Illinois Journal of Mathematics
Volume 51, Number 4, Winter 2007, Pages 1397–1407
S 0019-2082

A SPHERICAL INITIAL IDEAL FOR PFAFFIANS

JAKOB JONSSON AND VOLKMAR WELKER

Abstract. We determine a term order on the monomials in the variables X_{ij}, $1 \leq i < j \leq n$, such that corresponding initial ideal of the ideal of Pfaffians of degree r of a generic n by n skew-symmetric matrix is the Stanley-Reisner ideal of a join of a simplicial sphere and a simplex. Moreover, we demonstrate that the Pfaffians of the $2r$ by $2r$ skew-symmetric submatrices form a Gröbner basis for the given term order. The same methods and similar term orders as for the Pfaffians also yield squarefree initial ideals for certain determinantal ideals. Yet, in contrast to the case of Pfaffians, the corresponding simplicial complexes are balls that do not decompose into a join as above.

1. Introduction

The ideal $\mathrm{P}_{n,r}$ of Pfaffians of degree r of a generic $n \times n$ skew-symmetric matrix is one of the classical ideals in commutative algebra. Many of its properties are well understood. In particular, Herzog and Trung [11] have constructed a term order for which the standard generators of $\mathrm{P}_{n,r}$ constitute a Gröbner basis. Indeed, this term order has many nice properties, including the property that the corresponding initial ideal is squarefree and is the Stanley-Reisner ideal of a simplicial ball. In this paper, we determine a different term order $\preceq$ leading to the same Gröbner basis and a corresponding initial ideal that is squarefree and the Stanley-Reisner ideal of a simplicial ball. Yet, in contrast to Herzog and Trung's situation, our term order $\preceq$ satisfies the following conditions:

(S) The initial ideal $\mathrm{in}_{\preceq}(\mathrm{P}_{n,r})$ is the Stanley-Reisner ideal of a simplicial complex that decomposes into a join $\Delta * 2^{\Theta}$ of a simplicial sphere Δ and a full simplex 2^{Θ} on ground set Θ.

(M) The cardinality of the set Θ is the absolute value of the a-invariant of the quotient of the polynomial ring by $\mathrm{P}_{n,r}$.

Received May 17, 2006; received in final form March 5, 2007.

2000 *Mathematics Subject Classification.* Primary 13F10. Secondary 05E99.

The first author was supported by Graduiertenkolleg 'Combinatorics, Geometry, Computation', DFG-GRK 588/2. Both authors were supported by EU Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272.

If condition (S) is satisfied, then we say that $\preceq$ is a term order with *spherical* initial ideal. Note that if the set Θ from condition (S) is nonempty, then the simplicial complex $\Delta * 2^{\Theta}$ is a ball. As mentioned above, the initial ideal of Herzog and Trung's term order [11] is also the Stanley-Reisner ideal of a simplicial ball. However, this ball does not decompose into a join as postulated in (S).

Condition (M) assures a certain minimality of the initial ideal. Indeed, among all squarefree ideals occurring as initial ideals of $\mathrm{P}_{n,r}$, an ideal satisfying this condition is one that uses the least number of variables in its minimal generating set.

Recently, term orders satisfying (S) and (M) have been constructed for several classes of ideals [1], [3], [16], [17], [18]; see in particular [3] for a survey. In all these cases, the ideals were the defining ideals of affine semigroup rings or were reduced to this case by a flat deformation. This paper provides the first instance of a ring that is not an affine semigroup ring for which this construction can be performed.

In addition to the Pfaffian case, in Section 6 we study similar term orders for ideals generated by $r \times r$ minors of matrices whose generic entries form a stack polyomino shape. Again the initial ideals are squarefree, but this time the corresponding simplicial complexes do not decompose into a join of a simplicial sphere and a simplex.

