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MORSE INDEX OF CONSTANT MEAN CURVATURE TORI OF REVOLUTION IN THE 3-SPHERE

WAYNE ROSSMAN AND NAHID SULTANA

ABSTRACT. We compute lower bounds for the Morse index and nullity of constant mean curvature tori of revolution in the three-dimensional unit sphere. In particular, all such tori have index at least five, with index growing at least linearly with respect to the number of the surfaces' bulges, and the index of such tori can be arbitrarily large.

1. Introduction

The *Morse index* $\mathrm{Ind}(\mathcal{S})$ of a constant mean curvature (CMC) closed (compact without boundary) surface $\mathcal{S}$ is a measure of $\mathcal{S}$'s degree of instability with respect to area, in this sense: We define $\mathrm{Ind}(\mathcal{S})$ as the number of negative eigenvalues of $\mathcal{S}$'s Jacobi operator $\mathcal{L}$, where the function space is the C^∞ functions from $\mathcal{S}$ to the reals $\mathbb{R}$. This is one of two different definitions used in the literature, the other being the *weak index*, equal to the maximal dimension of a vector space of functions given by first derivatives of volume-preserving variations, all of whose nonzero members come from variations that reduce area (to second order). $\mathrm{Ind}(\mathcal{S})$ always equals or is one greater than the weak index ([1], [24]), and a CMC surface is *stable* exactly when the weak index is zero, implying $\mathrm{Ind}(\mathcal{S})$ is then ≤ 1. Various combinations of the two indices are possible: in the Euclidean 3-space $\mathbb{R}^3$, planes have both indices zero, spheres are stable with Morse index one, and catenoids and Enneper surfaces have both indices one [17] (for noncompact surfaces, definitions of the indices must be appropriately adjusted). Because the two indices differ by at most one, we choose to use only $\mathrm{Ind}(\mathcal{S})$ here (like in [32]), without significantly weakening our results.

The index of minimal surfaces in $\mathbb{R}^3$ has been well studied; see [8], [9], [19], [21] amongst numerous papers. In the 3-dimensional unit sphere $\mathbb{S}^3$, the totally geodesic spheres have index 1 [29], the minimal Clifford torus has index 5 and any other closed minimal surface has index ≥ 6 (Urbano [32]).

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The index of CMC surfaces is generally less accessible—certainly so in the case of $\mathbb{R}^3$ [24]. As for closed CMC surfaces in $\mathbb{S}^3$, it is perhaps known only that the stable CMC surfaces are precisely the round spheres [2]. Also, a straightforward computation gives the index of flat tori of revolution, which we do here. But an analogous result to that of [32] is not yet known for closed CMC surfaces in $\mathbb{S}^3$.

Even the Morse index of closed CMC surfaces of revolution in $\mathbb{S}^3$ is still unknown, so here we find lower bounds for the index of such surfaces. They come in three classes: (1) round spheres, (2) tori with two distinct axes of revolution, the *flat* CMC tori, (3) tori with only one axis of revolution, the *non-flat* CMC tori. The index in case (1) is trivial to find, and in case (2) is also easily found. Case (3) is more difficult, and we obtain lower bounds for that case.

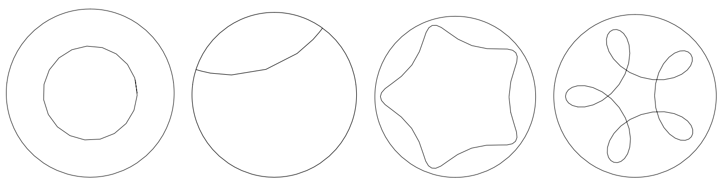

FIGURE 1. Profile curves of four CMC tori of revolution, in totally geodesic hemispheres having the rotation axis as boundary, are shown. The images are stereographic projections from $\mathbb{S}^3$ to $\mathbb{R}^3 \cup \{\infty\}$. The outer circle is the rotation axis, with profile curve inside. The first curve gives a flat torus, the second a round sphere, the third (resp. fourth) an embedded unduloidal (resp. non-embedded nodoidal) torus with five bulges and five necks.

To state our main result, we give notations: For a non-flat CMC torus of revolution $\mathcal{S}$, taking a totally geodesic hemisphere whose boundary is the axis of revolution, the hemisphere intersects $\mathcal{S}$ along a profile curve with an equal finite number of points of maximal and minimal distance from the axis [12], which we call *bulges* and *necks*, respectively. The nullity is the multiplicity of the zero eigenvalue of $\mathcal{L}$. We allow that the mean curvature H can be zero.

