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MINIMAL HOMEOMORPHISMS AND APPROXIMATE CONJUGACY IN MEASURE

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ABSTRACT. Let X be an infinite compact metric space with finite covering dimension. Let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Suppose that the range of K_0-groups of both crossed products are dense in the space of real affine continuous functions on the tracial state space. We show that α and β are approximately conjugate uniformly in measure if and only if they have affine homeomorphic invariant probability measure spaces.

1. Introduction

Let X be a compact metric space and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. If X has infinitely many points, then the associated crossed product C^*-algebras $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta \mathbb{Z}$ are unital separable simple C^*-algebras. It was proved by J. Tomiyama ([24]) that α and β are flip conjugate if there is a *-isomorphism ϕ from $C(X) \rtimes_\alpha \mathbb{Z}$ onto $C(X) \rtimes_\beta \mathbb{Z}$ which maps $C(X)$ onto $C(X)$. On the other hand, T. Giordano, I. Putnam and C. Skau ([6]) showed, among other things, that two minimal Cantor systems are topological orbit equivalent if and only if the tracial range $\rho(K_0(C(X) \rtimes_\alpha \mathbb{Z}))$ of $K_0(C(X) \rtimes_\alpha \mathbb{Z})$ is unital order isomorphic to that of $K_0(C(X) \rtimes_\beta \mathbb{Z})$. Both results show the strong connection between C^*-algebra theory and minimal dynamical systems. In this paper, we will also use C^*-algebra theory to study some particular relation among minimal dynamical systems. Fix a compact metric space X. Let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Denote by T_α and T_β the compact convex sets of α-invariant probability Borel measures and β-invariant probability Borel measures, respectively. Suppose that there is an affine homeomorphism r from T_α onto T_β. What can one say about (X, α) and (X, β)?

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Let $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $A_\beta = C(X) \rtimes_\beta \mathbb{Z}$. Suppose that X has finite covering dimension, Under the assumption that $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$ and $\rho(K_0((A_\beta)))$ is dense in $Aff(T(A_\beta))$ (see 2.1 (4), 2.5 and 2.1 below), we prove that if T_α and T_β are affine homeomorphic, then α and β are approximately conjugate uniformly in measure (see Theorem 5.6 below). By [16], the condition that $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$ is equivalent to A_α being real rank zero and also equivalent to A_α having tracial rank zero.

Some explanations of the result are in order. First we make a few comments on the assumption. When X has finite covering dimension and α is minimal, the dynamical system (X, α) has mean dimension zero (see 3.1 below). When X is the Cantor set, it is known that $\rho(K_0(A_\alpha))$ is always dense in $Aff(T(A_\alpha)))$. When X is a connected and (X, α) is unique ergodic, if the rotation number (defined by Exel in [5]) associated with α contains irrational values, then $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha)))$. In fact, the converse also holds, i.e., if $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha)))$, then the rotation number associated with α contains an irrational value when X is a connected finite CW complex (see [16]). We also note that, when $X = S^1$, α is minimal if and only if the rotation number is irrational.

Next, one should realize that the condition that there is an affine homeomorphism from T_α and T_β is a rather weak one. If both T_α and T_β have only finitely many extremal points, this condition simply says that T_α and T_β have the same number of extremal points. Therefore, one should not expect that a great deal of dynamical information can be recovered nor should one regard uniform approximate conjugacy in measure as a strong relation. To the contrary, we would like to emphasize that two minimal homeomorphisms could be easily approximately conjugate uniformly in measure. In particular, if both α and β are uniquely ergodic, then they are always approximately conjugate uniformly in measure. Given an affine homeomorphism $r : T_\alpha \to T_\beta$, Theorem 5.6 says that r can always be induced by a sequence of Borel equivalences $\{\gamma_n\}$ of X for which $\gamma_n^{-1}\alpha\gamma_n$ converges to β and $\gamma_n\beta\gamma_n^{-1}$ converges to α in measure uniformly (not just for each $\mu \in T_\alpha$ and $\nu \in T_\beta$). Moreover, some additional properties for $\{\gamma_n\}$ can also be required. It is the existence of those γ_n that we find interesting.

Roughly speaking, two minimal homeomorphisms α and β are approximately conjugate uniformly in measure if there exists a sequence of Borel isomorphisms γ_n: $X \to X$ such that $\gamma_n^{-1}\alpha\gamma_n$ converges to β and $\gamma_n\beta\gamma_n^{-1}$ converges to α in measure uniformly on the set of β-invariant measures and the set of α-invariant measures, respectively. We also require that $\{\gamma_n\}$ eventually preserves measures in a suitable sense. Moreover, $\{\gamma_n\}$ and $\{\gamma_n^{-1}\}$ should be continuous on some (eventually dense) open subsets of X. The precise definition is given in 5.2.

The paper is organized as follows. Section 2 lists a number of notations and facts used in this paper. Section 3 gives a version of the uniform Rohlin property for dynamical systems with mean dimension zero. Section 4 contains a number of technical lemmas which will be used in the proof of the main result of the paper. Section 5 discusses the notion of uniform approximate conjugacy in measure and presents the proof of the main result (Theorem 5.6). Finally, Section 6 gives a few concluding remarks.

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2. Preliminaries

2.1. (1) If k is a positive integer, M_k is the full matrix algebra over $\mathbb{C}$. Denote by Tr the standard trace on M_k and by tr the normalized trace on M_k.

(2) Let A be a C^*-algebra. Denote by $T(A)$ the tracial state space of A. If $\tau \in T(A)$, we will also use τ for $\tau \otimes Tr$ on $M_k(A)$, $i = 1, 2, \dots$

(3) Let $Aff(T(A))$ be the space of all real affine continuous functions on $T(A)$. Let $a \in A_{s.a.}$. Denote by $\hat{a}$ the real affine continuous function defined by $\hat{a}(\tau) = \tau(a)$ for $\tau \in T(A)$.