2. Basic facts and the main result

Let $S_n = k[X_{ij} : 1 \le i < j \le n]$ be the polynomial ring over the field k in the entries of a generic $n \times n$ skew-symmetric matrix

$$A_n = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n-1} & X_{1n} \\ -X_{12} & 0 & X_{23} & \cdots & X_{2n-1} & X_{2n} \\ -X_{13} & -X_{23} & 0 & \cdots & X_{3n-1} & X_{3n} \\ \cdots & \cdots & \cdots & \cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & . & . \\ -X_{1n-1} & -X_{2n-1} & -X_{3n-1} & \cdots & 0 & X_{n-1n} \\ -X_{1n} & -X_{2n} & -X_{3n} & \cdots & -X_{n-1n} & 0 \end{pmatrix}.$$

It is a well-known fact from linear algebra that $\det(A_n) = 0$ if n is odd and $\det(A_n) = p_{A_n}^2$ if n is even for a certain polynomial $p_{A_n} \in S_n$ of degree n. The polynomial p_{A_n} is called the *Pfaffian* of A_n. For $1 \le r \le \lfloor \frac{n}{2} \rfloor$ and indices $1 \le j_1 < \cdots < j_{2r} \le n$, we denote by $A_{j_1,\ldots,j_{2r}}$ the submatrix of A_n constructed by selecting the rows and columns indexed by $j_1, \ldots, j_{2r}$. Clearly, the matrices $A_{j_1,\ldots,j_{2r}}$ are the only skew-symmetric submatrices of A_n of size $2r$. We denote by $\mathrm{P}_{n,r}$ the ideal in S_n generated by the Pfaffians of the matrices $A_{j_1,\ldots,j_{2r}}$, $1 \le j_1 < \cdots < j_{2r} \le n$.

Before proceeding, we briefly mention some well-known definitions and facts about simplicial complexes and Gröbner bases. We refer to Bruns and Herzog [2, Section 5] for more details on simplicial complexes and to Fröberg [8] for more details on Gröbner bases.

Let $T = k[x_1, \ldots, x_\ell]$ be the polynomial ring in ℓ variables $x_1, \ldots, x_\ell$. A *monomial* in T is a product $m = \prod_{1=1}^{\ell} x_i^{a_i}$, where each a_i is a nonnegative integer. The monomial m is called *squarefree* if $a_i \in \{0,1\}$ for $1 \leq i \leq \ell$. An ideal I in T is called a *(squarefree) monomial ideal* if I is generated by a set of (squarefree) monomials. The *Stanley-Reisner ideal* $I_\Delta \subseteq T$ of a simplicial complex Δ over ground set $[\ell]$ is the ideal generated by all monomials $\underline{\mathbf{x}}_\sigma = \prod_{i\in\sigma} x_i$ such that $\sigma \notin \Delta$. Conversely, each squarefree monomial ideal in T is the Stanley-Reisner ideal of some simplicial complex on ground set $[\ell] := \{1, \ldots, \ell\}$.

If $f \in T$ and $\preceq$ is a term order, then we denote by $\mathrm{in}_\preceq(f)$ the leading monomial of f with respect to $\preceq$ (i.e., 0 if $f = 0$ and the largest monomial with respect to $\preceq$ occurring in f otherwise). For an ideal J, we write $\mathrm{in}_\preceq(J)$ for the *initial ideal* of J, i.e., the ideal generated by $\{\mathrm{in}_\preceq(f) : f \in J\}$. Clearly, $\mathrm{in}_\preceq(J)$ is a monomial ideal.

If Δ and Γ are simplicial complexes on disjoint ground sets, then the *join* $\Delta * \Gamma$ is the simplicial complex

$$\Delta * \Gamma := \{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}.$$

We are now in position to state the main results of this paper.

THEOREM 2.1. *Let $1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor$. Then there is a term order $\preceq$ on the monomials in S_n such that*

$$\mathrm{in}_\preceq(\mathrm{P}_{n,r}) = I_{\Delta_{n,r-1} * 2^{\Theta_{n,r-1}}},$$

where $\Delta_{n,r-1}$ is a simplicial sphere of dimension $(r-1)(n-2r+1)-1$ and $2^{\Theta_{n,r-1}}$ is the full simplex on a set $\Theta_{n,r-1}$ of size $n(r-1)$.

To describe the simplicial complex $\Delta_{n,r-1}$ appearing in the formulation of Theorem 2.1, we need to recall some more facts from the theory of simplicial complexes.

