \bigl|G/C_G(T)\bigr| = 3$ it follows that $[T, A] = [T, G]$ is the subgroup of T of order 3 and $S = T \cap Z = T^3$.

Our next aim is to prove that $G' = T$. Lemma 2.10 (iv) gives $|A| > 3^4$. Then we can apply Satz III.14.17 in [7][1] to show that A is metabelian and $C_A\big(Z_2(A)\big)$ is metacyclic. By Lemma 2.5 and Lemma 2.9 it follows that $\operatorname{Aut}_c G$ has rank 2. Now, $G/G'G^3 \simeq \operatorname{Hom}(G/G^3, S)$ embeds in $\operatorname{Aut}_c G$, as $S \leq G^3$. Since G/Z is not cyclic it follows that $Z \leq G'G^3$ and $\big|G/G'G^3\big| = 9$. As $TZ < M$, because M is not abelian, $TZ \leq G'G^3$ by Lemma 2.4 (i). Suppose that $G' < T$. Then $G' \leq T^3$. Hence $G/T^3 = (T/T^3) \times (V/T^3)$ for some $V \leq G$, and $G^3 \leq V$. But then $T = T \cap G'G^3 = G'(T \cap G^3) \leq G'(T \cap V) \leq T^3$, a contradiction. Therefore $G' = T$. Also, $\gamma_3(G) = [T, G]$ has order 3.

Let $u \in M \setminus G'G^3$ and $v \in G \setminus M$. Set $H := \langle u, v\rangle$. Then $G = HZ$, because $G'G^3 \lessdot M \lessdot G$. If $c = [u, v]$ then $G' = \langle c\rangle^G = \langle c\rangle$, because G' is cyclic. Next, u and v are not periodic, as $G' = T$; since $M = C_G(G')$ (see above) then $[c, u] = 1$. From now on let $3^n = |G'|$ and $q = 3^{n-1}$, the order of $G'/\gamma_3(G)$. Then $\gamma_3(G) = \langle c^q\rangle$, hence $[c, v] = c^{\varepsilon q}$, where $\varepsilon \in \{1, -1\}$. Finally, H/G' is not cyclic, otherwise G/ZG' would be cyclic and G/Z abelian. Hence u and v are independent modulo G', so $H = \big(\langle c\rangle \times \langle u\rangle\big) \rtimes \langle v\rangle$. We have already proved that $Z \leq G'G^3$. Since $G = HZ$ this shows that G/H is 3-divisible. By Lemma 1.6 the mapping $g \mapsto g^{3^n}$ is an endomorphism of G. Thus $L := G^{3^n} \simeq G/G[3^n] = G/G'$ is torsion-free. Hence $L \cap G' = 1$ and $L \leq Z$. By Lemma 1.3 (iii), $\operatorname{Aut} L$ is finite. Also, since G/L is finite by Lemma 1.3 (ii) and $G = HG'G^3$, we have $G = HL$.

Thus far we have proved the first part of the statement—that on the structure of G—apart from the fact that we may choose u and v in such a way that $\varepsilon = 1$, which will be settled shortly. That $p = 3$, and $\operatorname{Inn} G \lessdot A$, so $\big|A/\operatorname{Inn} G\big| = 3$, has also been shown in the first part of the proof, as well as the fact that A is metabelian. The previous paragraph contains a description of $\operatorname{Inn} G \simeq G/Z$, so (a.i) is also proved modulo the choice of ε. In particular, since $|\operatorname{Inn} G| = 3^{2n}$ and hence $|A| = 3^{2n+1}$ it follows that A has class $2n$. This gives (a.iii).

