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Q-DEGREES OF n-C.E. SETS

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ABSTRACT. In this paper we study Q-degrees of n-computably enumerable (n-c.e.) sets. It is proved that n-c.e. sets form a true hierarchy in terms of Q-degrees, and that for any $n \geq 1$ there exists a $2n$-c.e. Q-degree which bounds no noncomputable c.e. Q-degree, but any $(2n+1)$-c.e. non $2n$-c.e. Q-degree bounds a c.e. noncomputable Q-degree.

Studying weak density properties of n-c.e. Q-degrees, we prove that for any $n \geq 1$, properly n-c.e. Q-degrees are dense in the ordering of c.e. Q-degrees, but there exist c.e. sets A and B such that $A - B <_Q A \equiv_Q \emptyset'$, and there are no c.e. sets for which the Q-degrees are strongly between $A - B$ and A.

1. Introduction

In this paper we study Q-degrees of n-computably enumerable (n-c.e.) sets. Recall (Shoenfield [2]) that a set A is Q-reducible to a set B if there is a computable function f such that for every $x \in \omega$, $x \in A \Leftrightarrow W_{f(x)} \subseteq B$. In this case we say that $A \leq_Q B$ via f (or via a uniformly c.e. sequence of c.e. sets $U = \{U_x\}_{x\in\omega}$, if for all x $U_x = W_{f(x)}$).

The relation of Q-reducibility is transitive and reflexive, so that it generates a degree structure on 2^ω. It is not hard to show that in general Q-reducibility is incomparable with Turing (T-) reducibility, but in c.e. sets $A \leq_Q B$ implies $A \leq_T B$. Therefore, in c.e. sets the relation $\leq_Q$ is strictly stronger than $\leq_T$, since if $A \leq_Q B$, then $\omega - A$ is B-c.e.

A set A is *n-c.e.* if there is a computable function $f(s,x)$ such that for every x:

$$f(0,x) = 0,$$
$$A(x) = \lim_s f(s,x),$$
$$|\{s : f(s,x) \neq f(s+1,x)\}| \leq n.$$

The 2-c.e. sets are also known as the d-c.e. sets as they are differences of c.e. sets.

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A degree **a** is called n-c.e. degree for $n \geq 1$ if it contains an n-c.e. set, and it is called a *properly n-c.e. degree* if it contains an n-c.e. set but no m-c.e. set for any $m < n$.

We adopt the usual notational conventions, found, for instance, in Soare [3]. In particular, we write $[s]$ after functionals and formulas to indicate that every functional or parameter therein is evaluated at stage s. In particular, for an oracle X and a c.e. functional Φ, $\Phi(X; y, s)$ means only that at most s steps are allowed for the computation from the oracle X to converge, whereas $\Phi(X; y)[s]$ means also that the approximation X_s is used as the oracle, and may mean as well that some function-value $y(s)$ is being used as the argument for the computation. As usual, $\langle x_1, x_2, \ldots, x_n\rangle$ means the 1-1 enumeration of all n-tuples by integers.

2. Results

From our results it immediately follows that in n-c.e. sets (even for the case $n = 2$) T-reducibility is incomparable with Q-reducibility. Therefore, the development of the structural theory of Q-degrees of n-c.e. sets in comparison with their T-degrees becomes one of the interesting directions in the study of Q-degrees of n-c.e. sets.

We begin with some pathologies of the upper-semilattice of the n-c.e. Q-degrees relative to the n-c.e. Turing degrees. It is well-known that for any n-c.e. set ($n > 1$) A of properly n-c.e. degree there exists a $(n-1)$-c.e. set B such that $B <_T A$ (this is called Lachlan's Proposition). In Theorem 1 we establish a similar result in Q-degrees, but in the opposite direction.

Theorem 1. *Let $R_1 \supseteq R_2 \supseteq \ldots \supseteq R_{2n+1}$ be c.e. sets, $R_1 \neq \omega$, and let*

$$P_k = \bigcup_{1 \leq i \leq [\frac{k+1}{2}]} \{A_{2i-1} - A_{2i}\},\ k = 1, 2, \ldots, 2n+1,$$

where for all i, $A_i = R_i$, except when k is an odd number, in which case we have $A_i = R_i$ for $1 \leq i \leq k$, but $A_{k+1} = \emptyset$. Then, for all $k, k \geq 1$,

(a) $P_{2k} \leq_Q P_{2k-1}$,
(b) $P_{2k} \leq_Q P_{2k+1}$,
(c) $P_{2k} \leq_Q P_{2k+2}$.

In particular, for all c.e. sets A and B, $A - B \leq_Q A$.

The proof of this theorem immediately follows from the following proposition.

Proposition 2. *Let $X \subset \omega$ be a set, let A, B be c.e. sets, $A \supseteq B$, $X \cap A = \varnothing$ and $\overline{X \cup A} \neq \varnothing$. Then*

(1) $X \leq_Q X \cup (A - B)$,
(2) $X \cup (A - B) \leq_Q X \cup A$.

Proof. (1) Let f be a computable function such that for all x

$$W_{f(x)} = \begin{cases} \{x\} & \text{if } x \notin A, \\ \{x, b\} & \text{otherwise.} \end{cases}$$

Here b is a fixed element from $\overline{X \cup A}$.

If $x \in X$, then $x \notin A$ and $W_{f(x)} = \{x\} \to W_{f(x)} \subseteq X \cup (A - B)$.

If $x \notin X$, then either $x \in A$ or $x \notin A$. If $x \in A$, then $W_{f(x)} = \{x, b\} \to W_{f(x)} \nsubseteq X \cup (A - B)$.

If $x \notin A$, then $W_{f(x)} = \{x\} \to W_{f(x)} \nsubseteq X \cup (A - B)$. Therefore, $X \leq_Q X \cup (A - B)$.

(2) In this case we define the function f as follows:

$$W_{f(x)} = \begin{cases} \{x\} & \text{if } x \notin B, \\ \{x, b\} & \text{otherwise.} \end{cases}$$

Here again b is a fixed element from $\overline{X \cup A}$.

If $x \in X \cup (A - B)$, then $W_{f(x)} = \{x\} \to W_{f(x)} \subseteq X \cup A$. If $x \notin X \cup (A - B)$, then either $x \in B$ or $x \notin B$.

If $x \notin B$, then $W_{f(x)} = \{x\} \to W_{f(x)} \nsubseteq X \cup A$. If $x \in B$, then $W_{f(x)} = \{x, b\} \to W_{f(x)} \nsubseteq X \cup A$.