An element $\sigma \in \Delta$ of a simplicial complex is called a *face* of Δ. The *dimension* $\dim(\sigma)$ of σ is defined as $\dim(\sigma) := \#\sigma - 1$. The dimension of Δ is defined as $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \in \Delta\}$. If all inclusionwise maximal faces σ of Δ are of dimension $\dim(\Delta)$, then Δ is called *pure*.

Let $\Omega_n = \{(i,j) : 1 \leq i < j \leq n\}$. We can visualize the elements of Ω_n as edges and diagonals in a convex n-gon with vertices labelled in clockwise order $1, \ldots, n$. Clearly, Ω_n is in bijection with the variables of the polynomial ring S_n. Therefore, for any simplicial complex Δ on ground set Ω_n, we may consider I_Δ as an ideal in S_n. We introduce a distance function d on $[n]$ by setting

$$d_{ij} = \min\{|j-i|, |n+i-j|\}.$$

Equivalently, if we again regard i and j as vertices of a convex n-gon, then $d_{ij}-1$ is the minimum of the number of vertices on the left-hand side and the number of vertices on the right-hand side of the diagonal through i and j. We denote by $\Omega_{n,r}$ the set of elements $(i,j) \in \Omega_n$ such that $d_{ij} \geq r+1$.

Let $\Delta_{n,r}$ be the simplicial complex on ground set $\Omega_{n,r}$ whose simplices are those subsets $\sigma \subseteq \Omega_{n,r}$ for which there is no $\tau \subseteq \sigma$ such that $\#\tau = r+1$ and any two diagonals in τ intersect; by intersection we mean transversal intersection in the interior of the n-gon.

Proposition 2.2. *Let* $1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor$.

(i) [6] *The simplicial complex* $\Delta_{n,r-1}$ *triangulates a PL-sphere of dimension* $\dim(\Delta_{n,r-1}) = (r-1)(n-2r+1)-1$.

(ii) [12] *The number of faces of* $\Delta_{n,r-1}$ *of dimension* $(r-1)(n-2r+1)-1$ *is given by*

$$\prod_{1 \leq i \leq j \leq n-2r+1} \frac{2(r-1)+i+j}{i+j}.$$

We close this section by recalling some facts that link invariants of simplicial complexes and invariants of rings. If J is a finitely generated ideal in a ring T such that J is homogeneous with respect to the standard grading on monomials, then $R := T/J \cong \bigoplus_{i \geq 0} R_i$ is a standard graded ring, where R_i is the k-vectorspace of elements of degree i. In this situation, the Hilbert series $\mathrm{Hilb}(R,t) = \sum_{i \geq 0} \dim_k R_i t^i$ is a rational function

$$\mathrm{Hilb}(R,t) = \frac{h_R(t)}{(1-t)^d},$$

where $h_R(t)$ is a polynomial and d is the Krull dimension $\dim(R)$ of R. The *a-invariant* $a(R)$ of R is then defined as the difference of the degree of the polynomial $h_R(t)$ and d. If Δ is a simplicial complex, then $\dim(T/I_\Delta) = \dim(\Delta)+1$ [2, Theorem 5.1.4]. The number $h_R(1)$ is called the multiplicity $e(R)$ of R. If $J = I_\Delta$, then $e(T/I_\Delta)$ is the number of faces $\sigma \in \Delta$ such that $\dim(\sigma) = \dim(\Delta)$.

3. Consequences of Herzog and Trung's term order

Let $\Sigma_{n,r}$ be the simplicial complex on ground set Ω_n such that the minimal nonfaces of $\Sigma_{n,r}$ are those sets τ of size $r+1$ such that any two diagonals ab and cd in τ are nested, meaning that $a < c < d < b$ (assuming $a < b$, $c < d$, and $a \leq c$).