THEOREM 1.1. *Let $\mathcal{S}$ be a closed CMC H surface of revolution in $\mathbb{S}^3$. Then either*

- *$\mathcal{S}$ is a round sphere with Morse index 1 and nullity 3, or*
- *$\mathcal{S}$ is a flat torus, and with b equal to the greatest integer strictly less than $\sqrt{1+e^{2\,\mathrm{arcsinh}(|H|)}}$, the index is $3+2b$ and the nullity is either 4 or 6, or*
- *$\mathcal{S}$ is a non-flat torus with k bulges and k necks, and has index at least $\max(5, 2k+1)$ and nullity at least 5.*

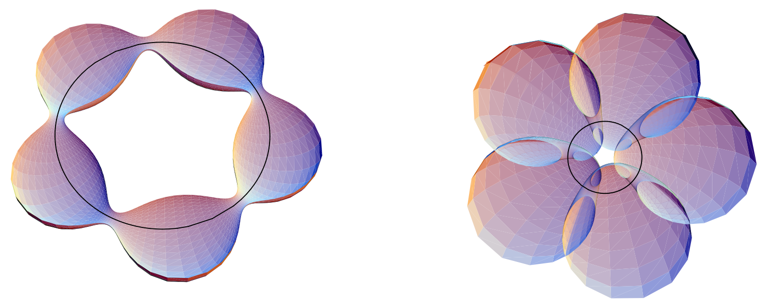

FIGURE 2. For the last two profile curves in Figure 1, half of the corresponding surfaces are shown.

In particular, the index of CMC tori of revolution will always be at least 5, and can indeed be 5 (when flat and when H is 0 or close to 0), and can also be arbitrarily large (when flat with $|H|$ large, for example).

A key ingredient in the proof of the third part of this result is an application of Courant's nodal domain theorem; see also [11], [24], [33] for similar applications. The nodal domain theorem can be used to show the index is at least $2k-1$ by a simpler proof than the one here, but the lower bound $\max(5, 2k+1)$ is much better, in light of [32] and the fact that the case $k=2$ actually occurs. The case $k=1$ has been recently shown to not occur [14]. When $H=0$, the Killing nullity was found in [13]. Also, the first of the three items in Theorem 1.1 is shown in [29].

Furthermore, when a CMC torus of revolution in $\mathbb{S}^3$ is nodoidal, or is unduloidal and wraps at least twice along its axis, the lower bound for the index can be improved as in Theorem 4.6 in Section 4.

2. CMC surfaces of revolution in $\mathbb{S}^3$

Let $\mathbb{S}^3 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}$ inherit its metric $\langle \cdot, \cdot \rangle_{\mathbb{S}^3}$ from the standard Euclidean 4-space $\mathbb{R}^4$. Up to rigid motions, all CMC surfaces of revolution in $\mathbb{S}^3$ can be conformally parametrized (following [30], [27], and secondarily [12], [20], and also [13], [22], [34] when $H=0$) by

$$\mathcal{S}_{s,t}(x, y) = (\mathrm{Re}(\mathcal{X}) \ , \ \mathrm{Im}(\mathcal{Y}) \ , \ \mathrm{Re}(\mathcal{Y}) \ , \ \mathrm{Im}(\mathcal{X})) : \mathcal{A} \to \mathbb{S}^3$$

on the annulus $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \,|\, (x,y) \equiv (x, y+2\pi)\}$ (the symbol "$\equiv$" denotes an equivalence relation), with

$$\mathcal{X} = 2e^{i\gamma}\bar{B}\left(\frac{c_-s_+M}{vA\sqrt{CD}} + c_-c_+\sqrt{\frac{\bar{A}\bar{C}}{\bar{B}C}} - s_-s_+\sqrt{\frac{BC}{A\bar{C}}}\right) \ ,$$

$$\mathcal{Y} = -\frac{c_-c_+M}{v\sqrt{ABCD}} - c_-s_+\sqrt{\frac{\bar{C}}{C}} + s_-c_+\sqrt{\frac{C}{\bar{C}}} \ ,$$

where $A = s + te^{2i\gamma}$, $B = se^{2i\gamma} + t$, $C = 4ste^{2i\gamma} + v^2$, $D = 4st + v^2e^{2i\gamma}$, $M = 2stv'(1 - e^{4i\gamma})$, and

$$c_\pm = \cosh\left(\tfrac{1}{2}(x + iy - g_\pm)\right) \ , \ \ s_\pm = \sinh\left(\tfrac{1}{2}(x + iy - g_\pm)\right) \ ,$$

and here

$$(1) \qquad (s+t)^2 - 4st\sin^2\gamma = 1/4 \ , \ \ g_\pm = \int_0^x \frac{2d\varrho}{1 + (4ste^{\pm 2i\gamma})^{-1}v^2(\varrho)} \ ,$$