Therefore, $x \in X \cup (A - B) \leftrightarrow W_{f(x)} \subseteq X \cup A$. □

It follows from Theorem 3 below that n-c.e. sets form a true hierarchy in terms of Q-degrees. We prove the existence of a 3-c.e. set $M = (A_1 - A_2) \cup A_3$ of properly 3-c.e. Q-degree. The proof easily generalizes to prove the existence of a properly n-c.e. Q-degrees for all $n > 1$.

THEOREM 3. *There exists a* 3*-c.e. set* $M = (A_1 - A_2) \cup A_3$ *of properly* 3*-c.e. Q-degree.*

Proof. We construct c.e. sets A_1, A_2 and A_3, $A_1 \supseteq A_2 \supseteq A_3$ such that the Q-degree of $M = (A_1 - A_2) \cup A_3$ does not contain d-c.e. sets.

To ensure that M is not of d-c.e. Q-degree, we meet for all $e, i, j \in \omega$ the requirements

$$\mathcal{R}_{e,i,j} : M \nleq_Q W_i - W_j \text{ via } \Theta_e \quad \vee \quad W_i - W_j \nleq_Q M \text{ via } \Phi_e \quad \vee W_i \nsupseteq W_j.$$

Here $\{(W_i, W_j, \Theta_e, \Phi_e,)\}_{e,i,j \in \omega}$ is some enumeration of all possible quadruples of c.e. sets W_i, W_j and partial computable functionals Θ and Φ.

The basic module for the requirement $\mathcal{R}_{e,i,j}$. For a convenience we again first rewrite the requirements $\mathcal{R}_{e,i,j}$ as follows:

$$\begin{gathered} \mathcal{R}_{e,i,j} : (\exists\, x)(x \notin M \& W_{\Theta_e(x)} \subseteq (W_i - W_j) \vee x \in M \& W_{\Theta_e(x)} \nsubseteq (W_i - W_j)) \\ \vee (\exists x)(x \notin (W_i - W_j) \& W_{\Phi_e(x)} \subseteq M \vee x \in (W_i - W_j) \& W_{\Phi_e(x)} \nsubseteq M) \\ \vee W_i \nsupseteq W_j. \end{gathered}$$

Now we proceeds as follows:

(1) Choose an unused candidate $x = x_{e,i,j}$ for $\mathcal{R}_{e,i,j}$ greater than any number mentioned in the construction thus far.
(2) Wait for a stage s such that $\Theta_e(x)\downarrow$, and for some (least) $y = y_{e,i,j}$ such that

$$y \in W_{\Theta_e(x)} - (W_i - W_j).$$

There are the following two possibilities:

Case 1. $y \in W_j$. Obviously, in this case the requirement $\mathcal{R}_{e,i,j}$ is satisfied via the witness x.

Case 2. $y \notin W_i$. In this case:

(3) Wait for a stage s' and for some (least) $z = z_{e,i,j}$ such that $\Phi_e(y)\downarrow$ and

$$z \in W_{\Phi_e(y_e)} - M.$$

Again, there are the following two possibilities:

Case a. $x \neq z$. In this case:

(4a) Put x into M.
(5a) Force y to enter into W_i.

(Otherwise the requirement $\mathcal{R}_{e,i,j}$ is satisfied.)

(6a) Protect z from other strategies from now on.
(7a) Wait for y to enter into W_j. (Now y is a permanent witness to the success of $\mathcal{R}_{e,i,j}$ because

$$y \in W_i - W_j \& z \in W_{\Phi_e(y_e)} - M,$$

which means that $W_i - W_j \not\leq_Q M$ via Φ_e.)

Case b. $x = z$. In this case:

(4b) Put x into M.
(5b) Force y to enter into W_i.
(6b) Remove $z(= x)$ from M.

Now there are following two possibilities:

Subcase (b_1). y enters into W_j. In this case:

$(7b_1)$ Enumerate x into M and stop.

Subcase (b_2). $y \in W_i - W_j$. In this case the requirement $\mathcal{R}_{e,i,j}$ is satisfied via the witness x.

The explicit construction and the remaining parts of the proof now straightforward, so we will not give them here. □

It is easy to see that if A is a c.e. noncomputable set, then $\omega - A$ in Q-degrees bounds no c.e. sets except computable sets. It immediately follows that the partial orderings of T- and Q- degrees of d-c.e. sets are elementarily non-equivalent, since by Lachlan's proposition each noncomputable d-c.e. set in T-degrees bounds some noncomputable c.e. set. Below in Proposition 4 we show that any $(2n+1)$-c.e. non $(2n)$-c.e. set in Q-degrees also bounds a noncomputable c.e. set for any $n \geq 1$.

PROPOSITION 4. *Let M be a $(2n+1)$-c.e. set which is not $(2n)$-c.e., and $M = (A_1 - A_2) \cup \ldots \cup A_{2n+1}$, where $A_1 \supseteq A_2 \ldots \supseteq A_{2n+1}$ are c.e. sets. Then there is a c.e. noncomputable set P which is Q-reducible to M.*

Proof. Let $A_{2n} = \{f(x) : x \in \omega\}$ for some computable function f, and let g be a computable function such that for any x, $W_{g(x)} = \{f(x)\}$. Define $P = f^{-1}(A_{2n+1})$. Then we have:

$$x \in P \rightarrow f(x) \in A_{2n+1} \rightarrow W_{g(x)} \subseteq M.$$
$$x \notin P \rightarrow f(x) \in A_{2n} - A_{2n+1} \rightarrow W_{g(x)} \nsubseteq M.$$

Therefore, $P \leq_Q M$.

If the set P is computable, then $A_{2n} - A_{2n+1}$ is c.e., since

$$A_{2n} - A_{2n+1} = \{x : (\exists y)((x = f(y)) \ \& \ y \notin P\}.$$

Therefore, M is a $(2n)$-c.e. set, since

$$(A_{2n-1} - A_{2n}) \cup A_{2n+1} = A_{2n-1} - (A_{2n} - A_{2n+1}).$$

Let $A'_{2n} = A_{2n} - A_{2n+1}$. Then A'_{2n} is c.e., $A'_{2n} \subseteq A_{2n-1}$ and $M = (A_1 - A_2) \cup \ldots \cup (A_{2n-1} - A'_{2n})$, which contradicts to the assumption of the theorem. □

Therefore, any n-c.e. for some odd $n \geq 2$ set which is not an m-c.e. set for some even $m < n$ bounds in Q-degrees some noncomputable c.e. set. As noted above, there are 2-c.e. sets which bound in Q-degrees no noncomputable c.e. sets. Generalizing this observation, we now prove that for any even $n > 2$ there is a n-c.e. set of properly n-c.e. degree which in Q-degrees does not bound any noncomputable c.e. sets.