PROPOSITION 3.1 ([11]). *Let $1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor$. Then there is a term order $\preceq'$ on the monomials in S_n such that*

$$\mathrm{in}_{\preceq'}(\mathrm{P}_{n,r}) = I_{\Sigma_{n,r-1}}.$$

Using the term order in Proposition 3.1, Herzog and Trung [11] derived a determinant formula for the multiplicity of $S_n/\,\mathrm{P}_{n,r}$. Using the same term order, Ghorpade and Krattenthaler [9] obtained a similar determinant formula, which they simplified to the following product formula via a determinant identity due to Krattenthaler [13, Th. 7]:

PROPOSITION 3.2 ([9]). *We have that*

$$e(S_n/\,\mathrm{P}_{n,r}) = \prod_{1\leq i\leq j\leq n-2r+1} \frac{2(r-1)+i+j}{i+j}.$$

Earlier, Harris and Tu [10] discovered a different determinant formula and also a different product formula for the same multiplicity.

By Proposition 3.2, the multiplicities of the two quotient rings $S_n/\,\mathrm{P}_{n,r}$ and $S_n/I_{\Delta_{n,r-1} * 2^{\Theta_{n,r-1}}}$ coincide; apply Proposition 2.2 (ii).

De Negri [4] derived a determinant formula for the Hilbert series of $S_n/\,\mathrm{P}_{n,r}$ and hence for the h-vector of $\Sigma_{n,r-1}$. Ghorpade and Krattenthaler [9] recovered this formula and also provided two alternative determinant formulas for the same Hilbert series.

By Proposition 3.1, Theorem 2.1 implies the following:

COROLLARY 3.3. *Let $1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor$. Then the two simplicial complexes $\Delta_{n,r-1}$ and $\Sigma_{n,r-1}$ have the same h-vector.*

4. The new term order

Using the distance function d from Section 2, we define a linear order $\preceq$ on the set of variables such that $X_{ij} \prec X_{kl}$ whenever $d_{ij} < d_{kl}$ and extend $\preceq$ to monomials using reverse lexicographic term order. Precisely, $m \prec m'$ if and only if either $\deg m < \deg m'$ or $\deg m = \deg m'$ and m is lexicographically smaller than m', where the variables in m and m' are arranged in *increasing* order. Note that the term order $\preceq$ does not depend on the parameter r.

For example, for $n = 5$, we may choose the order

$$X_{12} \prec X_{23} \prec X_{34} \prec X_{45} \prec X_{15} \prec X_{13} \prec X_{24} \prec X_{35} \prec X_{14} \prec X_{25}.$$

The order of the terms that appear in the Pfaffian of some A_{i_1,i_2,i_3,i_4} then becomes

$$\begin{aligned} &X_{12}X_{34} \prec X_{12}X_{45} \prec X_{12}X_{35} \prec X_{23}X_{45} \prec X_{23}X_{15} \\ \prec\, &X_{23}X_{14} \prec X_{34}X_{15} \prec X_{34}X_{25} \prec X_{45}X_{13} \prec X_{15}X_{24} \\ \prec\, &X_{13}X_{24} \prec X_{13}X_{25} \prec X_{24}X_{35} \prec X_{35}X_{14} \prec X_{14}X_{25}. \end{aligned}$$

Note that the monomials in the last row correspond to the minimal nonfaces of $\Delta_{5,1}$.

LEMMA 4.1. *Let* $1 \le r \le \frac{n}{2}$ *and set* $\Theta_{n,r} := \Omega_n \backslash \Omega_{n,r}$. *Then* $I_{\Delta_{n,r-1} * \Theta_{n,r-1}} \subseteq \mathrm{in}_{\preceq}(\mathrm{P}_{n,r})$.

Proof. The lemma is trivially true for $r = 1$; thus assume that $r > 1$. For every minimal nonface σ of $\Delta_{n,r-1}$, we need to prove that $m = \prod_{ij \in \sigma} X_{ij}$ is the leading term of some element in $\mathrm{P}_{n,r}$. Write

$$\sigma = \{j_1 j_{r+1}, j_2 j_{r+2}, \ldots, j_r j_{2r}\},$$

where $j_1 < j_2 < \ldots < j_{2r}$. It suffices to prove that m is maximal among all terms in the Pfaffian of $A_{j_1, j_2, \ldots, j_{2r}}$. Assume to the contrary that there is a term m' in the given Pfaffian such that $m \prec m'$.