(4) Denote by $\rho_A : K_0(A) \to Aff(T(A))$ the order homomorphism induced by $\hat{p}$ for projections $p \in M_k(A)$, $k = 1, 2, \dots$ We often use ρ if the C^*-algebra A is understood.

2.2. (5) Let X be a compact metric space. We say X has finite dimension if X has finite covering dimension.

(6) Let A be a unital C^*-algebra, let X be a compact metric space and let $h : C(X) \to A$ be a contractive positive linear map. Suppose that t is a positive linear functional of A. Then $t \circ h$ gives a positive linear functional of $C(X)$. We will use $\mu_{t\circ h}$ for the positive Borel measure on X induced by the positive linear functional $t \circ h$.

2.3. (7) Let X be a compact metric space and $\alpha : X \to X$ be a homeomorphism. Recall that α is minimal if α has no proper α-invariant closed subset, or, equivalently, for each $x \in X$, $\{\alpha^n(x) : n = 0, 1, 2, \dots\}$ is dense in X.

(8) Let X be a compact metric space and $x \in X$. The point x is said to be a *condensed point* if every open neighborhood of x contains uncountably many points of X.

(9) If X has infinitely many points and α is minimal, then the cross product $C(X) \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra. We will use A_α for $C(X) \rtimes_\alpha \mathbb{Z}$.

In this case, X has no isolated points and every point of X is condensed.

(10) Denote by $j_\alpha : C(X) \to A_\alpha$ the usual embedding. Denote by u_α the implementing unitary in A_α such that

$$u_\alpha^* j_\alpha(f) u_\alpha = j_\alpha(f \circ \alpha) \text{ for all } f \in C(X).$$

(11) Denote by T_α the space of all α-invariant probability Borel measures on X. If $\mu \in T_\alpha$, then it induces a tracial state τ_μ so that

$$\tau_\mu(j_\alpha(f)) = \int f d\mu$$

for all $f \in C(X)$. On the other hand, if $\tau \in T(A_\alpha)$, then $\mu_{\tau \circ j_\alpha}$ gives a measure in T_α. This measure will be denoted by μ_τ.

In fact, there is an affine homeomorphism between convex sets T_α and $T(A_\alpha)$ (see [2, VIII, 3.8], for example). The reader may notice that we do not always attempt to distinguish the convex sets T_α from $T(A_\alpha)$.

2.4. (12) Let A and B be two C^*-algebras. By a homomorphism $h : A \to B$, we mean a $*$-homomorphism from the C^*-algebra A to B. Suppose that both A and B are unital, exact and stably finite. We say that $r : Aff(T(A)) \to AffT(B))$ is a unital order homomorphism if r is an order homomorphism and $r(\hat{1_A}) = \hat{1_B}$. The homomorphism r is said to be an order isomorphism if r is a bijection and r^{-1} is an also order homomorphism.

Suppose that an affine continuous map $r : Aff(T(A)) \to Aff(T(A))$ is a unital order isomorphism. Denote by $r_\natural : T(B) \to T(A)$ the affine continuous map induced by $r_\natural(\tau)(a) = r(\hat{a})(\tau)$ for all $a \in A_{a.s}$ and $\tau \in T(B)$. If r is a unital order isomorphism, then $r_\natural$ is an affine homeomorphism.

On the other hand, if $\lambda : T(A_\beta) \to T(A_\alpha)$ is an affine homeomorphism, then one obtains a unital order isomorphism $\lambda^\sharp : Aff(T(A_\alpha)) \to Aff(T(A_\beta))$ by $\lambda^\sharp(a)(\tau) = a(\lambda(\tau))$ for all $a \in Aff(T(A_\alpha))$ and $\tau \in T(A_\alpha)$.

(13) If $\phi : A \to B$ is a homomorphism we will use $\phi_* : K_*(A) \to K_*(B)$ for the induced map on K-theory.

(14) Let A and B be two C^*-algebras and $\phi : A \to B$ be a contractive completely positive linear map. Suppose that $\mathcal{G}$ is a subset of A and $\delta > 0$. We say ϕ is $\mathcal{G}$-δ-multiplicative if

$$\|\phi(ab) - \phi(a)\phi(b)\| < \delta \text{ for all } a, b \in \mathcal{G}.$$

(15) Let $\phi : C(X) \to A$ be a homomorphism. We say that ϕ has finite dimensional range if the image of ϕ is contained in a finite dimensional C^*-subalgebra of A. If ϕ has finite dimensional range, then there are finitely many points $\{x_1, x_2, \ldots, x_m\} \subset X$ and pairwise orthogonal projections $p_1, p_2, \ldots, p_m$ in A such that

$$\phi(f) = \sum_{i=1}^m f(x_i)p_i \text{ for all } f \in C(X).$$

(16) Let A be a unital simple C^*-algebra. We write $TR(A) = 0$ if A has tracial rank zero. For the definition of tracial rank zero, we refer to [9] or 3.6.2 of [11]. A unital simple C^*-algebra with tracial rank zero has real rank zero, stable rank one and weakly unperforated $K_0(A)$ (see [9]).

2.5. (17) Let T be a convex set. Denote by $\partial_e(T)$ the set of extremal points of T.

(18) Let X be a compact metric space with infinitely many points and $\alpha : X \to X$ be a minimal homeomorphism. A Borel set $Y \subset X$ is said to be *universally null* if $\mu(Y) = 0$ for all $\mu \in T_\alpha$.

(19) Let A_α be the simple crossed product. A crucial assumption that we make in this paper is that $\rho(K_0(A_\alpha))$ (see (4) above) is dense in $Aff(T(A_\alpha))$.

We will use the following theorem ([16]).

THEOREM 2.1. *Let X be a finite dimensional compact metric space with infinitely many points and $\alpha : X \to X$ be a minimal homeomorphism. Then A_α has tracial rank zero if and only if $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$.*

Minimal dynamical systems whose crossed product C^*-algebras satisfy the above condition have been given and discussed in [16]. It should be mentioned that if (X, α) is a minimal Cantor system, then the condition in 2.1 is always satisfied. In the case when X is connected finite CW complex and (X, α) is uniquely ergodic, the condition in 2.1 is satisfied if and only if the rotation number associated with α has irrational values.