From now on let us write I for $\operatorname{Inn} G$ and, for every $x \in G$, let $\tilde{x}$ denote the inner automorphism of G induced by x. Since $C_A(G') \nleq I$ (see above) we may choose $\varphi \in C_A(G') \setminus I$. Then $A = I\langle\varphi\rangle$. To describe the structure of A we only need to describe the action of φ on G/Z (that is, on I) and work out φ^3, a generator of $I \cap \langle\varphi\rangle$. By Hilfssatz III.14.13 of [7] we have $\varphi^3 \in C_I(\varphi) = Z(A)$ (and $\varphi^9 = 1$). Thus $\big|\big\{[\tilde{g}, \varphi] \mid g \in G\big\}\big| = |I : Z(A)| = |I|/3$. Since $A' \lessdot I$ we have $[I, \varphi] \leq A'$ and $|A'| = |I|/3$. Therefore $A' = \big\{[\tilde{g}, \varphi] \mid g \in G\big\}$. On the other hand, $\tilde{M} := \{\tilde{x} \mid x \in M\}$ has index p^2 in A, as $|G : M| =$

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[1] Note that in the statement of this theorem, in [7], the hypothesis that the group has order more than 3^4 is omitted. However this hypothesis is explicitly used in the proof, and the example of the standard wreath product of two groups of order 3 shows that it is actually needed.

$|A : I| = p$, thus $\tilde{M} = A'$. Hence $\tilde{u} \in A'$. Therefore there exists $y \in G$ such that $\tilde{u}^{-1} = [\tilde{y}^{-1}, \varphi]$, which means that $y^{\varphi} = uys$ for some $s \in Z$. Moreover, since $\left|M/G'G^3\right| = 3$ and so $[M, \varphi] \le G'G^3$ (and $u \notin G'G^3$ by our choice) we have that $y \notin M$. Thus we may redefine v as y and u as us to get $v^{\varphi} = uv$, together with all the information already obtained (of course c, w and H also have to be redefined in relation to u and v). We can also make $\varepsilon = 1$. For,

$$\begin{aligned}[u^{-1}, v^{-1}] &= [u, v^{-1}]^{-1} = [u, v]^{v^{-1}} \\ &= c^{v^{-1}} = c[c, v^{-1}] = c[c, v]^{-1} = c^{1-\varepsilon q};\end{aligned} \tag{1}$$

by setting $u_1 := c^{-1}u^{-1}$ and $v_1 := v^{-1}$ we then have $v_1 \in G \setminus M$ and $u_1 \in M \setminus G'G^3$, and also $[u_1, v_1] = [c, v]\,[u^{-1}, v^{-1}] = c^{\varepsilon q}c^{1-\varepsilon q} = c$ and $[c, v_1] = [c, v]^{-1}$. Hence, if $\varepsilon = -1$, that is $[c, v] = c^{-q}$, we substitute u_1 and v_1 for u and v, respectively, to get $[c, v] = c^q$, i.e., $\varepsilon = 1$. Note that it remains the case that $v^{\varphi} = uv$ after this substitution.

Next, we shall work out u^{φ}. We have $\tilde{u}^{\varphi} = \tilde{u}^i\tilde{v}^j$ for some integers i and j. Now, A'/I^3 has order 3 and so is centralized by φ. Since $\tilde{u} \in A'$ and $I^3 = \langle \tilde{u}^3, \tilde{v}^3 \rangle$ then 3 divides j. Furthermore, $[v^j, u] = [v, u]^j$ by Lemma 1.6 (i), and this commutator lies in $\langle c^3 \rangle \le Z$; from this and since $c^{\varphi} = c$ commutes with u we have

$$c = [u^{\varphi}, v^{\varphi}] = [u^i v^j, uv] = [u, v]^i [v, u]^j = c^{i-j}.$$