THEOREM 5. *For any $n \geq 2$ there is a $(2n)$-c.e. set M of properly $(2n)$-c.e. Q-degree such that for any c.e. set W, if $W \leq_Q M$, then W is computable.*

Proof. For simplicity we will consider the case $n = 2$. The general case has the same proof with obvious changes.

We construct c.e. sets A_1, A_2, A_3 and A_4, $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$, such that the Q-degree of $M = (A_1 - A_2) \cup (A_3 - A_4)$ does not contain 3-c.e. sets, and for any c.e. set W, if $W \leq_Q M$, then W is computable.

To ensure that M is not of 3-c.e. Q-degree, we meet for all $e \in \omega$ the requirements

$$\mathcal{R}_e : M \not\leq_Q V_e \text{ via } \Theta_e \quad \vee \quad V_e \not\leq_Q M \text{ via } \Phi_e.$$

To ensure the last condition we meet for any $e \in \omega$ the following requirements:

$$\mathcal{S}_e : W_e \not\leq_Q M \text{ via } \Phi_e \quad \vee \quad W_e \;\; \textit{is} \;\; \textit{computable}.$$

Here $\{(W_e, \Phi_e,)\}_{e\in\omega}$ is some enumeration of all possible pairs c.e. sets W_e and partial computable functionals Φ.

The basic module for the requirement $\mathcal{R}_e$. This is similar to the appropriate module from Theorem 3. For a convenience again we first rewrite the requirements $\mathcal{R}_e$ as follows:

$$\mathcal{R}_e : (\exists\, x)(x \notin M \& W_{\Theta_e(x)} \subseteq V_e \vee x \in M \& W_{\Theta_e(x)} \not\subseteq V_e)$$
$$\vee (\exists y)(y \notin V_e \& W_{\Phi_e(y)} \subseteq M \vee y \in V_e \& W_{\Phi_e(y)} \not\subseteq M).$$

Now we proceed as follows:

(1) Choose an unused candidate $x = x_e$ for $\mathcal{R}_e$ greater than any number mentioned in the construction thus far.
(2) Wait for a stage s such that $\Theta_e(x)\downarrow$, and for some (least) $y = y_e$ such that
$$y \in W_{\Theta_e(x)} - V_e.$$
(3) Wait for a stage s' and for some (least) $z = z_e$ such that $\Phi_e(y)\downarrow$ and
$$z \in W_{\Phi_e(y_e)} - M.$$

Again, there are the following two possibilities:

Case a. $x \neq z$. In this case:

(4a) Put x into M, protect z from other strategies from now on.
(5a) Wait for y to enter into V_e.

Now the requirement is satisfied since $y \in V_e$ and $z \in W_{\Phi_e(y)} - M$. If later y leaves V_e, then the requirement is again satisfied since $x \in M$ and $y \in W_{\Theta_e(x)} - V_e$.

Case b. $x = z$. In this case:

(4b) Put x into M.
(5b) Force y to enter into V_e.
(6b) Remove $z (= x)$ from M.

Now there are the following two possibilities:

Subcase (b_1). y leaves V_e. In this case:

$(7b_1)$ Enumerate x into M.
$(8b_1)$ Force y to enter into V_e.
$(9b_1)$ Remove x from M.

Now we have $y \in V_e$ and $z \in W_{\Phi_e(y_e)} - M$, and the requirement $\mathcal{R}_e$ is satisfied.

Subcase (b_2). y remains in V_e. In this case we have $z \in W_{\Phi_e(y_e)} - M$, and again the requirement $\mathcal{R}_e$ is satisfied.

The basic module for the requirement $\mathcal{S}_e$. We use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) While $k \notin W_e$ wait either for k to enter into W_e or for some $y_k \in W_{\Phi_e(k)} - M$.
(2) Restrain y_k (if any) from being enumerated into M and stop.

Now suppose that $W_e \leq_Q M$ via Φ_e. To prove that in this case W_e is computable, for any $k \in \omega$ go through the k-th cycle of the strategy until a stage s is reached where either k enters into $W_{e,s}$ or some $y_k \notin M_s$ is enumerated into $W_{\Phi_e(k)}[s]$. Then $k \in W_e$ iff $k \in W_{e,s}$. Indeed, if $k \in W_e - W_{e,s}$, then we have $k \in W_e$ and $y_k \in W_{\Phi_e(k)} - M$, which means that $W_e \not\leq_Q M$ via Φ_e, a contradiction.

Interactions between the requirements. We only need to consider the case when the $\mathcal{S}$-strategies activity of higher priority interfere with the activity of $\mathcal{R}$-strategies of lower priority.

The only possible conflict in activities of these strategies is the following: an $\mathcal{S}$-strategy of higher priority restrains some integer y_k against M (at step 2 of the $\mathcal{S}$-module), but an $\mathcal{R}$-strategy of lower priority needs to enumerate y_k into M (at steps $4a$, $4b$ and $7b_1$). The obvious solution of this conflict is the following: enumerate y_k into M for the $\mathcal{R}$-strategy, wait for k to enter into W_e in the $\mathcal{S}$-strategy (if this never happens, then obviously this is okay for the $\mathcal{S}$-strategy), then remove y_k from M, satisfying the requirement $\mathcal{S}$, and possibly injuring the activity of the $\mathcal{R}$-strategy. For the latter we now choose a new witness and start the activity of the $\mathcal{R}$-strategy from the beginning. The crucial point here is that we construct a n-c.e. set M for an even n, and we can always remove the element y_k from M which was previously enumerated into M by the $\mathcal{R}$-strategy.

Construction. We order the requirements $\mathcal{R}_e, \mathcal{S}_e$ in an ω type list $\langle P_n \rangle$ and at stage s we consider the requirement P_n, $s = \langle n, k \rangle$, in our list.

Case 1. $P_n = R_e$ for some e.

If there is no witness associated with this requirement, we choose an integer x_e bigger than all integers so far mentioned during the construction as a witness associated with the requirement $\mathcal{R}_\rceil$ and go to the next stage. Otherwise, we check which step of the basic module for this requirement holds, and act accordingly. If for an integer m in this stage we have $M_s(m) \neq M_{s-1}(m)$, then we initialize all requirements of lower priority. (An $\mathcal{S}$-requirement is initialized by cancelling all its cycles. An $\mathcal{R}$-requirement is initialized by cancelling of its witness and, therefore, cancelling all its restraints.)