By symmetry, we may assume that $X_{j_1 j_{r+1}}$ is minimal among the variables in m and that $j_{r+1} - j_1 \le n + j_1 - j_{r+1}$. The latter property implies that $X_{j_k j_l} \preceq X_{j_1 j_{r+1}}$ whenever $1 \le k < l \le r+1$. By assumption, m' contains no such term $X_{j_k j_l}$, except possibly $X_{j_1 j_{r+1}}$ itself.

Recall that we may identify any term in the Pfaffian of $A_{j_1, j_2, \ldots, j_{2r}}$ with a perfect matching on the set $\{j_1, \ldots, j_{2r}\}$. We say that j_k is *matched* with j_l in such a term if the term contains the variable $X_{j_k j_l}$.

We identify two cases:

- m' does not contain $X_{j_1 j_{r+1}}$. Then each $i \in [r+1]$ must be matched with some element from $[r+2, 2r]$, because otherwise m' would contain variables that are strictly smaller than $X_{j_1 j_{r+1}}$. However, this is a contradiction, because $[r+2, 2r]$ has size only $r - 1$.
- m' does contain $X_{j_1 j_{r+1}}$. Consider the two monomials $\hat{m} = m / X_{j_1 j_{r+1}}$ and $\hat{m}' = m' / X_{j_1 j_{r+1}}$, which both appear in the Pfaffian of the matrix $A_{j_2, \ldots, j_r, j_{r+2}, \ldots, j_{2r}}$. By induction on r, we have that $\hat{m}' \preceq \hat{m}$, because $\sigma - j_1 j_{r+1}$ is a minimal nonface of $\Delta_{n,r-2}$. Yet, this implies that

$$m' = X_{j_1 j_{r+1}} \hat{m}' \preceq X_{j_1 j_{r+1}} \hat{m} = m,$$

which is a contradiction. □

5. Proof of the main theorem

To prove Theorem 2.1, we need the following simple lemma about the inclusion of monomial ideals. We are grateful to Ezra Miller for pointing out to us that various generalizations and disguises of Lemma 5.1 appear in the literature. Notably, [3, Lemma 4.2] together with Exercise 8.13 from [15] leads to the seemingly most general version (see the notes to Chapter 8 in [15] for further references).

LEMMA 5.1. *Let $T = k[x_1, \ldots, x_\ell]$ be the polynomial ring in ℓ variables. Suppose that $I \subseteq J$ are monomials ideals in T such that the following hold:*

(i) $\dim(T/I) = \dim(T/J)$.

(ii) $e(T/I) = e(T/J)$.

(iii) $I = I_\Delta$ *for a pure simplicial complex* Δ *on ground set* $[\ell]$.

Then $I = J$.

Proof. Since $\dim(T/I_\Delta) = \dim(\Delta) + 1$, it follows by (i) that $\dim(\Delta) = \dim(T/J) - 1$. Let J^{pol} be the polarization of J that lives in the polynomial ring T' in ℓ' variables. We refer the reader to Fröberg [7] for the definition and basics of polarization. We can assume that $T \subseteq T'$. It is well known [7] that $\dim(T'/J^{\mathrm{pol}}) = \dim(T/J) + (\ell' - \ell)$. Let I' be the ideal generated by I in T'. Then I' is the Stanley-Reisner ideal of a simplicial complex $\Delta' = \Delta * 2^\Theta$ for some set Θ of cardinality $\ell' - \ell$. In particular,

$$\dim(\Delta') = \dim(\Delta) + (\ell' - \ell) = \dim(T'/J^{\mathrm{pol}}) - 1.$$

Since J^{pol} is a squarefree monomial ideal, there is a simplicial complex Γ on ground set $[\ell']$ for which $J^{\mathrm{pol}} = I_\Gamma$. By $I' = I_{\Delta'} \subseteq J^{\mathrm{pol}} = I_\Gamma$, it follows that $\Gamma \subseteq \Delta'$. Moreover, $\dim(\Gamma) = \dim(\Delta')$.