$s \in \mathbb{R}^+$, $t \in (-s, s] \setminus \{0\}$, $\gamma \in (0, \pi/4]$, and $v = v(x)$ solves the ordinary differential equation

$$(v')^2 = -(v^2 - 4s^2)(v^2 - 4t^2) \ , \ v(0) = 2t \ , \quad v' = \frac{d}{dx}v \ .$$

We take v to be the nonconstant periodic solution with values between $2|t|$ and $2|s|$ when $|t| < s$, and we take v identically equal to $2s$ when $s = t$.

Setting $\rho = 16s^2t^2\sin^2(2\gamma)v^{-2}$, the metric ds^2 and Gauss curvature K and mean curvature H for $\mathcal{S}_{s,t}$ are

$$ds^2 = \rho(dx^2 + dy^2) \ , \ \ K = -\rho^{-1}(v^2 - 16s^2t^2v^{-2}) \ , \ \ H = \cot(2\gamma) \geq 0 \ .$$

When $t \neq s$, taking $\tau = \sqrt{1 - t^2/s^2} > 0$, we can write $v = 2t/\operatorname{dn}_\tau(2sx)$ explicitly via elliptic functions, with period (i.e., $v(x) = v(x + x_0)$ $\forall x \in \mathbb{R}$)

$$x_0 = \frac{1}{s}\int_0^1 \frac{d\varrho}{\sqrt{(1-\varrho^2)(1-\tau^2\varrho^2)}} \ .$$

By (1), we have $st \in \big(-(16\sin^2\gamma)^{-1}, 0\big) \cup \big(0, (16\cos^2\gamma)^{-1}\big]$. When st degenerates to zero, then $t = 0$ and the profile curve of $\mathcal{S}_{s,t}$ intersects the axis of revolution of $\mathcal{S}_{s,t}$, resulting in a round sphere [12]. Also by [12], when $st \neq 0$, we know that $\mathcal{S}_{s,t}$ alternates periodically between points of maximum distance (bulges) and minimum distance (necks) from the axis of revolution. In analogy to Delaunay surfaces to $\mathbb{R}^3$, we have unduloidal surfaces when $st > 0$ and either nodoidal or unduloidal surfaces when $st < 0$. We get the minimal Clifford torus when $s = t = 1/(2\sqrt{2})$ and $\gamma = \pi/4$, and other flat CMC tori of revolution when $s = t$ and $\gamma < \pi/4$.

Taking $\mu_+ = \sqrt{1 + 16st\sin^2\gamma}$ and $\mu_- = \sqrt{1 - 16st\cos^2\gamma}$, and defining

$$X_+ = \frac{-2\mu_-\sin^2\gamma + 2\mu_+\cos^2\gamma}{2 + (\mu_- + \mu_+)\sin(2\gamma)} \ , \quad X_- = \frac{2\mu_-\sin^2\gamma + 2\mu_+\cos^2\gamma}{2 - (\mu_- - \mu_+)\sin(2\gamma)} \ ,$$

and letting r_+ and r_- be the distances in $\mathbb{S}^3$ from the axis of revolution to the bulges and necks of $\mathcal{S}_{s,t}$, we have $r_+ + r_- = 2\gamma$. In particular, $r_\pm = \frac{\pi}{2} - 2\,\arctan(X_\pm)$. For $st < 0$, $\mathcal{S}_{s,t}$ is nodoidal exactly when $r_+ \leq \pi/2$.