3. Uniform Rohlin Tower Lemma and mean dimension zero

DEFINITION 3.1. Let X be a compact metric space and let $\alpha : X \to X$ be a homeomorphism. We say that (X, α) has the *small-boundary property* if for every point $x \in X$ and every open neighborhood of x there exists an open neighborhood $V \subset U$ such that $\mu(\overline{V} \setminus V) = 0$ for all $\mu \in T_\alpha$.

By a result of Lindenstrauss and Weiss (see [19, §5]), if (X, α) has the small boundary property, then (X, α) has mean dimension zero (see [19] for the definition of mean dimension zero). The converse is also true, for example, if (X, α) is minimal (see Theorem 6.2 of [18]).

It is also shown in [19] that *if X has finite covering dimension, then any minimal system (X, α) has mean dimension zero.*

The following is an easy lemma.

LEMMA 3.2. *Let X be a compact metric space with infinitely many points and let $\alpha : X \to X$ be a homeomorphism. Suppose that $\partial_e(T_\alpha)$ is countable. Then (X, α) has the small boundary property. Consequently (X, α) has mean dimension zero.*

More precisely, given any point $x \in X$ and $\delta > 0$, there is an open ball of X with center at x and radius $\delta/2 < r < \delta$ such that

$$\mu(\{y \in X : \mathrm{dist}(x, y) = r\}) = 0$$

for all $\mu \in T_\alpha$.

Proof. Let $\partial_e(T_\alpha) = \{\mu_1, \mu_2, \dots, \mu_n, \dots\}$. Given a point $x \in X$ and $\delta/2 < r < \delta$ define

$$R = \{y \in X : \delta/2 < \mathrm{dist}(y, x) < \delta\} \text{ and}$$
$$C_r = \{y \in X : \mathrm{dist}(y, x) = r\}.$$

Since

$$\mu(R) = \mu\left(\bigcup_{\delta/2 < r < \delta} C_r\right)$$

and $\mu(R) \le 1$ for all $\mu \in T_\alpha$, there are uncountably many $r \in (\delta/2, \delta)$ such that

$$\mu_n(C_r) = 0, \;\; n = 1, 2, \dots.$$

Let r be one of them. It follows that

$$\mu(C_r) = 0$$

for all $\mu \in T_\alpha$. □

The Rohlin Tower Lemma is well known in ergodic theory. The following two lemmas are some uniform versions of it, which will be used later.

LEMMA 3.3. *Let X be a compact metric space with infinitely many points, let $\alpha : X \to X$ be a minimal homeomorphism, and let T_α be the set of α invariant probability measures. Suppose that (X, α) has mean dimension zero. Then, for any integer $n \ge 1$, there exist finitely many open subsets $G_1, G_2, \dots, G_m \subset X$ such that*

(i) *$\alpha^j(G_i)$ are mutually disjoint for $0 \le j \le h(i) - 1$, $0 \le i \le m$,*
(ii) *$h(i) \ge n$ for each i,*
(iii) *$\mu(X \setminus \bigcup_{i=1}^m \bigcup_{j=0}^{h(i)-1} \alpha^j(G_i)) = 0$ for all $\mu \in T_\alpha$.*

Proof. We start with a non-empty open subset $\Omega \subset X$ such that the $\alpha^j(\overline{\Omega})$ are pairwise disjoint for $0 \le j \le n-1$. This is possible since α is minimal. By 3.2 and 3.1, we may assume that $\mu(\partial(\Omega)) = 0$ for all $\mu \in T_\alpha$.

Let $Y = \overline{\Omega}$. For each $y \in Y$, define

$$r(y) = \min\{m > 1 : \alpha^m(y) \in Y\}.$$

It follows from Theorem 2.3 of [16] (see also p. 299 of [17]) that $\sup_{y \in Y} r(y) < \infty$. Let $n(0) < n(1) < \cdots < n(l)$ be distinct values in the range of r, and for each $0 \le k \le l$, set

$$Y_k = \overline{\{y \in Y : r(y) = n(k)\}} \text{ and } Y_k^o = \mathrm{int}\{y \in Y : r(y) = n(k)\}.$$

Set

$$X_k = \{y \in Y : r(y) \le n(k)\}.$$

Since Y is closed, so is X_k. Moreover, $Y_0 = X_0$. Then

$$Y_0 = X_0, Y_1 = \overline{X_1 \setminus X_0}, \dots, Y_l = \overline{X_l \setminus X_{l-1}}.$$

Note that $n(0) \geq n$.

Set $\Omega_0 = \mathrm{int}(Y)$. Note that $\Omega \subset \Omega_0$. Therefore $\overline{\Omega_0} = Y$. Put

$$S_1 = \alpha^{n(0)}(\Omega_0) \cap \Omega_0.$$

Then S_1 is open and

$$\begin{aligned} (3.1) \qquad & (\alpha^{n(0)}(Y) \cap Y) \setminus S_1 \\ & = \Big[(\alpha^{n(0)}(Y) \cap Y) \setminus \alpha^{n(0)}(\Omega_0)\Big] \bigcup \Big[(\alpha^{n(0)}(Y) \cap Y) \setminus \Omega_0\Big] \\ & \subset \alpha^{n(0)}(\partial(\Omega_0)) \bigcup \partial(\Omega_0). \end{aligned}$$