The number of faces of Δ in top dimension $\dim(\Delta)$ coincides with the number of faces of Δ' in top dimension $\dim(\Delta')$ and is equal to the multiplicity $e(T/I) = e(T'/I')$ of T/I and T'/I'. Since the multiplicity does not change under polarization [7], it follows that $e(T/J) = e(T'/J^{\mathrm{pol}})$. Since $e(T'/J^{\mathrm{pol}})$ equals the number of faces of Γ in top dimension $\dim(\Gamma) = \dim(\Delta')$, we may conclude by (i) and (ii) that Γ and Δ' have the same number of faces in this dimension. By $\Gamma \subseteq \Delta'$, it follows that these faces are indeed the same. From the fact (iii) that Δ is pure, we obtain that any face of Δ is contained in a top-dimensional face. Since purity of Δ implies purity of Δ', the same holds for Δ'. But then $\Gamma \subseteq \Delta'$ implies $\Gamma = \Delta'$. Therefore, the minimal monomial generating set of $J^{\mathrm{pol}} = I_\Gamma$ uses only the variables from T. Thus $J = J^{\mathrm{pol}} \cap T = I_\Delta = I$. $\square$

Proof of Theorem 2.1: Let S_n, $\mathrm{P}_{n,r+1}$, $\Delta_{n,r}$, Ω_n, $\Omega_{n,r}$, $\Theta_{n,r}$ and $\preceq$ be as in Sections 2 and 4.

We set $I := I_{\Delta_{n,r} * \Theta_{n,r}}$ and $J := \mathrm{in}_\preceq(\mathrm{P}_{n,r+1})$.

By Lemma 4.1, we know that $I \subseteq J$.

By Proposition 2.2 (i), we know that $\Delta_{n,r}$ and hence $\Delta_{n,r} * \Theta_{n,r}$ are pure simplicial complexes. Moreover, the dimension of $\Delta_{n,r}$ is $r(n-2r-1)-1$, which yields that

$$\dim(\Delta_{n,r} * 2^{\Theta_{n,r}}) = r(n-2r-1) + \#\Theta_{n,r} - 1.$$

Simple enumeration shows that $\#\Theta_{n,r} = nr$; thus

$$\dim(\Delta_{n,r} * 2^{\Theta_{n,r}}) = r(2n-2r-1) - 1.$$

Hence $\dim(S_n/I) = r(2n-2r-1)$, which coincides [11] with the dimension $\dim(S_n/\mathrm{P}_{n,r+1})$. Since by general Gröbner basis theory

$$\dim(S_n/\mathrm{P}_{n,r+1}) = \dim(S_n/\mathrm{in}_{\preceq}(\mathrm{P}_{n,r+1})) = \dim(S_n/J),$$

it follows that $\dim(S_n/I) = \dim(S_n/J)$.

The number of maximal faces of $\Delta_{n,r}$ coincides with the number of maximal faces of $\Delta_{n,r} * 2^{\Theta_{n,r}}$ and therefore with $e(S_n/I)$. Again general Gröbner basis theory says that

$$e(S_n/\mathrm{P}_{n,r+1}) = e(S_n/\mathrm{in}_{\preceq}(\mathrm{P}_{n,r+1})) = e(S_n/J).$$

By Proposition 2.2 (ii) and Proposition 3.2, it follows that $e(S_n/I) = e(S_n/J)$.

Therefore, the ideals I and J satisfy the assumptions of Lemma 5.1. Thus $I = J$, which concludes the proof. □

Recently, in a more general setting, Krattenthaler [14] obtained a degree-preserving bijection between the sets of monomials in the two quotient rings $S_n/I_{\Delta_{n,r-1}*2^{\Theta_{n,r-1}}}$ and $S_n/I_{\Sigma_{n,r-1}}$. The bijection is based on a variation of the Robinson-Schensted-Knuth correspondence. Using this bijection, one may prove Theorem 2.1 without using Propositions 2.2 (ii) and 3.2.

6. Determinantal ideals

Using Lemma 5.1, we prove that certain simplicial complexes related to a given determinantal ideal have the same h-vector, thereby generalizing enumerative results due to the first author [12] presented in Proposition 6.2 below. This time, the derived initial ideals are not spherical in general.