For non-flat $\mathcal{S}_{s,t}$ to close, i.e., to become well defined on a torus

$$\mathcal{T} = \{(x,y) \in \mathbb{R}^2 \,|\, (x,y) \equiv (x, y+2\pi) \equiv (x+x_1, y)\}$$

for some $x_1 \in \mathbb{R}^+$, we need

$$\arctan\left(\frac{2(t+s\cos(2\gamma))}{\mu_+\cos^2\gamma - \mu_-\sin^2\gamma}\cdot\tan\left(\int_0^{x_0}\frac{8stv^2(\varrho)\sin(2\gamma)d\varrho}{v^4(\varrho)+16s^2t^2+8stv^2(\varrho)\cos(2\gamma)}\right)\right)$$

to be a rational multiple of 2π. Then x_1 will be a positive integer multiple of x_0. In the flat case, $\mathcal{S}_{t,t}$ closes when $x_1 = 2\pi/\tan\gamma$.

3. The Jacobi operator

Endow $\mathcal{T}$ with a metric $ds^2 = \rho(dx^2+dy^2)$ for C^∞ $\rho = \rho(x,y) : \mathcal{T} \to \mathbb{R}^+$ (in fact, ρ is real-analytic in our application), and let $\mathcal{S} : \mathcal{T} \to \mathbb{S}^3$ be an isometric (hence conformal) immersion with mean curvature H and Gauss curvature K. When H is constant, $\mathcal{S}$ is critical for a variation problem ([1], [2], [4], [15], [28]) whose associated *Jacobi operator*, or *stability operator*, is

$$\mathcal{L} = -\Delta - 4 - 4H^2 + 2K\ ,$$

where Δ is the Laplace-Beltrami operator of ds^2. The function space of $\mathcal{L}$ is taken to be the C^∞ functions from $\mathcal{T}$ to $\mathbb{R}$. As $-\mathcal{L}$ is elliptic and $\mathcal{T}$ is compact, it is well known ([3], [5], [18], [29], [31]) that the eigenvalues are real, discrete with finite multiplicities, and diverge to $+\infty$. Since Δ, K and H are all independent of how the surface $\mathcal{S}(\mathcal{T})$ is parametrized, clearly the same holds for the following $\mathrm{Ind}(\mathcal{S})$ and $\mathrm{Null}(\mathcal{S})$:

DEFINITION 3.1. The *index* $\mathrm{Ind}(\mathcal{S})$ of $\mathcal{S}$ is the sum of the multiplicities of the negative eigenvalues of $\mathcal{L}$, and the *nullity* $\mathrm{Null}(\mathcal{S})$ is the multiplicity of the zero eigenvalue of $\mathcal{L}$.

Defining $\hat{\mathcal{L}} = \rho\mathcal{L}$, the eigenvalues of $\hat{\mathcal{L}}$ (like for $\mathcal{L}$) form a discrete sequence

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \uparrow +\infty$$

(each considered with multiplicity 1) whose first eigenvalue λ_1 is simple, and the corresponding eigenfunctions

$$\phi_1, \phi_2, \phi_3, \dots \in C^\infty(\mathcal{T})\ , \quad \hat{\mathcal{L}}\phi_j = \lambda_j\phi_j\ , \quad j = 1,2,3,\dots\ ,$$

can be chosen to form an orthonormal basis with respect to the standard L^2 inner product $\langle f, g\rangle_{L^2} = \int_{\mathcal{T}} fg\,dxdy$ for functions $f,g : \mathcal{T} \to \mathbb{R}$.

Furthermore, Courant's nodal domain theorem applies [6], so the number of nodal domains of any eigenfunction associated to the eigenvalue λ_j is at most j.

$\hat{\mathcal{L}}$ and $\mathcal{L}$ will have different eigenvalues, but Rayleigh quotient characterizations ([3], [18], [31]) for the eigenvalues show that these two operators will give the same index. This fact was used by Ritoré and Ros [23] in their classification of stable constant mean curvature tori in three space forms. Furthermore, as the eigenfunctions associated to the zero eigenvalue are the same, $\hat{\mathcal{L}}$ and $\mathcal{L}$ will also give the same nullity. So henceforth we can use either $\mathcal{L}$ or $\hat{\mathcal{L}}$. In the case that $\mathcal{S} = \mathcal{S}_{s,t}$, we have

$$\hat{\mathcal{L}} = -\partial_x\partial_x - \partial_y\partial_y - 2v^2 - 32s^2t^2v^{-2} \ ,$$

with function space the C^∞ real-valued functions on $\mathcal{T}$. Later, we also use

$$\hat{\mathcal{L}}_0 = -\partial_x\partial_x - 2v^2 - 32s^2t^2v^{-2} \ ,$$

now with domain the C^∞ functions from the loop $\mathcal{T}_0 = \{x \in \mathbb{R} \,|\, x \equiv x + x_1\}$ to $\mathbb{R}$. The spectrum $\lambda_{1,0} < \lambda_{2,0} \leq \lambda_{3,0} \leq \cdots \uparrow +\infty$ of $\hat{\mathcal{L}}_0$ has all the analogous properties as for $\hat{\mathcal{L}}$, and Courant's nodal domain theorem also applies to $\hat{\mathcal{L}}_0$.