Therefore $i \equiv j + 1 \pmod{3^n}$ and $\tilde{u}^{\varphi} = \tilde{u}^{j+1}\tilde{v}^j$. Also, $\tilde{v}^{\varphi^2} = (\tilde{u}\tilde{v})^{\varphi} = \tilde{u}^{j+1}\tilde{v}^j\tilde{u}\tilde{v} = \tilde{u}^{j+2}\tilde{v}^{j+1}$, because $\tilde{v}^j \le I^3 \le Z(I)$. Since $|I'| = 3$ it also follows that $(\tilde{u}\tilde{v})^{j+1} = \tilde{u}^{j+1}\tilde{v}^{j+1}$ and so

$$\begin{aligned}\tilde{v}^{\varphi^3} &= (\tilde{u}^{j+1}\tilde{v}^j)^{j+2}(\tilde{u}\tilde{v})^{j+1} = \tilde{u}^{(j+1)(j+2)}\tilde{v}^{j(j+2)}\tilde{u}^{j+1}\tilde{v}^{j+1} \\ &= \tilde{u}^{(j+1)(j+3)}\,\tilde{v}^{j(j+3)+1}.\end{aligned}$$

But $\varphi^3 \in Z(A)$, as we said above, hence $\tilde{v}^{\varphi^3} = \tilde{v}$ and so $\tilde{u}^{(j+1)(j+3)} = 1$. Then 3^n divides $(j+1)(j+3)$. Since 3 divides j it follows that 3^n divides $j+3$. Thus $j \equiv -3 \pmod{3^n}$. Hence $\tilde{u}^{\varphi} = \tilde{u}^{j+1}\tilde{v}^j = \tilde{u}^{-2}\tilde{v}^{-3}$. Therefore $u^{\varphi} = u^{-2}v^{-3}z$ for some $z \in Z$, as required in (a.ii). To complete the proof for (a.ii) we still have to compute φ^3 and check that z is fixed by φ.

Since $\langle \tilde{c} \rangle = I' \lhd A$ and $\left|\langle \tilde{c} \rangle\right| = 3$ (or by Lemma 2.10) we have that $Z(A) = \langle \tilde{c} \rangle$. Thus $\varphi^3 \in \langle \tilde{c} \rangle$, so $\varphi^3 = \tilde{c}^{\lambda}$ for some $\lambda \in \{-1, 0, 1\}$. (Note that the three different values for λ give rise to three non-isomorphic groups.) We have $v^{\varphi^3} = u^{\varphi^2}u^{\varphi}uv$ and $v^{c^{\lambda}} = v[c, v]^{-\lambda} = vc^{-\lambda q} = c^{-\lambda q}v$, thus

$$u^{\varphi^2}u^{\varphi}u = c^{-\lambda q}. \tag{2}$$

We shall make use of the following rule for calculating cubes in G. Since $[M, G'] = 1$, for all $x \in G$ and $m \in M$,

$$(3) \qquad \begin{aligned}(xm)^3 &= x^3 m^{x^2} m^x m = x^3 \bigl(m[m, x^2]\bigr)\bigl(m[m, x]\bigr)m \\ &= x^3 m^3 [m, x]^2 [m, x, x]\, [m, x] \\ &= x^3 m^3 [m, x]^3 [m, x, x].\end{aligned}$$

As $v^3 \in G^3 \le Z_2(G)$ and by Lemma 1.6 (i) we have that

$$(v^3u^2)^2 = v^6u^4[u^2, v^3] = c^6v^6u^4.$$

Also, $[u^{-1}, v^{-1}] = c^{1-q}$ by (1). By using these equalities together with (3) and Lemma 1.6 (i), and remembering that $c^3 \in Z$, we obtain

$$\begin{aligned} u^{\varphi^2} &= \bigl(u^\varphi\bigr)^{-2}(v^\varphi)^{-3}z^\varphi = \bigl(v^3u^2\bigr)^2 z^{-2}(v^{-1}u^{-1})^3 z^\varphi \\ &= c^6v^6u^4v^{-3}u^{-3}[u^{-1}, v^{-1}]^3[u^{-1}, v^{-1}, v^{-1}]z^{\varphi-2} \\ &= c^6v^3uv^3[v^3, u]u^3v^{-3}u^{-3}c^{3(1-q)}[c^{1-q}, v^{-1}]z^{\varphi-2} \\ &= c^6v^3uv^3c^{-3}u^3v^{-3}u^{-3}c^3[c, v]^{-1}z^{\varphi-2} \\ &= c^6v^3u[v^{-3}, u^{-3}]c^{-q}z^{\varphi-2} = c^6v^3uc^{-9}c^{-q}z^{\varphi-2} = v^3uc^{-3-q}z^{\varphi-2}, \end{aligned}$$