Let

$$M = \begin{pmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

be a generic matrix indexed by $\mathbb{P}^2$, where $\mathbb{P} = \{1, 2, 3, \ldots\}$. Let Λ be a finite subset of $\mathbb{P}^2$ such that the following hold:

- If $ab := (a, b) \in \Lambda$, then $cb \in \Lambda$ for $1 \le c \le a$.
- If $ab_1, ab_2 \in \Lambda$ and $b_1 \le b_2$, then $ad \in \Lambda$ for $b_1 \le d \le b_2$.

We refer to Λ as a *stack polyomino.* If in addition $a1 \in \Lambda$ whenever $ab \in \Lambda$ for some b, then Λ is a *Ferrers diagram.*

Define $M(\Lambda) = (m_{ab})$ by

$$m_{ab} = \begin{cases} X_{ab} & \text{if } ab \in \Lambda; \\ 0 & \text{otherwise.} \end{cases}$$

For $r \geq 1$ and index sets $\alpha = \{a_1, \dots, a_r\}$ and $\beta = \{b_1, \dots, b_r\}$ of size r, we denote by $M_{\alpha,\beta}(\Lambda)$ the submatrix of $M(\Lambda)$ constructed by selecting the rows indexed by α and the columns indexed by β. Define

$$D_{\alpha,\beta}(\Lambda) = \begin{cases} \det M_{\alpha,\beta}(\Lambda) & \text{if } a_i b_j \in \Lambda \text{ for } 1 \leq i, j \leq r; \\ 0 & \text{otherwise.} \end{cases}$$

Let $R_\Lambda = k[X_{ab} : ab \in \Lambda]$ be the polynomial ring over the field k in the variables indexed by Λ. We denote by $\mathrm{D}_{\Lambda,r}$ the ideal in R_Λ generated by the polynomials $D_{\alpha,\beta}(\Lambda)$ for all possible choices of index sets α and β of size r. Note that all but finitely many of these polynomials are zero.

Two elements ab and cd form a 2-*diagonal* in Λ if $a < c$, $b < d$, and the 2×2 square $\{ab, ad, cb, cd\} = \{a, c\} \times \{b, d\}$ is a subset of Λ. More generally, $a_1b_1, \dots, a_rb_r$ form an r-*diagonal* in Λ if the following hold:

- $a_1 < a_2 < \dots < a_r$ and $b_1 < b_2 < \dots < b_r$
- The $r \times r$ square $\{a_ib_j : i, j \in [1, r]\} = \{a_1, \dots, a_r\} \times \{b_1, \dots, b_r\}$ is a subset of Λ.

For $r \geq 1$, define $\Delta_{\Lambda,r-1}$ as the family of subsets σ of Λ such that σ does not contain any set forming an r-diagonal in Λ. Clearly, $\Delta_{\Lambda,r-1}$ is a simplicial complex.

Define a linear order on the set of variables in Λ such that $X_{ij} \prec X_{kl}$ whenever $i > k$ and also whenever $i = k$ and $j < l$. Extend $\preceq$ to monomials using reverse lexicographic term order in the same manner as in Section 4. Again, the term order $\preceq$ does not depend on r.

Lemma 6.1. *Let $r \geq 1$. Then $I_{\Delta_{\Lambda,r-1}} \subseteq \mathrm{in}_{\preceq}(\mathrm{D}_{\Lambda,r})$.*

Proof. The lemma is trivially true for $r = 1$; thus assume that $r > 1$. As in the proof of Lemma 4.1, we need to prove that $m = \prod_{ab\in\sigma} X_{ab}$ is the leading term of some element in $\mathrm{D}_{n,r}$ for every minimal nonface σ of $\Delta_{\Lambda,r-1}$. Write

$$\sigma = \{a_1b_1, a_2b_2, \dots, a_rb_r\},$$

where $a_1 < a_2 < \dots < a_r$ and $b_1 < b_2 < \dots < b_r$. Let $\alpha = \{a_1, \dots, a_r\}$ and $\beta = \{b_1, \dots, b_r\}$. It suffices to prove that m is maximal among all terms in $D_{\alpha,\beta}(\Lambda)$. Now, each monomial in $D_{\alpha,\beta}(\Lambda)$ contains one variable from each row in α and one variable from each column in β. As a consequence, the smallest variable in any term of $D_{\alpha,\beta}(\Lambda)$ appears in row a_r. In particular, the maximal term of $D_{\alpha,\beta}(\Lambda)$ contains the element $X_{a_rb_r}$. Proceeding by induction on r with $D_{\alpha\setminus\{a_r\},\beta\setminus\{b_r\}}(\Lambda)$ in the same manner as in the proof of Lemma 6.1, we conclude that m is indeed maximal in $D_{\alpha,\beta}(\Lambda)$. □