4. Proof of the main results

For a conformal immersion of a CMC H sphere $\mathcal{S}$ in $\mathbb{S}^3$ with Laplace-Beltrami operator Δ, the Gauss curvature is $K = 1 + H^2$ and the Jacobi operator is $\mathcal{L} = -\Delta - 2(1 + H^2)$. By the canonical correspondence [16], the immersion of $\mathcal{S}$ is isometric to a conformal immersion of a sphere in $\mathbb{R}^3$ of radius $1/\sqrt{1 + H^2}$, whose first two eigenvalues of Δ are well known to be 0 and $2(1 + H^2)$, with multiplicities 1 and 3 respectively (see [5], for example). This shows the first item of Theorem 1.1.

When $\mathcal{S}_{t,t}$ is a flat CMC H torus as in Section 2, $\hat{\mathcal{L}} = -\partial_x\partial_x - \partial_y\partial_y - (\cos\gamma)^{-2}$ with eigenvalues

$$\lambda_{m,n} = m^2 + \alpha n^2 - 1 - \alpha \ , \quad \alpha = (\sqrt{1 + H^2} - H)^2 \in (0, 1] \ ,$$

for integers $m, n \geq 0$, with multiplicity 4, resp. 2, 1, when $mn > 0$, resp. $m + n > 0$ and $mn = 0$, $m = n = 0$. The region $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + \alpha y^2 < 1 + \alpha, x > 0, y > 0\}$ is bounded by two line segments and part of an ellipse centered about $(0, 0)$. Furthermore, $\mathcal{R}$ contains no points of $\mathbb{R}^2$ with integer coordinates, and the points $(1, 1)$, $(0, \sqrt{(1 + \alpha)/\alpha})$, $(\sqrt{1 + \alpha}, 0)$ and $(0, 0)$ lie in its boundary. Now, by summing the multiplicities of the negative $\lambda_{m,n}$, we have the index as in the second item of Theorem 1.1. The nullity is 6 when $\sqrt{(1 + \alpha)/\alpha}$ is an integer, and is 4 otherwise.

The third item of Theorem 1.1 is shown by the following three lemmas:

LEMMA 4.1. *Let $\mathcal{S}(x, y) : \mathcal{A} \to \mathbb{S}^3$ be any conformal immersion of revolution with axis $\ell_1 \subset \mathbb{S}^3$, where for each $y_0 \in \mathbb{R}$, the curve $\hat{c}_{y_0}(x) = \mathcal{S}(x, y_0)$ lies in a unique totally geodesic sphere $\mathcal{P}_{y_0}$ of $\mathbb{S}^3$ with $\ell_1 \subset \mathcal{P}_{y_0}$. By conformality, the angle between any $\mathcal{P}_{y_0}$ and $\mathcal{P}_{y_1}$ along ℓ_1 is $y_0 - y_1$, and for* *each $x_0 \in \mathbb{R}$ the curve $\check{c}_{x_0}(y) = \mathcal{S}(x_0, y)$ is a circle with center in ℓ_1. Let $\mathcal{P}_1 = \mathcal{P}_{\pi/2} \supset \hat{c}_{\pm\pi/2}(x)$. Let $\vec{N}$ be a unit normal vector to $\mathcal{S}(x,y)$.*

Let $\mathcal{P}_2$ be a totally geodesic sphere perpendicular to both $\mathcal{P}_1$ and ℓ_1, and let ℓ_2 be a geodesic circle in $\mathcal{P}_2$ intersecting both $\mathcal{P}_1$ and ℓ_1 perpendicularly. For a Killing field $\mathcal{K}$ on $\mathbb{S}^3$ produced by constant-speed rotation in $\mathbb{S}^3$ about ℓ_2, $f(x,y) = \langle \mathcal{K}, \vec{N} \rangle_{\mathbb{S}^3}$ satisfies both