hence (2) gives

$$c^{-\lambda q} = u^{\varphi^2}u^\varphi u = v^3uc^{-3-q}z^{\varphi-2}u^{-2}v^{-3}zu = [v^{-3}, u]c^{-3-q}z^{\varphi-1} = c^{-q}[z, \varphi].$$

Therefore

$$(4) \qquad [z, \varphi] = c^{(1-\lambda)q}.$$

Now recall that $L \le Z$ and $G = LH$, hence $Z = LZ(H) = LH^{3^n}\langle c^3, w\rangle = L\langle c^3, w\rangle$; we also recall that $w = cu^{-q}$. Since $w^3 \in \langle c^3\rangle L$ we have $z = c^{3t}g^{3^n}w^\mu = c^{3t+\mu}g^{3^n}u^{-\mu q}$ for some $g \in G$ and integers $t \in \mathbb{N}$ and $\mu \in \{-1, 0, 1\}$. To compute $[z, \varphi]$ we observe first that from (3) and from $\bigl|(G^3)'\bigr| = |\langle c^9\rangle| = q/3$ it follows that $(v^3u^2)^q = (v^9u^6[u^2, v]^9)^{q/3} = v^{3^n}u^{2q}$. Also, Lemma 1.6 (ii) yields $[g^{3^n}, \varphi] = [g, \varphi]^{3^n}$. Then

$$\begin{aligned} [z, \varphi] &= [g^{3^n}u^{-\mu q}, \varphi] = [g^{3^n}, \varphi]\,[u^{-\mu q}, \varphi] = [g, \varphi]^{3^n}u^{\mu q}\bigl(u^{-2}v^{-3}z\bigr)^{-\mu q} \\ &= [g, \varphi]^{3^n}u^{\mu q}v^{\mu 3^n}u^{2\mu q}z^{-\mu q} = [g, \varphi]^{3^n}u^{\mu 3^n}v^{\mu 3^n}c^{-\mu^2 q}g^{-\mu q 3^n}u^{\mu^2 q^2} \\ &= \bigl([g, \varphi]g^{-\mu q}u^{\mu+\mu^2 q/3}v^\mu\bigr)^{3^n}c^{-\mu^2 q}. \end{aligned}$$

Since $L = G^{3^n}$ is torsion-free, (4) implies that $([g, \varphi]g^{-\mu q}u^{\mu+\mu^2 q/3}v^\mu)^{3^n} = 1$ and $c^{(1-\lambda)q} = [z, \varphi] = c^{-\mu^2 q}$. Hence $[g, \varphi]g^{-\mu q}u^{\mu+\mu^2 q/3}v^\mu \in T$. Now, $[g, \varphi]$, $g^{-\mu q}$, $u^{\mu+\mu q/3} \in M$ and $T \le M$, hence $v^\mu \in M$, so $\mu = 0$. Therefore $[z, \varphi] = 1$ and $\lambda = 1$. By definition of λ we have $\varphi^3 = \tilde{c}$. Thus (a.ii) is proved.

Finally, we shall prove that A splits over I and obtain (a.ii′). Since $I'I^3 \leq Z(I)$ we have

$$\begin{aligned}(\varphi\tilde{v})^3 &= \varphi^3\tilde{v}^{\varphi^2}\tilde{v}^{\varphi}\tilde{v} = \tilde{c}(\tilde{u}\tilde{v})^{\varphi}\tilde{v}^{\varphi}\tilde{v}\\ &= \tilde{c}\tilde{u}^{-2}\tilde{v}^{-3}(\tilde{u}\tilde{v})^2\tilde{v} = \tilde{c}\tilde{u}^{-2}\tilde{v}^{-3}\tilde{u}^2\tilde{v}^2[\tilde{v},\tilde{u}]\tilde{v} = \tilde{c}\tilde{c}^{-1} = 1,\end{aligned}$$