Case 2. $P_n = S_e$ for some e.

(a) If for each $k \leq s$ such that $k \notin W_{e,s}$ either there is an integer y_k such that y_k is associated with k in a stage $s' < s$ or $\neg\exists y(y \in W_{\Phi_e(k)} - M)[s]$, then go to stage $s+1$.
(b) Otherwise, but for some (least) $k \leq s$ we have $k \notin W_{e,s}$. Then let $y_k = \mu y(y \in W_{\Phi_e(k)} - M)[s]$. Associate y_k with k, restrain y_k from other strategies from now on, and go to stage $s+1$.
(c) If there is a (least) $k \in W_{e,s}$ such that an associated with k in a stage $s' < s$ integer y_k is enumerated into M by a requirement $R_{e'}, e' > e$, then remove y_k from M, and initialize the requirement $R_{e'}$.
(d) Otherwise, go to stage $s+1$.

This ends the construction.

Verification.

Lemma 6. *Each requirement P_e is satisfied.*

Proof. Let P_n be the first requirement which is not satisfied and let s be the first stage after which no requirement initializes it. (It follows from the choice of n that there is such a stage s.)

Case 1. $P_n = R_e$ for some e. In this case the requirement P_n is satisfied with the first witness x_e chosen after the stage s.

Case 2. $P_n = S_e$ for some e. Suppose that $W_e \leq_Q M$ via Φ_e. To effectively compute $W_e(k)$ for an (arbitrary) k, continue the construction until a stage $s' > s$ such that either $k \in W_{s'}$ or an integer y_k is associated with k at stage s'. Then $k \in W_e$ iff $k \in W_{e,s'}$. Indeed, suppose that $k \in W_e - W_{e,s'}$. Since $W_e \leq_Q M$, we have $y_k \in M$, which means that a requirement $\mathcal{R}_i$ of lower priority enumerates y_k into M. But since k enters into W_e after the stage s', by the construction we remove y_k from M, a contradiction. □

The first significant result concerning the partial-ordering of the c.e. Q-degrees was provided by Downey, LaForte and Nies [1]. They proved that the c.e. Q-degrees form a dense partial order, just as in the case of the c.e. T-degrees.

For the n-c.e. degrees, the density problem has some variants. Namely, studying so-called *weak density* properties one may investigate the existence of n-c.e. degrees $\mathbf{a} < \mathbf{b}$ such that there is no m-c.e. degree $\mathbf{c}$ between $\mathbf{a}$ and $\mathbf{b}$ for any $m < n$.

We have mentioned already that it follows from Proposition 2 that, in particular, for all c.e. sets A and B, we have $A - B \leq_Q A$. In Theorem 7 we construct c.e. sets A and B such that the strong reducibility $A - B <_Q A$ holds with several additional properties. Here we note that for c.e. sets A and B the strong reducibility $A - B <_Q A$ is the most prevailing case. First note, as can easily be shown, that if A is computable and infinite, then for any c.e. noncomputable subset $B \subseteq A$ and any noncomputable c.e. set C we have $C \not\leq_Q A - B$. Further, let A be any infinite noncomputable c.e. set and B its c.e. subset such that $A - B$ is immune (obviously, for any such A there exists such a B). Then $A \not\leq_Q A - B$. Indeed, if $A \leq_Q A - B$ via some computable function f, then the c.e. set $\{\cup W_{f(x)} : x \in A\}$ must be finite. Let $\{a_0, \ldots, a_n\}$ be all its elements. Then for any x, $x \in \omega - A$ if and only if $\exists s, y (\forall i \leq n)(y \in W_{f(x),s} \;\&\; y \neq a_i)$. Therefore, $\omega - A$ is a Σ^0_1-set and A is computable, a contradiction.

Below combining weak density questions with the above mentioned property of n-c.e. sets we prove the following result:

THEOREM 7. *There exists a d-c.e. set $A_1 - A_2$ such that $A_1 - A_2 <_Q A_1$, and for every c.e. set W, if $A_1 - A_2 \leq_Q W$, then $A_1 \leq_Q W$.*

Proof. We construct c.e. sets $A_1, A_2, A_2 \subseteq A_1$, such that $A = A_1 - A_2 \leq_Q A_1$, the Q-degree of A does not contain c.e. sets, and $(\forall\ c.e.W)(A \leq_Q W \rightarrow A_1 \leq W)$. Obviously, this is enough to prove the theorem.

To ensure that A is not of c.e. Q-degree we meet for all $e \in \omega$ the following requirements:

$$\mathcal{R}_e : A \not\leq_Q W_e \text{ via } \Theta_e \quad \vee \quad W_e \not\leq_Q A \text{ via } \Phi_e.$$

To satisfy the second property we meet the following requirements for all $e \in \omega$:

$$\mathcal{S}_e : A_1 - A_2 \leq_Q W_e \text{ via } \Phi_e \Rightarrow$$

$$\Rightarrow (\exists \text{ uniformly c.e. sequence of c.e. sets } U_e)(A_1 \leq_Q W_e \text{ via } U_e).$$

Here $\{(W_e, \Theta_e, \Phi_e,)\}_{e \in \omega}$ is an effective enumeration of all possible triples of c.e. sets W and partial computable functionals Θ and Φ. □

The basic module for the requirement $\mathcal{R}_e$. This is similar to the appropriate modules from Theorems 3 and 5. Again for a convenience we rewrite the requirements $\mathcal{R}_e$ as follows:

$$\mathcal{R}_e : (\exists\ x)(x \notin A \& W_{\Theta_e(x)} \subseteq W_e \vee x \in A \& W_{\Theta_e(x)} \not\subseteq W_e)$$
$$\vee(\exists x)(x \notin W_e \& W_{\Phi_e(x)} \subseteq A \vee x \in W_e \& W_{\Phi_e(x)} \not\subseteq A).$$

Now we proceed as follows:

(1) Choose an unused candidate x_e for $\mathcal{R}_e$ greater than any number mentioned in the construction thus far.
(2) Wait for a stage s such that $\Theta_e(x_e)\downarrow$, and for some least y_e such that $y_e \in W_{\Theta_e(x_e)} - W_e$. (If this never happens, then x_e is a witness to the success of $\mathcal{R}_e$).
(3) Wait for a stage s' such that $\Phi_e(y_e)\downarrow$, and for some least z_e such that $z_e \in W_{\Phi_e(y_e)} - A$. (Again, if this never happens, then y_e is a witness to the success of $\mathcal{R}_e$.)