It follows that

$$\mu((\alpha^{n(0)}(Y) \cap Y) \setminus S_1) = 0$$

for all $\mu \in T_\alpha$. Note that $\alpha^{-n(0)}(\alpha^{n(0)}(Y) \cap Y)) = Y_0$. By the continuity of α, we also have

$$\alpha^{-n(0)}(S_1) = Y_0^o.$$

It follows that

$$\mu(X_0 \setminus \mathrm{int}\, X_0) = \mu(Y_0 \setminus Y_0^o)) = 0 \tag{3.2}$$

for all $\mu \in T_\alpha$. For $k > 0$, let

$$S_k = \alpha^{n(k)}(\Omega_0) \cap \Omega_0.$$

Then S_k is open and, as above,

$$\mu((\alpha^{n(k)}(Y) \cap Y) \setminus S_k)) = 0 \tag{3.3}$$

for all $\mu \in T_\alpha$ and $1 \leq k \leq l$. We have

(3.4)

$$\alpha^{-n(k)}(\alpha^{n(k)}(Y) \cap Y) \setminus X_{k-1} = X_k \setminus X_{k-1} \text{ and } \alpha^{-n(k)}(S_k) \setminus X_{k-1} = Y_k^o.$$

Moreover, for $k > 0$, by (3.4),

(3.5)

$$\begin{aligned} X_k \setminus \mathrm{int}(X_k) & \subset [(X_k \setminus X_{k-1}) \setminus Y_k^o] \bigcup (X_{k-1} \setminus \mathrm{int}(X_{k-1})) \\ & \subset \Big(\alpha^{-n(k)}(\alpha^{n(k)}(Y) \cap Y) \setminus X_{k-1}\Big) \setminus \Big(\alpha^{-n(k)}(S_k) \setminus X_{k-1}\Big) \\ & \qquad \bigcup (X_{k-1} \setminus \mathrm{int}(X_{k-1})) \\ & \subset \Big(\alpha^{-n(k)}(\alpha^{n(k)}(Y) \cap Y) \setminus \alpha^{-n(k)}(S_k)\Big) \\ & \qquad \bigcup (X_{k-1} \setminus \mathrm{int}(X_{k-1})). \end{aligned}$$

By induction on k, combing the above with (3.2) and with (3.3), we conclude that

$$\mu(X_k \setminus \mathrm{int}(X_k)) = 0 \tag{3.6}$$

for all $\mu \in T_\alpha$, $1 \le k \le l$.

We also have

$$\begin{aligned} Y_k \setminus Y_k^o &\subset \overline{X_k \setminus X_{k-1}} \setminus (\alpha^{-n(k)}(S_k) \setminus X_{k-1}) \\ &\subset \Big(\alpha^{-n(k)}(\alpha^{n(k)}(Y) \cap Y) \setminus \mathrm{int}(X_{k-1})\Big) \\ &\qquad \setminus \Big(\alpha^{-n(k)}(S_k) \setminus X_{k-1}\Big) \\ &\subset \Big(\alpha^{-n(k)}(\alpha^{n(k)}(Y) \cap Y) \setminus \alpha^{-n(k)}(S_k)\Big) \\ &\qquad \bigcup (X_{k-1} \setminus \mathrm{int}(X_{k-1})). \end{aligned} \tag{3.7}$$

From this, by (3.3) and (3.6), we have

$$\mu(Y_k \setminus Y_k^o) = 0 \ \text{ for all } \mu \in T_\alpha. \tag{3.8}$$

It follows from Theorem 2.3 of [16] (see also p. 299 of [17]) that

(i) $\alpha^j(Y_k^o)$ are pairwise disjoint for $1 \le j \le n(k)$, $0 \le k \le l$;

(ii) $\bigcup_{k=0}^{l} \bigcup_{j=0}^{n(k)} \alpha^j(Y_k) = X$.

Moreover,

$$\mu\left(X \setminus \bigcup_{k=0}^{l} \bigcup_{j=0}^{n(k)} \alpha^j(Y_k^o)\right) \le \sum_{k=0}^{l} \sum_{j=0}^{n(k)} \mu(\alpha^j(Y_k \setminus Y_k^o)) = 0$$

for all $\mu \in T_\alpha$. Define $G_k = \alpha(Y_k^o)$, $k = 0, 1, \ldots, l$. With $m = l + 1$ and $h(k) = n(k) + 1$, we see that the lemma follows. $\square$

LEMMA 3.4. *Let X be a compact metric space with infinitely many points, let $\alpha : X \to X$ be a minimal homeomorphism and let T_α be the set of α-invariant probability measures. Suppose that (X, α) has mean dimension zero. Let $\{y_1, y_2, \ldots, y_k\}$ be an $\eta/3$-dense subset of X for some $\eta > 0$.*

Then, for any integer $n \ge 1$, there exists an open subset $G \subset X$ containing a subset $\{x_1, x_2, \ldots, x_k\}$ which is η-dense in X with $\mathrm{dist}(x_i, y_i) < \eta/3$ *($1 \le i \le k$) such that $\alpha^i(G)$ are mutually disjoint for $0 \le i \le n-1$ and $\mu(\bigcup_{i=0}^{n-1} \alpha^i(G)) > 1 - \varepsilon$ for all $\mu \in T_\alpha$.*

Moreover,

$$\mu(\partial(G)) = 0$$

for all $\mu \in T_\alpha$.

Proof. Choose an integer $K > 0$ such that $1/K < \varepsilon$. Let $N = nK$. By 3.3, we obtain finitely many open subsets $G_1, G_2, \ldots, G_m$ such that

(i) $\alpha^j(G_i)$ are pairwise disjoint for $1 \le i \le m$, $0 \le j \le h(i)$;
(ii) $h(i) \ge N$, $1 \le i \le m$;
(iii) $\mu(X \setminus \bigcup_{i=1}^m \bigcup_{j=0}^{h(i)-1} \alpha^j(G_i)) = 0$ for all $\mu \in T_\alpha$.

Write $h(i) = L(i)n + r(i)$, where $L(i) \ge 1$ and $n > r(i) \ge 0$ are integers, $i = 1, 2, \ldots, m$. Define, for each i,

$$U(i,1) = \alpha^n(G_i),\ U(i,2) = \alpha^{2n}(G_i), \ldots,\ U(i, L(i)-1) = \alpha^{(L(i)-1)n}(G_i).$$

Note that

$$\mu(G_i) \le \frac{1}{nK}\mu\left(\bigcup_{j=0}^{h(i)-1} \alpha^j(G_i)\right),\ \ 1 \le i \le m, \tag{3.9}$$

for all $\mu \in T_\alpha$.