so $\psi := \varphi\tilde{v}$ has order 3. Also, $\tilde{c} = \tilde{u}^q$, hence

$$\begin{aligned}\tilde{u}^{\psi} &= \left(\tilde{u}^{-2}\tilde{v}^{-3}\right)^{\tilde{v}} = \tilde{u}^{-2}[\tilde{u},\tilde{v}]^{-2}\tilde{v}^{-3} = \tilde{u}^{-2-2q}\tilde{v}^{-3} = \tilde{u}^{q-2}\tilde{v}^{-3}\\ \tilde{v}^{\psi} &= (\tilde{u}\tilde{v})^{\tilde{v}} = \tilde{u}[\tilde{u},\tilde{v}]\tilde{v} = \tilde{u}^{q+1}\tilde{v},\end{aligned}$$

which proves (a.ii′). □

The description of the group A in (a.ii) makes it clear that A is isomorphic to the group X_n appearing in the introduction. An alternative presentation for A is suggested by the semidirect product decomposition in (a.ii′):

$$\begin{aligned}A \simeq \big\langle x, y, t \;\big|\; & x^{3^n} = y^{3^n} = t^3 = 1,\ [x,y] = x^{3^{n-1}},\\ & x^t = x^{3^{n-1}-2}y^{-3},\ y^t = x^{3^{n-1}+1}y\big\rangle.\end{aligned}$$

3. Examples

In this section we shall complete the proof of the Theorem in the introduction, by showing that groups isomorphic to the groups X_n (that is, to those described as A in Theorem 2.11) actually are realized as the full automorphism group of some (infinite) group, for every possible choice of the parameter n.

Let n be an integer greater than 1. As in the previous proof, we set $q := 3^{n-1}$. Let $H := \big(\langle c\rangle \times \langle u\rangle\big) \rtimes \langle v\rangle$, where u and v have infinite order and c has order 3^n, and the action of v on $\langle c, u\rangle$ is defined by $[u,v] = c$ and $[c,v] = c^q$. Then $H' = \langle c\rangle$ and $\gamma_3(H) = \langle c^q\rangle$ has order 3. Lemma 1.6 may be applied to check that H^{3^n}, $\langle c^3\rangle$ and cu^{-q} lie in $Z(H)$ (as a matter of fact they generate $Z(H)$), and $H/Z(H)$ is the semidirect product of the images modulo $Z(H)$ of $\langle u\rangle$ and $\langle v\rangle$, both of order 3^n. Thus $\big|H/Z(H)\big| = 3^{2n}$.

For every $i \in \mathbb{N}$ let p_i be a prime congruent to 1 modulo 3^n, chosen in such a way that $p_i \neq p_j$ if $i \neq j$. For each $i \in \mathbb{N}$ the polynomial $x^3 - 1$ has three roots in $\mathbb{Z}_{p_i}$, hence there exists an integer λ_i such that $\lambda_i^2 + \lambda_i + 1 \equiv 0 \pmod{p_i}$. Let $h_i := uv^{1-\lambda_i}$.

Define a sequence $(H_i)_{i\in\mathbb{N}_0}$ of groups by letting $H_0 = H$ and, for each $i \in \mathbb{N}$, by letting H_i be a central product $H_{i-1}\langle z_i\rangle$ where $z_i^{p_i} = h_i^{1-p_i}$. This amalgamation makes sense because $h_i^{1-p_i} \in H^{3^n} \leq Z(H)$. Let $G = \bigcup_{i\in\mathbb{N}} H_n$, the direct limit of the groups H_i. Clearly G is a central product of H and $\langle z_i \mid i \in \mathbb{N}\rangle \leq Z(G)$. Hence $G' = H' = \langle c\rangle$ and $\gamma_3(G) = \gamma_3(H) = \langle c^q\rangle \leq Z(G)$.