There are following two possibilities:

Case a. $x_e \neq z_e$. In this case:

(4a) Put x_e into A_1.
(5a) Force y_e to enter into W_e. (If this never happens, then x_e is a witness to the success of $\mathcal{R}_e$.)
(6a) Protect z_e from other strategies from now on. (Now y_e is a permanent witness to the success of $\mathcal{R}_e$ because $y_e \in W_e \& z_e \in W_{\Phi_e(y_e)} - A$, which means that $W_e \not\leq_Q A$ via Φ_e.)

Case b. $x_e = z_e$. In this case:

(4b) Put x_e into A_1.
(4b) Force y_e to enter into W_e. (If this never happens, then x_e is a witness to the success of $\mathcal{R}_e$).
(6b) Put $z_e(= x_e)$ into A_2. (Now again y_e is a permanent witness to the success of $\mathcal{R}_e$ because $y_e \in W_e \& z_e \in W_{\Phi_e(y_e)} - A$, which means that $W_e \not\leq_Q A$ via Φ_e.)

The basic module for the requirement $\mathcal{S}_e$. Again, for convenience we first rewrite the requirements S_e as follows:

$$\mathcal{S}_e : (\exists\ x)(x \notin A \& W_{\Phi_e(x)} \subseteq W_e \vee x \in A \& W_{\Phi_e(x)} \not\subseteq W_e)$$
$$\vee(\forall x)(x \in A_1 \leftrightarrow U_{e,x} \subseteq W_e).$$

Now the strategy proceeds as follows: we use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) While $k \notin A_1$ wait for $\Phi_e(k)\downarrow$ and some $u_k \in W_{\Phi_e(k)} - W_e$. (It is clear that otherwise the requirement is satisfied via the cycle k.)
(2) Enumerate u_k into $U_{e,k}$, open cycle $k+1$.

(3) Wait for a stage s when k enters A_1. (If between steps (2) and (3) u_k enters W_e, then close all cycles $> k$ and go to step (1) for a new u_k.)
(4) Close all cycles $> k$ and wait for a stage s' when u_k enters W_e. (If there is no such stage s', then again the requirement $\mathcal{S}_e$ satisfied via the cycle k.)
(5) Open cycles $> k$ and close the cycle k.

The module has the following possible outcomes:

(A) Some (least) cycle k eventually waits either at step (1), or at step (4) forever. This means that we were successful in satisfying $\mathcal{S}_e$ through the cycle k since in this case $A \not\leq_Q W_e$ via Φ_e.
(B) Some cycle k loops from step (3) to step (1) infinitely often. This means that $k \notin A$ and $W_{\Phi_e(x)} \subseteq W_e$, and again we were successful in satisfying $\mathcal{S}_e$ through the cycle k.
(C) Otherwise, for each cycle k, either it eventually waits at step (3) forever, or proceeds through step (5). This obviously means that for all k, $k \in A_1 \leftrightarrow U_{e,k} \subseteq W_e$. Indeed, for each k there are following two possibilities:

Case 1. $k \notin A_1$. Then cycle k eventually waits at step (3), which means that $k \notin A_1$ and $u_k \in U_{e,k} - W_e$.

Case 2. $k \in A_1$. Then cycle k achieves step (5), which means that $U_{e,k} \subseteq W_e$.

Interactions between the requirements. Note that we enumerate integers into A_1 and A_2 only by the $\mathcal{R}$-strategy. But nevertheless the $\mathcal{S}$-strategies activity interferes with the $\mathcal{R}$-strategies. How do we get $U_{e,k} \subseteq W_e$, which is needed in cycle k of the basic module for $\mathcal{S}_e$, when $k \in A_1$, if, by a $\mathcal{R}_i$-strategy, we enumerate k into A_2, and if u_k never enters W_e?

There are following two possibilities:

Case 1. $i \leq e$. In this case the $\mathcal{R}_i$-requirements have higher priority, and in the $\mathcal{S}_e$-strategy we simply close this cycle k. Since there are only finitely many $\mathcal{R}$-requirements of higher priority, this is enough for the $\mathcal{S}_e$-requirement to be satisfied: if $A \leq_Q W_e$, then $k \in A_1 \leftrightarrow U_{e,k} \subseteq W_e$ for all except finitely many k.

Case 2. $e < i$. First note that by the $\mathcal{R}_i$- strategy we may enumerate k into A_2 only in step (6b) of case b. This means that at step (3) of the $\mathcal{R}_i$-strategy we first obtain $z_e = x_e (= k)$ and then at step (6b) enumerate it into A_2. This lack of co-ordination between $\mathcal{R}$- and $\mathcal{S}$-strategies can be avoided by inserting between steps (5b) and (6b) of $\mathcal{R}$-strategy the following additional step.

(5.5b) Wait for a stage t such that for all $\mathcal{S}_i$-strategies of higher priority for which some $k = x_e = z_e$ with $k \in A_1$, $W_{\Phi_i(k)} \subseteq W_i[t]$.

If there is no such stage t, then this means that a requirement $\mathcal{S}_i$ of higher priority is satisfied diagonalizing its left part (i.e., $A \not\leq_Q W_i$). Since there are only finitely many $\mathcal{S}$-requirements of higher priority, then for the success of the $\mathcal{R}$-requirement it is now enough to choose a new witness x_e and proceed.

Construction. We order the requirements $\mathcal{R}_e, \mathcal{S}_e$ in an ω-type list $\langle P_n \rangle_{n\in\omega}$ and at stage s we consider the requirement P_n, $s = \langle n, k\rangle$, in our list.

Case 1. $P_n = R_e$ for some e.

If there is no witness associated with this requirement, we choose an integer x_e bigger than all integers so far mentioned during the construction as a witness associated with the requirement $\mathcal{R}_\rceil$ and go to the next stage. Otherwise, for each witness x_e of this requirement we check which step of the basic module for this requirement holds, and act accordingly. For step (5.5b), if for each witness x_e there is an $\mathcal{S}$-requirement of higher priority $\mathcal{S}_i, i < e$, such that $x_e = k \in A_1$, but $u_k \notin W_i$, then for the requirement R_e we choose a new witness $x_{e'}$ and go to the next stage. Otherwise, we enumerate k into A_2 and go to the next step.