So

$$\mu\left(\bigcup_{j=L(i)}^{h(i)-1} \alpha^j(G_i)\right) = r(i)\mu(G_i) \le \frac{1}{K}\mu\left(\bigcup_{j=0}^{h(i)-1} \alpha^j(G_i)\right) \tag{3.10}$$

for all $\mu \in T_\alpha$ and $1 \le i \le m$.

Let $G = \bigcup_{i=1}^m G_i \bigcup (\bigcup_{i=1}^m \bigcup_{s=1}^{L(i)-1} U(i,s))$. Then

(1) $\alpha^j(G)$ are pairwise disjoint for $0 \le j \le n-1$,

and, by (iii) and by (3.10),

(2) $\mu(\bigcup_{j=0}^{n-1} \alpha^j(G)) > 1 - \sum_{i=1}^m \mu(\bigcup_{j=L(i)}^{h(i)-1} \alpha^j(G_i)) > 1 - \frac{1}{K} > 1 - \varepsilon$ for all $\mu \in T_\alpha$.

Now let $\{y_1, y_2, \ldots, y_k\}$ be an $\eta/3$-dense set. Define $y_i' = \alpha^{-1}(y_i)$, $i = 1, 2, \ldots, k$. Choose $\delta > 0$ such that

$$\mathrm{dist}(\alpha(x), \alpha(y)) < \eta/9$$

whenever $\mathrm{dist}(x,y) < \delta$.

Choose $z_1 = y_1'$. Since y_2' is a condensed point, there is $z_2 \in O(y_2')$, where $O(y_2') = \{x \in X : \mathrm{dist}(y_2, x) < \delta\}$, such that $z_2 \notin \{\alpha^n(x_1) : n \in \mathbb{Z}\}$. We then choose $z_3 \notin \{\alpha^n(x_1), \alpha^n(x_2) : n \in \mathbb{Z}\}$ such that $\mathrm{dist}(z_3, y_2) < \delta$. By induction, we obtain $\{z_1, z_2, \ldots, z_k\} \subset X$ such that none of z_i lies in the orbit of z_j if $i \ne j$. We note that $\{\alpha(z_1), \alpha(z_2), \ldots, \alpha(z_k)\}$ is $4\eta/9$-dense in X. So we may start with an open subset Ω which contains $\{z_1, z_2, \ldots, z_k\}$ at the beginning of the proof of 3.3.

Note that, by the proof of 3.3, $G_k = \alpha(Y_k^o)$, $k = 0, 1, \ldots, l$. In the proof of 3.3,

$$\bigcup_{k=0}^{l} Y_k \supset Y = \overline{\Omega}.$$

It follows that

$$\alpha(Y) \setminus \bigcup_{k=0}^{l} G_k \subset \bigcup_{k=0}^{l} \alpha(Y_k \setminus Y_k^o).$$

Since

$$\mu\left(Y_k \setminus Y_k^o\right) = 0$$

for all $\mu \in T_\alpha$, and since α is minimal, for each i,

$$U(\alpha(z_i)) \cap \bigcup_{k=1}^{m} G_k \neq \emptyset,$$

where $U(\alpha(z_i)) = \{x \in X : \operatorname{dist}(\alpha(z_i), x) < \eta/9\}$. Choose a point $x_i \in U(\alpha(z_i)) \cap \bigcup_{k=1}^{l} G_k$, $1 \le i \le k$. Then the above proof shows that

$$x_i \in G, \;\; i = 1, 2, \ldots, k.$$

Note that $\operatorname{dist}(x_i, y_i) < \eta/3$ $i = 1, 2, \ldots, k$ and $\{x_1, x_2, \ldots, x_k\}$ is η-dense in X. $\square$

Let X be a compact metric space and let A be a unital C^*-algebra. Suppose that $\phi : C(X) \to A$ is a homomorphism. Then ϕ can be extended to a homomorphism from $\mathcal{B}(X)$, the algebra of all bounded Borel functions, to the enveloping von-Neumann algebra A^{**} (see 4.5.11 of [22]).

LEMMA 3.5. *Let X be a compact metric space and $\phi : C(X) \to A$ be a unital monomorphism from $C(X)$ into a unital simple C^*-algebra A. Suppose that G is an open subset of X such that*

$$\mu_\tau\left(\overline{G} \setminus G\right) = 0$$

for all $\tau \in T(A)$, where μ_τ is the measure induced by $\tau \circ \phi$.

Then $\phi(\chi_G)$ (in A^{**}) is continuous function on $T(A)$, or equivalently, for any $\varepsilon > 0$, there exists $f \in C(X)$, with $0 \le f(t) \le 1$ for all $t \in X$ and $f(t) = 0$ if $t \in X \setminus G$ such that

$$|\tau(\phi(f)) - \mu_\tau(G)| < \varepsilon$$

for all $\tau \in T(A)$.

Proof. Let h be a continuous function on X defined by

$$h(x) = \frac{1}{1 + \operatorname{dist}(x, \bar{G} \setminus G)} \text{ for all } x \in X.$$

Note that $0 \le h(x) \le 1$. Let $g_n(x) = h(x)^n$ for $x \in X$. Then $g_n \in C(X)$. The condition that $\mu_\tau(\bar{G} \setminus G) = 0$ and the fact that $0 \le g_n \le 1$ imply that $\widehat{\phi(g_n)}(\tau) = \int_X g_n d\mu_\tau$ converges to zero pointwise on $T(A)$. Hence, by the Dini Theorem, $\widehat{\phi(g_n)}$ converges uniformly to zero on $T(A)$. Put $f(x) = \chi_{\bar{G}}(x) - g_n(x)$ for $x \in G$ and $f(x) = 0$ for $x \in X \setminus G$. It is easy to check that

$f \in C(X)$. Moreover, $0 \le f \le 1$. One sees, with sufficiently large n, that f meets the requirements of the lemma. $\square$

The author would like to thank the referee for the suggestion of this simple proof which replaces the original longer proof.