Lemma 3.1. $\operatorname{tor} G = G'$.

Proof. Clearly H/G' is torsion-free and G/H is periodic. For each $i \in \mathbb{N}$ the factor $\langle z_i \rangle H/H$ has order p_i and is the p_i-primary component of G/H. Thus $G/H = \mathrm{Dr}_{i\in\mathbb{N}}\langle z_i\rangle H/H$. If $\operatorname{tor} G \neq G'$ then $z_i h$ is periodic—of order p_i modulo G'—for some $i \in \mathbb{N}$ and some $h \in H$. In this case $h_i^{1-p_i}h^{p_i} = z_i^{p_i}h^{p_i} = (z_i h)^{p_i} \in G'$. This is impossible, as $h_i \notin G'H^{p_i}$. □

Therefore the abelianized factor group G_{ab} is torsion-free of rank 2. A generating set for G_{ab} is given by $\{uvG', v^{-1}G', h_i z_i G' \mid i \in \mathbb{N}\}$; since $(h_i z_i)^{p_i} = h_i = (uv)v^{-\lambda_i}$ for every $i \in \mathbb{N}$ this shows that G_{ab} is isomorphic to the subgroup $(\mathbb{Z}\oplus\mathbb{Z}) + \langle p_i^{-1}(1,\lambda_i) \mid i \in \mathbb{N}\rangle$ of $\mathbb{Q}\oplus\mathbb{Q}$. The automorphism group of this latter group (and hence that of G_{ab}) is cyclic of order 6 (see [1], vol. II, p. 272, Example 2). Now we shall see that only the automorphisms of G_{ab} of order 3 (and the identity) lift to automorphisms of G.

Lemma 3.2. *No automorphism of G induces the inversion automorphism on G_{ab}.*

Proof. Suppose that $\varphi \in \operatorname{Aut} G$ and that $u^\varphi \equiv u^{-1}$ and $v^\varphi \equiv v^{-1} \pmod{G'}$. Then, modulo $\gamma_3(G) = \langle c^q\rangle$ we have $c^\varphi \equiv [u^{-1}, v^{-1}] \equiv c$. So $c^\varphi = cc^{qt}$ for some integer t. This implies that $c^{q\varphi} = c^q$ on the one hand, but also that $c^{q\varphi} = [c,v]^\varphi = [c^\varphi, v^\varphi] = [c, v^{-1}] = c^{-q}$, a contradiction. □

We may apply Lemma 1.6 (i) to get that $[u^{-2}v^{-3}, uv] = [u,v]^{-2}[v,u]^{-3v} = c^{-2}c^3 = c$. So H has an automorphism φ_0 defined by

$$\varphi_0 : \left\{ \begin{array}{l} u \longmapsto u^{-2}v^{-3} \\ v \longmapsto uv \\ (c \longmapsto c) \end{array} \right. .$$

We shall extend φ_0 to an automorphism of G. To this end, first note[2] that $(1-\lambda_i)\lambda_i = \lambda_i - \lambda_i^2 \equiv \lambda_i + (1+\lambda_i) = 2\lambda_i + 1 \pmod{p_i}$ for every $i \in \mathbb{N}$, and so

$$\begin{aligned} h_i^{\lambda_i\varphi_0} &\equiv (u^{-2}v^{-3})^{\lambda_i}(uv)^{(1-\lambda_i)\lambda_i} \\ &\equiv (u^{-2}v^{-3})^{\lambda_i}(uv)^{2\lambda_i+1} \equiv uv^{1-\lambda_i} = h_i \pmod{G'H^{p_i}}. \end{aligned}$$