We may identify any given stack polyomino Λ with the sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ defined by $\lambda_j = \max\{i : (i,j) \in \Lambda\}$. Note that λ_j is the number of elements in column j in Λ. The *content* of Λ is the Ferrers diagram obtained by arranging the elements $\lambda_1, \lambda_2, \lambda_3, \dots$ in decreasing order.

Proposition 6.2 (Jonsson [12]). *Let Λ be a stack polyomino and let Λ' be its content. Then $\Delta_{\Lambda,r-1}$ and $\Delta_{\Lambda',r-1}$ are pure complexes of the same dimension with the same number of maximal faces.*

Proposition 6.3 (Herzog and Trung [11]). *Let Λ be a Ferrers diagram. Then $I_{\Delta_{\Lambda,r-1}} = \mathrm{in}_{\preceq}(\mathrm{D}_{\Lambda,r})$.*

Theorem 6.4. *Let Λ be a stack polyomino. Then $I_{\Delta_{\Lambda,r-1}} = \mathrm{in}_{\preceq}(\mathrm{D}_{\Lambda,r})$.*

Proof. Set $I := I_{\Delta_{\Lambda,r-1}}$ and $J := \mathrm{in}_{\preceq}(\mathrm{D}_{\Lambda,r})$.

By Lemma 6.1, we know that $I \subseteq J$.

By Proposition 6.2, we have that $\dim(R_\Lambda/I) = \dim(R_{\Lambda'}/I')$, where Λ' is the content of Λ and $I' := I_{\Delta_{\Lambda',r-1}}$. By Proposition 6.3, it follows that

$$\dim(R_{\Lambda'}/I') = \dim(R_{\Lambda'}/\,\mathrm{D}_{\Lambda',r}).$$

Since $R_\Lambda/\,\mathrm{D}_{\Lambda,r}$ and $R_{\Lambda'}/\,\mathrm{D}_{\Lambda',r}$ are isomorphic, we conclude that this equals

$$\dim(R_\Lambda/\,\mathrm{D}_{\Lambda,r}) = \dim(R_\Lambda/J);$$

hence $\dim(R_\Lambda/I) = \dim(R_\Lambda/J)$.

By Proposition 6.2, we have that $e(R_\Lambda/I) = e(R_{\Lambda'}/I')$. Another application of Proposition 6.3 yields that

$$e(R_{\Lambda'}/I') = e(R_{\Lambda'}/\,\mathrm{D}_{\Lambda',r}) = e(R_\Lambda/\,\mathrm{D}_{\Lambda,r}) = e(R_\Lambda/J).$$

Summarizing, we deduce that $e(R_\Lambda/I) = e(R_\Lambda/J)$.

Applying Lemma 5.1, we obtain that $I = J$, and we are done. □

Corollary 6.5. *Let Λ and Λ' be stack polyominoes with the same content. Then $\Delta_{\Lambda,r-1}$ and $\Delta_{\Lambda',r-1}$ have the same f-vector.*

The bijection of Krattenthaler [14] mentioned at the end of Section 5 applies also to this situation for the special case that Λ is a Ferrers diagram and Λ' is the (horizontal) reflection of Λ. In particular, Corollary 6.5 is a consequence of this bijection for the given special case.

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Jakob Jonsson, Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge MA 02139, USA

Current address: KTH, Department of Mathematics, SE-10044 Stockholm, Sweden

E-mail address: `jakobj@kth.se`

Volkmar Welker, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, 35032 Marburg, Germany

E-mail address: `welker@mathematik.uni-marburg.de`