4. Perturbations

The following lemma is well-known (note that finite dimensional C^*-algebras are semiprojective (see 0.4 of [20]) and their unit balls are compact).

LEMMA 4.1. *Let F be a finite dimensional C^*-algebra. Then for any $\varepsilon > 0$ there exist a finite subset $\mathcal{G} \subset F$ and $\delta > 0$ satisfying the following: For any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\phi : F \to A$, where A is any C^*-algebra, there exists a homomorphism $h : F \to A$ such that*

$$\|h - \phi\| < \varepsilon.$$

LEMMA 4.2 (Lemma 4.1 of [12]). *Let A be a unital C^*-algebra. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following:*

If B is another unital C^*-algebra, $\phi : A \to B$ is a unital contractive completely positive linear map which is $\mathcal{G}$-δ- multiplicative and $\tau \in T(B)$, then there exists a tracial state $t \in T(A)$ such that

$$|\tau \circ \phi(a) - t(a)| < \varepsilon$$

for all $a \in \mathcal{F}$.

LEMMA 4.3. *Let X be a compact metric space with infinitely many points and let $\alpha : X \to X$ be a minimal homeomorphism. Let $G_1, G_2, \dots, G_L$ be finitely many open subsets with the property that $\mu(\overline{G_i} \setminus G_i) = 0$ for all $\mu \in T_\alpha$.*

For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset $\mathcal{G}_1 \subset C(X)$ and $\eta > 0$ satisfying the following:

If there exists a projection $p \in A_\alpha$ and a unital homomorphism $\phi_0 : C(X) \to pA_\alpha p$ with finite dimensional range such that

(1) $\|pj_\alpha(f) - j_\alpha(f)p\| < \eta$ *for all* $f \in \mathcal{G}_1$,
(2) $\|pj_\alpha(f)p - \phi_0(f)\| < \eta$ *for all* $f \in \mathcal{G}_1$,
(3) $\tau(1 - p) < \eta$ *for all* $\tau \in T(A_\alpha)$,

and if $\phi : A_\alpha \to M_k$ is a unital $\mathcal{G}_2$-δ-multiplicative contractive completely positive linear map (for some $k > 0$) , where $\mathcal{G}_2$ is a finite subset of A_α and $\delta > 0$, both of which depend on the image of ϕ_0, $\mathcal{G}_1$, η, ε, as well as $G_1, G_2, \dots, G_L$, then there is $\tau \in T(A_\alpha)$, such that

$$|\operatorname{tr} \circ \phi \circ j_\alpha(g) - \tau(g)| < \varepsilon/2 \text{ and } |\operatorname{tr} \circ \phi \circ \phi_0(g) - \tau(g)| < \varepsilon$$

for all $g \in \mathcal{F}$, there are $\{y_1, y_2, \ldots, y_K\} \subset X$ and mutually orthogonal rank one projections in M_k such that

$$\left\| \sum_{i=1}^{K} f(y_i)p_i - \phi \circ \phi_0 \circ (f) \right\| < \varepsilon$$

for all $f \in \mathcal{F}$ and

$$\mu_\tau(G_j) + \varepsilon > \frac{N_j}{k} > \mu_\tau(G_j) - \varepsilon,$$

where N_j is the number of y_is in G_j. Moreover, $\frac{k-K}{k} < \varepsilon$.

Proof. To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C(X)$.

Let

$$\gamma_0 = \inf \{\mu_\tau(G_j) : \mu \in T(A), j = 1, 2, \ldots, L\}.$$

Since A_α is simple, one has $\gamma_0 > 0$. By Lemma 3.5, choose $g_j \in C(X)$ with $0 \le g_j \le 1$, $g_j(x) = 0$ if $x \notin G_j$, and

$$\mu_\tau(G_j) < \tau(j_\alpha(g_j)) - \min(\gamma_0/4, \varepsilon/8) \tag{4.1}$$

for all $\tau \in T(A)$ and $j = 1, 2, \ldots, L$.

Let $\mathcal{F}_1 = \mathcal{F} \cup \{g_j : 1 \le j \le L\}$. Let $\eta_1 > 0$ be such that

$$|f(x) - f(x')| < \varepsilon/4$$

if $\mathrm{dist}(x, x') < \eta_1$ for all $f \in \mathcal{F}_1$. Let $\eta = \min\{\gamma_0/32, \varepsilon/64, \eta_1/32\}$. Let $\mathcal{G}_1 = \mathcal{F}_1$. Suppose that $p \in A_\alpha$ and $\phi_0 : C(X) \to pA_\alpha p$ is a homomorphism with finite dimensional range which satisfies (1)–(3) as described in the statement (for the above $\mathcal{G}_1$ and η).

Put $\mathcal{F}_2 = j_\alpha(\mathcal{F}_1) \cup \phi_0(\mathcal{F}_1) \cup \{p, 1-p\} \cup \{pj_\alpha(f)p : f \in \mathcal{F}_1\}$.

Let $\mathcal{G} \subset A_\alpha$ be a finite subset and $\delta > 0$ be a positive number given by Lemma 4.2 corresponding to $\mathcal{F}_2$ and η. Let C be the image of ϕ_0, which is a finite dimensional C^*-algebra. Choose a smaller δ required by 4.1 and a larger $\mathcal{G}$ which contains a finite subset required by 4.1 for C and η.

Let $\mathcal{G}_2 = \mathcal{G} \cup \mathcal{F}_2$. Now let $\phi : A_\alpha$ be a unital $\mathcal{G}_2$-δ-multiplicative contractive completely positive linear map from $A_\alpha \to M_k$ (for some $k > 0$).