- *$f(x,y) = u(x) \sin y$ for some function $u(x)$ depending only on x, and*
- *if the metric, mean and Gauss curvatures are $\rho(dx^2 + dy^2)$, H and K, and if $\hat{\mathcal{L}} = -\rho(\Delta + 4 + 4H^2 - 2K)$, then $\hat{\mathcal{L}}(f(x,y))$ is identically zero.*

Proof. Let us use the stereographic projection $\mathbb{S}^3 = \{(y_1, y_2, y_3) \,|\, y_j \in \mathbb{R}\} \cup \{\infty\}$, with metric $ds^2 = 4(\sum_{j=1}^3 dy_j^2)/(1 + \sum_{j=1}^3 y_j^2)^2$. (We use this model for $\mathbb{S}^3$ only in this proof.) By a rigid motion of $\mathbb{S}^3$, we may assume ℓ_j is the y_j-axis and $\mathcal{P}_j$ is the $y_j y_3$-plane for $j = 1, 2$. Then $\mathcal{S}$ can be parametrized as

$$\mathcal{S}(x,y) = (\phi(x), \psi(x) \cos y, \pm\psi(x) \sin y)$$

for functions $\psi(x) \neq 0$ and $\phi(x)$ depending only on x. Then $\vec{N}$ is of the form

$$\vec{N} = (a(x), b(x) \cos y, \pm b(x) \sin y)$$

for functions $a(x)$, $b(x)$ of x with $a^2 + b^2 = (1 + \phi^2 + \psi^2)^2/4$. So

$$f = u(x) \cdot \sin y \;, \quad u(x) = c(b\phi - a\psi)(1 + \phi^2 + \psi^2)^{-2} \;, \quad c \in \mathbb{R} \setminus \{0\} \;,$$

and $\hat{\mathcal{L}}(f) = 0$ (see Prop. 2.12 of [2], for example, or [7] for when $H = 0$). $\square$

Lemma 4.2. *Let $\mathcal{S}_{s,t}$ be a non-flat CMC torus of revolution as in Section 2, with k bulges and associated operator $\hat{\mathcal{L}}_0$. Then $\hat{\mathcal{L}}_0$ has eigenvalue -1 with multiplicity two.*

Proof. The axis of $\mathcal{S}_{s,t}$ is $\mathbb{S}^3 \cap \{x_3 = x_4 = 0\}$, and the point $\mathcal{S}_{s,t}(0,0) \in \mathbb{S}^3 \cap \{x_2 = x_3 = 0\}$ is a bulge of $\mathcal{S}_{s,t}$. Taking two Killing fields of constant-speed rotations of $\mathbb{S}^3$ about the geodesic circles $\mathbb{S}^3 \cap \{x_j = x_3 = 0\}$ for $j = 1, 2$ respectively, two corresponding eigenfunctions $u_j(x) \sin y$ ($j = 1, 2$) of $\hat{\mathcal{L}}$, both with associated eigenvalue zero, are produced as in Lemma 4.1. Thus $\hat{\mathcal{L}}_0(u_j(x)) = -u_j(x)$ for $j = 1, 2$. Since $\mathcal{S}_{s,t}(0,0)$ is a bulge, we have $u_2(0) = 0$ and $u_1(0) \neq 0$. Thus $u_1(x)$ and $u_2(x)$ must be independent functions of x, and the eigenvalue -1 of $\hat{\mathcal{L}}_0$ has multiplicity ≥ 2. Because $\hat{\mathcal{L}}_0(u_j) = -u_j$ is a linear second-order ordinary differential equation on the domain $\mathcal{T}_0$, -1 has multiplicity ≤ 2, and hence exactly 2. $\square$

Remark 4.3. When $\mathcal{S}_{s,t}$ is nodoidal with $k \geq 2$ bulges, it is not difficult to see that u_2 in the above proof must have at least four zeros. So, by Courant's nodal domain theorem, $\lambda_{j,0} < -1$ for $j = 1, 2, 3$. When $\mathcal{S}_{s,t}$ is unduloidal and the projection of a profile curve to the axis circle has wrapping number $w \geq 2$ about the circle (see Definition 4.5 below), it follows that u_2 must have at least $2w$ zeros, so $\lambda_{j,0} < -1$ for all $j \leq 2w - 1$.

LEMMA 4.4. *Let $\mathcal{S}_{s,t}$ be a non-flat CMC torus of revolution with k bulges as in Section 2. Then* $\mathrm{Ind}(\mathcal{S}_{s,t}) \geq \max(5, 2k+1)$ *and* $\mathrm{Null}(\mathcal{S}_{s,t}) \geq 5$.