Since $h_i = (z_i h_i)^{p_i} \in H_i^{p_i}$ and p_i does not divide λ_i it follows that $h_i^{\varphi_0} \in G'H_i^{p_i}$. Thus there exist $g \in H_i$ and $t \in \mathbb{N}$ such that $z_i^{p_i\varphi_0} = h_i^{(1-p_i)\varphi_0} = g^{p_i}c^t$. Let $r_i := gc^t$. By Lemma 1.6 (ii), since $p_i \equiv 1 \pmod{3^n}$ and $G^{3^n} \leq Z(G)$, the mapping $x \in G \mapsto x^{p_i} \in G$ is an endomorphism, hence $r_i^{p_i} = g^{p_i}c^{tp_i} = g^{p_i}c^t = z_i^{p_i\varphi_0}$. Moreover, $r_i \equiv r_i^{p_i}$ modulo $G^{3^n} \leq Z(G)$, hence $r_i \in Z(G)$.

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[2] What follows explains the choice of the integers λ_i and the elements h_i. The point is that the h_i must be eigenvectors for φ_0 modulo $H'H^{p_i}$.

Now suppose that $i \in \mathbb{N}$ and φ_{i-1} is an automorphism of H_{i-1} extending φ_0. The above discussion shows that φ_{i-1} can be extended to an automorphism φ_i of $H_i = H_{i-1}\langle z_i \rangle$ by setting $z_i^{\varphi_i} = r_i$. A straightforward induction and direct limit argument now proves the existence of an automorphism φ of G extending φ_0. Since φ does not centralize G_{ab} and $|\operatorname{Aut} G_{\mathrm{ab}}| = 6$, by Lemma 3.2 we have:

Lemma 3.3. $\bigl|\operatorname{Aut} G / C_{\operatorname{Aut} G}(G_{\mathrm{ab}})\bigr| = 3.$

Next we have:

Lemma 3.4. $C_{\operatorname{Aut} G}(G_{\mathrm{ab}}) = \operatorname{Inn} G.$

Proof. Let $\mathit{\Gamma} = C_{\operatorname{Aut} G}(G_{\mathrm{ab}})$ and let $\Delta = C_{\operatorname{Aut} G}(\bar{G})$, where $\bar{G} = G/\gamma_3(G)$. Then $\Delta \lhd \mathit{\Gamma}$. Since $\Delta \leq \operatorname{Aut}_c G$, then Δ centralizes G'. Thus Δ embeds in $\operatorname{Hom}\bigl(G_{\mathrm{ab}}, \gamma_3(G)\bigr)$. Since G_{ab} has rank 2 and $\bigl|\gamma_3(G)\bigr| = 3$ this group has order 9. Also, $\mathit{\Gamma}/\Delta$ is isomorphic to a subgroup of $\mathit{\Gamma}_1 := C_{\operatorname{Aut} \bar{G}}(\bar{G}_{\mathrm{ab}})$. As with the above step, all elements of $\mathit{\Gamma}_1$ are central automorphisms, and as such they centralize $\bar{G}'$. Therefore $\mathit{\Gamma}_1$ embeds in $\operatorname{Hom}(\bar{G}_{\mathrm{ab}}, \bar{G}')$, which has order q^2. Thus $|\mathit{\Gamma}| \leq 9q^2 = 3^{2n} = |\operatorname{Inn} G|$. But clearly $\operatorname{Inn} G \leq \mathit{\Gamma}$, so the lemma is proved. $\square$

The two previous lemmas show that $\operatorname{Aut} G = (\operatorname{Inn} G)\langle \varphi \rangle$ and that $\varphi^3 \in \operatorname{Inn} G$. Therefore $\operatorname{Aut} G$ is a finite 3-group. To conclude that $\operatorname{Aut} G$ has maximal class it will be enough to show that $C_{\operatorname{Inn} G}(\varphi)$ has order 3 (see [7], Satz III.14.23). Let $\tilde{u}$ and $\tilde{v}$ be the inner automorphisms of G induced by u and v respectively. Then $H/Z(H) \simeq \operatorname{Inn} G = \langle \tilde{u} \rangle \rtimes \langle \tilde{v} \rangle$, where $\tilde{u}$ and $\tilde{v}$ have order 3^n and $[\tilde{u}, \tilde{v}] = \tilde{u}^q$. So $(\operatorname{Inn} G)'$ is central and has order 3. Assume that $[\tilde{u}^i \tilde{v}^j, \varphi] = 1$ for some $i, j \in \mathbb{Z}$. Then