By 4.1 (and the choice of $\mathcal{G}$ and δ), we may also assume that there is a homomorphism, $\phi_{00} : C(X) \to EM_kE$ (for some projection E), such that

$$\|\phi_{00}(f) - \phi \circ \phi_0(f)\| < \eta$$

for all $f \in \mathcal{F}_1$.

By the choice of $\mathcal{G}$ and δ, applying 4.2, there is a tracial state $\tau \in T(A)$ such that

$$|\tau(a) - \mathrm{tr} \circ \phi(a)| < \eta$$

for all $f \in \mathcal{F}_2$. In particular,

$$|\tau(1-p) - \mathrm{tr} \circ \phi(1-p)| < \eta.$$

It follows that

$$\operatorname{tr} \circ \phi(1-p) < 2\eta < \varepsilon/4. \tag{4.2}$$

Moreover,

$$|\tau(j_\alpha(f)) - \operatorname{tr} \circ \phi_{00}(f)| < 3\eta$$

for all $f \in \mathcal{F}_1$.

Write $\phi_{00}(f) = \sum_{i=1}^{K} f(y_i)p_i$ for all $f \in C(X)$, where $y_i \in X$ and $\{p_1, p_2, \ldots, p_K\}$ is a set of mutually orthogonal rank one projections in M_k, and $0 < K < k$.

On the other hand,

$$|\operatorname{tr}(\phi_{00}(g_i)) - \tau(j_\alpha(g_i))| < 3\eta \tag{4.3}$$

for $i = 1, 2, \ldots, L$. It follows from (4.1) and (4.3) that

$$\mu_\tau(G_j) + \varepsilon/2 > \frac{N_j}{k} > \mu_\tau(G_j) - \varepsilon/2,$$

where N_j is the number of y_j's which lie in G_j, $j = 1, 2, \ldots, L$.

By (4.2), we compute that

$$\frac{k-K}{k} < \varepsilon/4 < \varepsilon. \qquad \square$$

LEMMA 4.4. *Let X be a finite dimensional compact metric space with infinitely many points and $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha)$. Then, for any $\varepsilon > 0$, $\sigma > 0$ and finite subset $\mathcal{F} \subset C(X)$, there are mutually orthogonal projections $\{p_1, p_2, \ldots, p_m\} \subset A_\alpha$ and $\{x_1, x_2, \ldots, x_m\} \subset X$ such that*

(1) *$\|pj_\alpha(f) - j_\alpha(f)p\| < \varepsilon$ for $f \in \mathcal{F}$, where $p = \sum_{k=1}^{m} p_k$,*
(2) *$\|pj_\alpha(f)p - \sum_{k=1}^{m} f(x_i)p_k\| < \varepsilon$ for all $f \in \mathcal{F}$,*
(3) *$\tau(1-p) < \sigma$ for all $\tau \in T(A_\alpha)$.*

Several versions of Lemma 4.4 are known. By 2.1, A_α has tracial rank zero. Lemma 4.4 then follows from the definition of tracial rank zero and Lemma 6.27 of [11].

LEMMA 4.5. *Let X be a finite dimensional compact metric space with infinitely many points and let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$.*

Let $G_1, G_2, \ldots, G_L$ be finitely many open subsets with the property that $\mu(\overline{G_i} \setminus G_i) = 0$ for all $\mu \in T_\alpha$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there are a (specially selected) projection $p \in A_\alpha$ with $\tau(1-p) < \varepsilon/2$ for all $\tau \in T(A_\alpha)$, and a finite subset $\mathcal{G} \subset A_\alpha$ and $\delta > 0$ satisfying the following property:

If $\phi : A_\alpha \to M_k$ is a unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear map (for some $k > 0$), then there is $\tau \in T(A_\alpha)$ such that

$$|\operatorname{tr} \circ \phi \circ j_\alpha(g) - \tau(g)| < \varepsilon/2 \text{ and } |\operatorname{tr} \circ \phi(pgp) - \tau(g)| < \varepsilon$$

for all $g \in \mathcal{F}$, and there are $\{y_1, y_2, \ldots, y_K\} \subset X$ and mutually orthogonal rank one projections $\{p_1, p_2, \ldots, p_k\}$ in M_k such that

$$\left\| \sum_{i=1}^{K} f(y_i)p_i - \phi \circ (pfp) \right\| < \varepsilon$$

for all $f \in \mathcal{F}$ and

$$\mu(G_j) + \varepsilon > \frac{N_j}{k} > \mu(G_j) - \varepsilon,$$

where N_j is the number of y_i's in G_j and μ is the probability measure induced by τ. Moreover, $\frac{k-K}{k} < \varepsilon$.

Proof. To prove this lemma, we combine 4.3 and 4.4. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\mathcal{G}_1 \subset C(X)$ be a finite subset and $\eta > 0$ given by 4.3. By applying 4.4, we obtain a projection $p \in A_\alpha$ and a unital homomorphism $\phi_0 : C(X) \to pAp$ with finite dimensional range which satisfies (1)–(3) in 4.3. We then apply 4.3 to obtain this lemma. □

LEMMA 4.6. *Let A be a unital simple C^*-algebra with the following property: Any two projections p and q in A with $\tau(p) = \tau(q)$ for all $\tau \in (A)$ are equivalent.*

Let X be a compact metric space and $h_1, h_2 : C(X) \to A$ be two unital monomorphisms. Suppose that

$$\tau \circ h_1(f) = \tau \circ h_2(f) \tag{4.4}$$

for all $\tau \in T(A)$ and all $f \in C(X)$. Suppose also that, for any $r > 0$, there are finitely many pairwise disjoint open subsets $U_1, U_2, \ldots, U_m$ whose diameters are less than r such that $X = \bigcup_{i=1}^m \overline{U_i}$ and

$$\mu_{\tau \circ h_1}\left(\bigcup_{i=1}^{m} (\overline{U_i} \setminus U_i) \right) = 0$$

for all $\tau \in T(A)$.