Proof. Recalling the $v(x)$ in Section 2, a direct computation shows that the function $u_0(x) = v'/v$ is an eigenfunction of either $\hat{\mathcal{L}}$ or $\hat{\mathcal{L}}_0$ with eigenvalue zero. (Geometrically, u_0 is the oriented length of the normal projection of a Killing field produced by constant-speed rotation of $\mathbb{S}^3$ about the geodesic of distance $\pi/2$ from the axis of $\mathcal{S}_{s,t}$.)

As u_0 is independent of y and has $2k$ nodal domains on $\mathcal{T}_0$, Courant's nodal domain theorem implies $\hat{\mathcal{L}}_0$ has at least $2k - 1$ negative eigenvalues.

By Lemma 4.2, we get two independent eigenfunctions $u_j(x)$ of $\hat{\mathcal{L}}_0$, $j = 1, 2$, both with eigenvalue -1. Thus $\lambda_{1,0} < -1$. Let $u_3(x)$ be an eigenfunction of $\hat{\mathcal{L}}_0$ with eigenvalue $\lambda_{1,0}$. Hence u_3, $u_3 \cos y$, $u_3 \sin y$, u_1 and u_2 are five independent eigenfunctions of $\hat{\mathcal{L}}$ with negative eigenvalues, so $\mathrm{Ind}(\mathcal{S}_{s,t}) \geq 5$.

Furthermore, as $\hat{\mathcal{L}}_0$ has at least $2k - 1$ negative eigenvalues, all with associated eigenfunctions independent of y, this is true of $\hat{\mathcal{L}}$ as well. And as $u_3 \cos y$ and $u_3 \sin y$ are two more eigenfunctions of $\hat{\mathcal{L}}$ with negative eigenvalues, $\mathrm{Ind}(\mathcal{S}_{s,t}) \geq 2k + 1$.

Since u_0, $u_1 \cos y$, $u_1 \sin y$, $u_2 \cos y$ and $u_2 \sin y$ are five independent eigenfunctions of $\hat{\mathcal{L}}$ with eigenvalue zero, $\mathrm{Null}(\mathcal{S}_{s,t}) \geq 5$. □

Consider a closed curve in a totally geodesic two-sphere in $\mathbb{S}^3$, and take a point in the two-sphere so that both it and its antipodal point in that two-sphere are disjoint from the curve. Then we can consider the winding number of the curve with respect to the point, just as we do for disjoint closed curves and points in the Euclidean 2-plane. (See [10], for example.) Then, as in Remark 4.3, we can define the wrapping number $w \in \mathbb{Z}^+$ of $\mathcal{S}_{s,t}$ as follows:

DEFINITION 4.5. The *wrapping number* $w \in \mathbb{Z}^+$ of $\mathcal{S}_{s,t}$ is the winding number about the axis (with respect to a center point of the axis, i.e., a center of the axis circle in some geodesic two-sphere containing the axis) of the orthogonal projection to the axis of a profile curve of the full surface.

By Remark 4.3, when either $\mathcal{S}_{s,t}$ in Lemma 4.4 is nodoidal with $k \geq 2$ bulges or is unduloidal with wrapping number $w \geq 2$, we can easily strengthen the proof of Lemma 4.4 to obtain:

THEOREM 4.6. *Let $\mathcal{S}$ be a closed CMC H surface of revolution in $\mathbb{S}^3$.*

- *If $\mathcal{S}$ is nodoidal with $k \geq 2$ bulges, then* $\mathrm{Ind}(\mathcal{S}) \geq \max(11, 2k+5)$.

- *If $\mathcal{S}$ is non-flat and unduloidal with wrapping number $w \geq 2$ along its axis, and with k bulges, then* $\mathrm{Ind}(\mathcal{S}) \geq \max(6w-1, 2k+4w-3)$.