If in this stage for an integer m we have $A_s(m) \neq A_{s-1}(m)$, then we initialize all $\mathcal{R}$-requirements of lower priority. (An $\mathcal{R}$-requirement is initialized by cancelling its witness and, therefore, cancelling all its restraints.)

Case 2. $P_n = S_e$ for some e.

(a) Let $k_0 \leq s$ be the greatest integer such that for any $k \leq k_0$, $\Phi_{e,s}(k)$ is defined. (If there is no such k_0, then go to stage $s+1$.)
(b) For each $k \leq k_0$ such that $k \notin A_{1,s}$ and $U_{e,k,s} \subseteq W_{e,s}$, if there is an (least) integer $u_k \in W_{\Phi_{e,s}(k),s} - W_{e,s}$, then enumerate u_k into $U_{e,k,s+1}$.
(c) Go to stage $s+1$.

This ends the construction.

Verification. Let $U_{e,k} = \bigcup_{s\in\omega} U_{e,k,s}$. It is clear that there is a computable function f such that $U_{e,k} = W_{f(e,k)}$.

The proof that the Q-degree of $A_1 - A_2$ does not contain c.e. sets is similar to the appropriate claim of Theorem 5: For the sake of contradiction suppose that $\mathcal{P}_n$ is the first requirement which is not satisfied and $\mathcal{P}_n = \mathcal{R}_e$ for some e. Let s be the least stage after which no $\mathcal{R}$-requirement of higher priority enumerates elements into A_1 or A_2. Let x_e be the first witness chosen for the $\mathcal{R}_e$-requirement after stage s.

If $x_e \notin A_1$, then it follows immediately from the construction that $\mathcal{R}_e$ is satisfied by the witness x_e. Now let $x_e \in A_1$. We assumed that all $\mathcal{S}$-requirements of higher priority are also satisfied. This means that there is a stage $s' \geq s$ such that for each requirement $\mathcal{S}_i, i \leq e$, if there is $k \in A$ but $u_k \in W_{\Phi_i(k)} - W_i$, then for some such k we have $u_k \in W_{\Phi_{i,s'}(k)} - W_{i,s'}$. Now by the construction the first witness x_e which is chosen at stage s' or later, satisfies the requirement $\mathcal{R}_e$.

Further, it follows from Theorem 1 that $A_1 - A_2 \leq_Q A_1$. Therefore, we have $A_1 - A_2 <_Q A_1$.

Now suppose that $\mathcal{P}_n = \mathcal{S}_e$ for some e. To prove that $\mathcal{S}_e$ is satisfied we assume that $A_1 - A_2 \leq_Q W_e$ via Φ_e, i.e., Φ_e is total and for any k, $k \in A_1 - A_2$ if and only if $W_{\Phi_e(k)} \subseteq W_e$. Let s be the least stage such that the $\mathcal{R}$-requirements of higher priority after stage s do not enumerate elements into A_1 or A_2.

We prove that for each k, k enters into A_1 after stage s if and only if $U_{e,k} = W_{f(e,k)} \subseteq W_e$.

If $k \notin A_1$, then by construction $U_{e,k}$ contains an element $u_k \in W_{\Phi_e(k)} - W_e$ (otherwise we have $k \notin A$ and $W_{\Phi_e(k)} \subseteq W_e$, which contradicts to our assumption $A_1 - A_2 \leq_Q W_e$ via Φ_e). Therefore, $U_{e,k} \not\subseteq W_e$.

Now suppose that k enters into A_1 at a stage $s_0 \geq s$. By construction we have $U_{e,k} \subseteq W_{\Phi_e(k)}$. If $U_{e,k} \not\subseteq W_e$, then there is an element u_k such that $u_k \in U_{e,k} - W_e$. By construction this means that $u_k \in W_{\Phi_e(k)} - W_e$. But we have $k \notin A_2$ since we enumerate k into A_2 only if all such u_k are enumerated already into W_e. Therefore, $k \in A_1 - A_2$ and $W_{\Phi_e(k)} \not\subseteq W_e$, a contradiction. $\square$

Now in Theorem 8 below we prove that adding to the construction of Theorem 7 a variant of a permitting argument for Q-reducibility, we can achieve that the Q-degree of A_1 coincides with the Q-degree of the creating set K.

Theorem 8. *There exists a d-c.e. set $A_1 - A_2$ such that $A_1 - A_2 <_Q K$, and for every c.e. set W, if $A_1 - A_2 \leq_Q W$, then $K \leq_Q W$.*

Proof. We describe the modifications needed in the construction of the previous theorem. We have to ensure $K \leq_Q A_1$ through a variant of permitting argument for Q-reducibility. For this we construct (let us denote this strategy by $\mathcal{P}$) a uniformly c.e. sequence of c.e. sets V_e such that $(\forall k)(k \in K \leftrightarrow V_k \subseteq A_1)$.

First let us agree that in the previous theorem witnesses for $\mathcal{R}$-requirements we choose only among even numbers. Now for any $k \in \omega$, we have:

The basic module for the requirement $\mathcal{P}$.

- Choose a big (bigger than all numbers mentioned so far) odd number v_k as a witness for k, enumerate v_k into V_k.
- Keep it out of A_1 until k enters K.
- Enumerate v_k into A_1 and stop.

Obviously $K \leq_Q A_1$ via $V = \{V_k\}_{k \in \omega}$. Since in Theorem 7 we choose witnesses for $\mathcal{R}$-strategies only among even numbers, and $\mathcal{S}$-strategies involve only numbers enumerated into A_1 or A_2 by $\mathcal{R}$-strategies, this new modified strategy does not interfere with the activity of $\mathcal{R}$- and $\mathcal{S}$-strategies, except for the following possibility: we first choose some odd number v_k as a witness for k (in the $\mathcal{P}$-strategy), then at step (3) of its basic module an $\mathcal{R}$-strategy obtains v_k as some $z_e \in W_{\Phi_e(y_e)} - A$, and later at step (6a) restrains it from other strategies (to keep it out of $A_1 - A_2$), finally this new $\mathcal{P}$-strategy (having $k \in K$) enumerates v_k into A_1. To avoid this conflict between strategies we could enumerate $v_k = z_e$ simultaneously into A_1 and A_2, but some $\mathcal{S}$-strategy in its $(k' = v_k)$-cycle may wait (having also an appropriate integer $u_{k'}$, see step 3 of the basic module of $\mathcal{S}$-strategy) for k' to enter into A_1. Now enumerating v_k simultaneously into A_1 and A_2 means that $k' = v_k \notin A$, and if $u_{k'} \notin W_e$ this action kills the $\mathcal{S}$-strategy. This difficulty can be avoided by a priority ordering of the $\mathcal{R}$- and $\mathcal{S}$-requirements (the $\mathcal{P}$-requirement is the global requirement and does not participate in this priority ordering of requirements). Then:

- If $\mathcal{R}$ has higher priority, then we enumerate v_k into A_1 and A_2, and meet the $\mathcal{R}$-requirement and initialize the $\mathcal{S}$-requirement.
- If $\mathcal{S}$ has higher priority, then we enumerate $v_k = k'$ into A_1, wait for $u_{k'}$ to enter into W_e (if this never happens, then the $\mathcal{S}$-requirement is satisfied), and then enumerate v_k into A_2.