$$\tilde{u}^i \tilde{v}^j = (\tilde{u}^{-2} \tilde{v}^{-3})^i (\tilde{u} \tilde{v})^j = \tilde{u}^{-2i} \tilde{v}^{-3i} \tilde{u}^j \tilde{v}^j \tilde{u}^{-qj(j-1)/2} = \tilde{u}^{-2i + j(1 + q(j-1))} \tilde{v}^{j-3i},$$

because $j(j-1)/2 \equiv j(1-j) \pmod 3$. Hence $j - 3i \equiv j$ and $-2i + j\bigl(1 + q(j-1)\bigr) \equiv i \pmod{3^n}$. It follows that $3i \equiv 0$ and $j \equiv 0 \pmod{3^n}$. This shows that $C_{\operatorname{Inn} G}(\varphi) = \langle \tilde{u}^q \rangle$. So, this centralizer has order 3, as required. Therefore $\operatorname{Aut} G$ is a 3-group of maximal class. By Theorem 2.11, then $\operatorname{Aut} G \simeq X_n$. $\square$

With this last result the proof of the Theorem in the introduction is complete.

References

[1] M.R. Dixon and M.J. Evans, *Periodic divisible-by-finite automorphism groups are finite,* J. Algebra **137** (1991), 416–424.

[2] J. Dyer, *A remark on automorphism groups,* Contribution to group theory (K.I. Appel, J.G. Ratcliffe and P.E. Schupp, eds.), Contemp. Math., vol. 33, Amer. Math. Soc., Providence, 1984; pp. 208–211.

[3] R. Faudree, *A note on the automorphism group of a p-group,* Proc. Amer. Math. Soc. **19** (1968), 1379–1382.
[4] T.A. Fournelle, *Finite groups of automorphisms of infinite groups, I,* J. Algebra **70** (1981), 16–22.
[5] ———, *Finite groups of automorphisms of infinite groups, II,* J. Algebra **80** (1983), 106–112.
[6] L. Fuchs, *Infinite abelian groups, vol. 2,* Academic Press, New York, 1973.
[7] B. Huppert, *Endliche Gruppen, Bd. I,* Springer, Berlin, 1967.
[8] A. Mann, *Some questions about p-groups,* J. Austral. Math. Soc. (Ser. A) **67** (1999), 356–379.
[9] U. Martin, *Almost all p-groups have automorphism group a p-group,* Bull. Amer. Math. Soc. **15** (1986), 78–82.
[10] T. Matumoto, *Any group is represented by an outerautomorphism group,* Hiroshima Math. J. **19** (1989), 209–219.
[11] V.T. Nagrebeckiĭ, *On the periodic part of a group with a finite number of automorphisms,* Dokl. Akad. Nauk SSSR **205** (1972), 519–521; English transl., Soviet Math. Dokl. **13** (1972), 953–956.
[12] D.J.S. Robinson, *A contribution to the theory of groups with finitely many automorphisms,* Proc. London Math. Soc. (3) **35** (1977), 34–54.
[13] ———, *Infinite torsion groups as automorphism groups,* Quart. J. Math. Oxford (2) **30** (1979), 351–364.
[14] ———, *Applications of cohomology to the theory of groups,* Groups—St. Andrews 1981 (C.M. Campbell and E.F. Robertson, eds.), London Math. Soc. Lecture Note Ser., vol. 71, Cambridge Univ. Press, Cambridge, 1982, pp. 46–80.
[15] ———, *A course in the theory of groups,* Springer, Berlin, 1982.
[16] U. Stammbach, *Homology in group theory,* Lecture Notes in Mathematics, vol. 359, Springer, Berlin, 1973.

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