Remark 4.7. Using the methods in [25], one can numerically compute the eigenvalues of $\hat{\mathcal{L}}_0$, and thus the index of, any given non-flat torus $\mathcal{S}_{s,t}$, and results of that are shown in Table 1. (See [26] for more details.)

surf-ace $\mathcal{S}_{s,t}$	s	t	k	w	$\lambda_{1,0}$	$\mathcal{B}_-$	$\mathcal{B}_+$	numerical value of $\mathrm{Ind}(\mathcal{S}_{s,t})$	Theorem 1.1's lower bound for $\mathrm{Ind}(\mathcal{S}_{s,t})$	Theorem 4.6's lower bound for $\mathrm{Ind}(\mathcal{S}_{s,t})$
A	0.4078	0.1583	2	1	-1.28	1	1	6	5	–
B	0.4392	0.0811	3	1	-1.08	1	3	8	7	–
C	0.4352	0.0757	4	1	-1.04	1	5	10	9	–
D	0.4275	0.0796	5	1	-1.02	1	7	12	11	–
E	0.4431	0.0881	5	2	-1.12	3	5	16	11	15
F	0.4561	0.0559	7	2	-1.04	3	9	20	15	19
G	0.4738	0.0527	7	3	-1.11	5	7	24	15	23
H	0.4829	0.0408	9	4	-1.09	7	9	32	19	31
I	0.5112	-0.050	3	1	-1.26	3	1	12	7	11
J	0.5061	-0.089	3	1	-1.40	3	1	12	7	11
K	0.5291	-0.068	4	1	-1.43	5	1	18	9	13
L	0.5256	-0.155	4	1	-1.85	5	1	18	9	13
M	0.5501	-0.095	5	1	-1.66	7	1	24	11	15
N	0.5199	-0.087	7	2	-1.47	9	3	32	15	19
O	0.5210	-0.051	11	3	-1.30	15	5	52	23	27

Table 1. Numerical results on the index of the CMC non-flat tori of revolution shown in Figures 3 and 4. There are two $\lambda_{j,0}$ equal to -1, and here $\mathcal{B}_-$ and $\mathcal{B}_+$ denote the number of $\lambda_{j,0} < -1$ and the number of $\lambda_{j,0} \in (-1,0)$ respectively, found numerically. By the numerical value of $\mathrm{Ind}(\mathcal{S}_{s,t})$ we mean the value of $\mathrm{Ind}(\mathcal{S}_{s,t}) = 3\mathcal{B}_- + 2 + \mathcal{B}_+$ confirmed with numerics, but not yet proven with mathematical rigor. Note that $\mathrm{Ind}(\mathcal{S}_{s,t})$ equaling $3\mathcal{B}_- + 2 + \mathcal{B}_+$ depends on knowing that $\lambda_{1,0} > -2$.

Remark 4.8. In the proof of Lemma 4.4, if the eigenvalue of u_3 for $\hat{\mathcal{L}}_0$ were strictly less than $-n^2$ for some integer $n \geq 2$, then eigenvalues of $\hat{\mathcal{L}}$ associated to $u_3 \cos(jy)$ and $u_3 \sin(jy)$, for integers $j \leq [0, n]$, would be negative, allowing us to further strengthen the above results. However, numerical evaluation shows that $\lambda_{1,0} > -2$ for at least the fifteen $\mathcal{S}_{s,t}$ shown in Figures 3 and 4. See Table 1. This is related to a result in [20] about the first eigenvalue of the Laplacian on minimal examples.

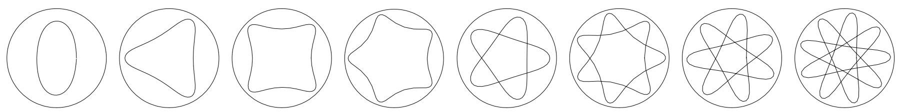

Figure 3. Profile curves for the unduloidal surfaces A, B, C, D, E, F, G and H in Table 1.

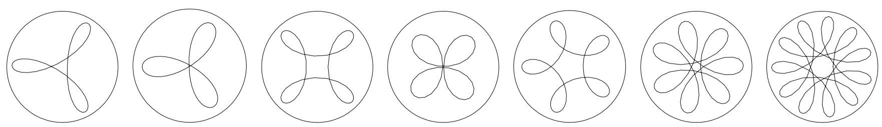

Figure 4. Profile curves for the nodoidal surfaces I, J, K, L, M, N and O in Table 1.

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Wayne Rossman, Department of Mathematics, Faculty of Science, Kobe University, Kobe 657-8501, Japan

E-mail address: `wayne@math.kobe-u.ac.jp`

Nahid Sultana, Department of Mathematics, Faculty of Science, Kobe University, Kobe 657-8501, Japan

E-mail address: `nahid@math.kobe-u.ac.jp`