Obviously this refinement of the $\mathcal{P}$-strategy solves this problem. □

THEOREM 9. *Let V be a c.e. set such that $V <_Q K$. Then there exist c.e. sets A and B such that $V <_Q A - B <_Q K$ and the Q-degree of $A - B$ does not contain c.e. sets.*

Proof. We will construct c.e. sets A and B so that $A \supseteq B$ and the Q-degree of $V \oplus (A - B)$ have the desired property. For convenience we suppose without loss of generality that V contains only even numbers and will construct A and B as subsets of the set of odd numbers. Then obviously $V \oplus (A - B) \equiv_Q V \cup (A - B)$.

This is ensured by the following requirements. Let $n = \langle e, i, j \rangle$.

$$\mathcal{R}_n : \ A - B \not\leq_Q W_e \text{ via } \Phi_i \text{ or } W_e \not\leq_Q V \oplus (A - B) \text{ via } \Phi_j.$$

We rewrite the requirement $\mathcal{R}_n$ as follows:

$$\begin{aligned}
&\exists x \notin A - B \ \& \ W_{\Phi_i(x)} \subseteq W_e, \quad \text{or} \\
&\exists x \in A - B \ \& \ W_{\Phi_i(x)} \not\subseteq W_e, \quad \text{or} \\
&\exists y \notin W_e \ \& \ W_{\Phi_j(y)} \subseteq V \cup (A - B), \quad \text{or} \\
&\exists y \in W_e \ \& \ W_{\Phi_j(y)} \not\subseteq V \cup (A - B).
\end{aligned}$$

Basic module for the $\mathcal{R}_n$-strategy in isolation. We use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) Pick an unused odd witness x which is larger than all integers mentioned so far and keep it out of A.
(2) Wait for $\Phi_i(x)\downarrow$.
(3) Wait for some $y \in W_{\Phi_i(x)} - W_e$.
(4) Wait for $\Phi_j(y)\downarrow$.
(5) Wait for some $z \in W_{\Phi_j(y)} - \{V \cup (A-B)\}$.
(6a) If z is an odd number, then enumerate x into A, restrain z from being enumerated into A and B by the requirements of lower priority. Force y to enter into W_e (otherwise the requirement is satisfied). If $x = z$, then enumerate z into B, otherwise (if $x \neq z$) keep z out of A and B.

(Now the requirement $\mathcal{R}_n$ is satisfied, since $y \in W_e$ and $z \in W_{\Phi_j(y)} - \{V \cup (A-B)\}$.)

(6b) If z is an even number, then start cycle $k+1$ to run simultaneously.
(7) Wait for $k \searrow K$.

(If before step (7) y enters into W_e, then go to step (3) for a new y. If y does not enter into W_e, but between steps (6) and (7) z enters into V, then return to step (5) for a new z. In both cases stop all cycles $> k$ and remove all restraints of cycle k.)

(8) Enumerate x into A, stop all cycles $> k$.
(9) Wait for $y \searrow W_e$.
(10) Wait for $z \searrow V$.
(11) Open cycles $> k$.
(12) Wait for some $z' \in W_{\Phi_j(y)} \cap A$.
(13) Enumerate z' into B and stop.

The module has the following possible outcomes:

(A) Some (least) cycle k eventually waits either at steps (2)–(5) or at steps (9)–(10) forever. This means that we were successful in satisfying $\mathcal{R}_e$ through the cycle k since in this case either $A - B \not\leq_Q W_e$ via Φ_i or $W_e \not\leq_Q V \oplus (A-B)$ via Φ_j.
(B) Otherwise, some (least) cycle k comes to step (3) infinitely often. This means that $x \notin A-B$, but $W_{\Phi_i(x)} \subseteq W_e$. Therefore we are successful in satisfying $\mathcal{R}_e$ through the cycle k.
(C) Otherwise, some (least) cycle k comes to step (5) infinitely often. This means that $\exists y \notin W_e$ & $W_{\Phi_j(x)} \subseteq V \cup (A-B)$, and again we are successful in satisfying $\mathcal{R}_e$ through the cycle k.
(D) Some cycle k reaches the step (13). This means that we were successful in satisfying $\mathcal{R}_e$ through the cycle k since in this case we have $y \in W_e$ but $z' \in W_{\Phi_j(y)} - \{V \cup (A-B)\}$.
(E) Otherwise, for each cycle k, either it eventually waits at step (7) forever, or proceeds through step (7) but then (the only remaining possibility) it also proceeds through step (11) and never comes to

step (12). Obviously this means that

$$k \notin K \to W_{\Phi_j(y)} - V \neq \emptyset$$

(since it contains z),

$$k \in K \to W_{\Phi_j(y)} \subseteq V$$

(since all elements of $W_{\Phi_j(y)}$ enter also into V). Since for any given k the integer $\Phi_j(y)$ is computed effectively before step (7), this means that $K \leq_Q V$ via the computable function $f(k) = \Phi_j(y)$, contrary to hypothesis.

Interactions between the requirements $\mathcal{R}_n$ and $\mathcal{R}_m$ for $n \neq m$. The only conflict between the requirements $\mathcal{R}_n$ and $\mathcal{R}_m$ for $n \neq m$ is the following: $\mathcal{R}_n$ wants to enumerate some x into A at step (8) which is restrained by $\mathcal{R}_m$ at step (1), or $\mathcal{R}_n$ wants to enumerate some x into B which is restrained by $\mathcal{R}_m$ at step (8).

As usual, we settle this conflict by a priority ordering of the requirements: if $n < m$, then we simply close the appropriate cycles of the $\mathcal{R}_m$-strategy, by cancelling all its restraints. If $m < n$, then we close the cycle k of the $\mathcal{R}_n$-strategy, cancelling all its restraints.

Construction. At stage s we consider the requirement $\mathcal{R}_n$, where $s = \langle n, t\rangle$ for some $t \geq 0$. Let $n = \langle e, i, j\rangle$ and $k = (t)_0$.

If there is no number which is $\mathcal{R}_n$-associated with k and for each $e < k$ some number x_e^n is $\mathcal{R}_n$-associated with e, then $\mathcal{R}_n$-associate with k the least number x_k^n which is greater than all numbers so far mentioned during the construction, and go to stage $s+1$. If some $e < k$ have no $\mathcal{R}_n$-associated number, then directly go to stage $s+1$.

Otherwise, suppose x_k^n is associated with k.

Case 1. $x_k^n \notin A_s$. Consider following two subcases:

Subcase 1.1. There is a y such that

(a) $(y \in W_{\Phi_i(x_k^n)} - W_e)[s]$,
(b) $\Phi_j(y) \downarrow [s]$,
(c) there is an integer z which is greater than all higher priority restraints such that $z \in W_{\Phi_j(y)} - \{V \cup (A - B)\}[s]$.

Let y_k^n be the least such y, and z_k^n be the least z from c) for this y_k^n.

If z_k^n is an even number and $k \notin K_s$, then set $A_{s+1} = A_s$, $B_{s+1} = B_s$. Otherwise (i.e., if z_k^n is an odd number or if z_k^n is an even number and $k \in K_s$) set $A_{s+1} = A_s \cup \{x_k^n\}$, $B_{s+1} = B_s$. Initialize all requirements of lower priority. (An $\mathcal{R}$-requirement is initialized by cancelling its associated numbers and cancelling all its restraints.)

Go to stage $s+1$.

Subcase 1.2. Otherwise. Set $A_{s+1} = A_s$, $B_{s+1} = B_s$, and go to stage $s + 1$.

Case 2. $x_k^n \in A_s$. It follows from the construction that we enumerate integers into A only in case 1. Therefore, in this case integers y_k^n and z_k^n are already defined.

If $y_k^n \in W_{e,s}$, $x_k^n = z_k^n$ and z_k^n is an odd number, then set $A_{s+1} = A_s$, $B_{s+1} = B_s \cup \{z_k^n\}$ and initialize all requirements of lower priority. If z_n^k is an odd number and either $y_k^n \notin W_{e,s}$ or $x_k^n \neq z_k^n$, then set $A_{s+1} = A_s$, $B_{s+1} = B_s$ and restrain z_k^n by priority $\mathcal{R}_n$ from being enumerated into A and B by requirements of lower priority. If z_n^k is an even number, $y_k^n \in W_{e,s}$, $z_k^n \in V_s$, and there is $z' \in W_{\Phi_j(y)} \cap (A - B)[s]$, then set $A_{s+1} = A_s$, $B_{s+1} = B_s \cup \{z'\}$. If z_k^n is an even number and either $y_k^n \notin W_{e,s}$ or $z_k^n \notin V_s$, or $W_{\Phi_j(y)} \cap (A - B)[s] = \emptyset$, then set $A_{s+1} = A_s$, $B_{s+1} = B_s$. Go to stage $s + 1$.

This ends the construction.

Verification: Let $A = \bigcup_{s\in\omega} A_s$ and $B = \bigcup_{s\in\omega} B_s$. We prove that $A - B$ have the desired properties.

LEMMA 10. *Each $\mathcal{R}$-requirement restrains only finitely many odd numbers.*

Proof. Let $\mathcal{R}_{n=\langle e,i,j\rangle}$ be the first requirement which restrains infinitely many odd numbers and let s be the first stage after which no requirement of higher priority restrains new odd numbers. By construction, only odd numbers z_k^n, which correspond to an associated with some k number x_k^n, can be restrained.

After stage s the requirement $\mathcal{R}_n$ restrains at most one number z_k^n. Indeed, if $\mathcal{R}_n$ restrains an integer z_k^n, then by case 1 of the construction this means that at a stage $s' \geq s$ we enumerated x_k^n into A having $(y_k^n \in W_{\Phi_i(x_k^n)} - W_e)[s']$, $\Phi_j(y_k^n) \downarrow [s']$, and $z_k^n \in W_{\Phi_j(y)} - \{V \cup (A - B)\}[s']$. If $y_k^n \in W_{\Phi_i(x_k^n)} - W_e$, then the requirement is satisfied since $x_k^n \in A$ and $W_{\Phi_i(x_k^n)} \not\subseteq W_e$. If later y_k^n enters into W_e, then by case 2 of the construction at a stage $> s'$ we enumerate z_k^n into B and again the requirement is satisfied, since in this case we have $y_k^n \in W_e$ and $W_{\Phi_j(y)} \not\subseteq V \cup (A - B)[s']$. $\square$

LEMMA 11. *Each requirement $\mathcal{R}_n$ is satisfied.*

Proof. Let $\mathcal{R}_{n=\langle e,i,j\rangle}$ be the first requirement which is not satisfied and let s be the first stage after which no requirement of higher priority restrains any new number. (It follows from Lemma 10 that there is a such stage s.) If $x_{k_0}^{n_0}$ and $x_{k_1}^{n_1}$ are two numbers which are associated with k_0 and k_1 according to the $\mathcal{R}_{n_0}$- and $\mathcal{R}_{n_1}$-requirements, then by the construction we have $x_{k_0}^{n_0} \neq x_{k_1}^{n_1}$. This means that any restraint of $x_{k_0}^{n_0}$ does not hinder our work with $x_{k_1}^{n_1}$. (By construction any requirement $\mathcal{R}_m$ of lower priority enumerates into A only associated with some k numbers x_k^m which are not equal to $x_{k_1}^{n_1}$, and $\mathcal{R}_m$ may enumerate into B an integer z_k^m only if $z_k^m = x_k^m$. Therefore, we again have $x_{k_0}^{n_0} \neq x_{k_1}^{n_1}$.)

This means that either for some k the first integer x_k^n which is associated with k after stage s satisfies the requirement $\mathcal{R}_n$, diagonalizing $A-B$ against W_e or diagonalizing W_e against $V \oplus (A-B)$, or for each k, either $k \notin K$ and there are integers y and z such that $z \in W_{\Phi_j(y)} - V$, or $k \in K$ and $W_{\Phi_j(y)} \subseteq V$. Since in this case for each k the integer $y = y(k)$ is computed effectively, and the function $\Phi_j(y(k))$ is total, we have $K \leq_Q V$, a contradiction. $\square$

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