Then, for any $\eta > 0$, there exist a finite subset $\mathcal{F}_0 \subset C(X)$, $\mathcal{F} \subset A$ and $\delta > 0$ satisfying the following: for any $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $\phi : A \to B$ and any homomorphism $\psi_1, \psi_2 : C(X) \to B$ for some unital stably finite C^*-algebra B with

$$\|\phi \circ h_i(f) - \psi_i(f)\| < \delta$$

for all $f \in \mathcal{F}_0$, $i = 1, 2$, one has

$$\mu_{t \circ \psi_1}(S) \leq \mu_{t \circ \psi_2}(B_\eta(S)) \text{ and } \mu_{t \circ \psi_2}(S) \leq \mu_{t \circ \psi_1}(B_\eta(S))$$

for any $t \in T(B)$ and any closed subset $S \subset X$, where $B_\eta(S) = \{x \in X : \mathrm{dist}(x, S) < \eta\}$.

Proof. Fix $\eta > 0$. Let $X = \sum_{j=1}^N X_i$, where each X_i is a clopen set which is $\eta/4$-connected, i.e., for any two points $x, y \in X_i$, there are $x_1, x_2, \dots, x_m \in X_i$ such that $\mathrm{dist}(x, x_1) < \eta/4$, $\mathrm{dist}(x_i, x_{i+1}) < \eta/4$ and $\mathrm{dist}(x_m, y) < \eta/4$.

Let $U_1, U_2, \dots, U_m$ be pairwise disjoint non-empty open subsets whose diameters are less than $\eta/8$, such that $X = \bigcup_{i=1}^m \overline{U_i}$ and

$$\mu_{\tau\circ h_1}\left(\bigcup_{i=1}^m (\overline{U_i} \setminus U_i)\right) = 0$$

for all $\tau \in T(A)$.

Let

$$d = \inf\{\mu_{\tau\circ h_1}(U_i) : 1 \le i \le m, \tau \in T(A)\}.$$

Since A is simple, $d > 0$.

Let $e_1 = h_1(\chi_{X_i})$ and $f_i = h_2(\chi_{X_i})$, where χ_{X_i} is the characteristic function on the clopen set X_i, $i = 1, 2, \dots, N$. Then, for any $\tau \in T(A)$,

$$\tau(e_i) = \tau(f_i) \tag{4.5}$$

for all $\tau \in T(A)$. By the assumption on A, there is a partial isometry $u_i \in A$ such that

$$u_i^* u_i = e_i \text{ and } u_i u_i^* = f_i \quad i = 1, 2, \dots, N. \tag{4.6}$$

Let Λ be a subset of $\{1, 2, \dots, m\}$. By 3.5, for each Λ, there exists a $g_\Lambda \in C(X)$ with $0 \le g_\Lambda \le 1$, $g_\Lambda(x) = 1$ if $x \in \bigcup_{i\in\Lambda} U_i$ and $g_i(x) = 0$ if $\mathrm{dist}(x, \bigcup_{i\in\Lambda} U_i) > \eta/128$ such that

$$\tau(h_1(g_\Lambda)) - \frac{d}{8} < \mu_{\tau\circ h_1}\left(\bigcup_{i\in\Lambda} U_i\right) \tag{4.7}$$

for all $\tau \in T(A)$, $i = 1, 2, \dots, m$.

Let $\mathcal{F}_0 = \{g_\Lambda : \Lambda \subset \{1, 2, \dots, m\}\}$, $\mathcal{F} = \{u_i, u_i^* : 1 \le i \le N\} \bigcup_{i=1}^2 h_i(\mathcal{F}_0)$. Let $\mathcal{G}$ be a finite subset and $\delta > 0$ be given by 4.2 for the above A, $\mathcal{F}$ and $d/8$. We may assume that $\delta < d/4$.

Now suppose that $\phi : A \to B$ is a $\mathcal{G}$-$\delta/4$-multiplicative contractive completely positive linear map and $\psi_i : C(X) \to B$ is (for each i) a homomorphism such that

$$\|\psi_i(f) - \phi \circ h_i(f)\| < \delta/4 \tag{4.8}$$

for all $f \in \mathcal{F}$.

Hence

$$\|\psi_1(\chi_{X_i}) - \phi_1(u_i)\phi(u_i)^*\| < \delta \text{ and } \|\psi_2(\chi_{X_i}) - \phi(u_i)^*\phi(u_i)\| < \delta \tag{4.9}$$

for $i = 1, 2, \ldots, N$. With $\delta < d/4 < 1$, it follows (for example, from 2.5.3 of [11]) that $\psi_1(\chi_{X_i})$ is equivalent to $\psi_2(\chi_{X_i})$ in B, $i = 1, 2, \ldots, N$.

In particular,

$$t(\psi_1(\chi_{X_i})) = t(\psi_2(\chi_{X_i})) \tag{4.10}$$

for all $t \in T(B)$, $i = 1, 2, \ldots, N$.

By the choice of $\mathcal{G}$ and δ, applying 4.2, we have, for each $t \in T(B)$, that there is $\tau \in T(A)$ such that

$$|\tau(h_1(g_\Lambda)) - t \circ \psi_j(g_\Lambda)| < d/8 \tag{4.11}$$

for $j = 1, 2$ and $\Lambda \subset \{1, 2, \ldots, m\}$.

For any closed subset $S \subset X$, if S is a union of some of X_i, then, by (4.10),

$$\mu_{t\circ\psi_1}(S) = \mu_{t\circ\psi_2}(S). \tag{4.12}$$

Suppose that S is a closed subset of X which is not a finite union of some X_i's. Then there must be a point $\xi \in B_{5\eta/16}(S) \setminus B_{\eta/4}(S)$. But $\mathrm{dist}(\xi, U_j) = 0$ for some j. Since the diameter of U_j is less than $\eta/8$,

$$U_j \subset B_{7\eta/16}(S) \subset B_{\eta/2}(S). \tag{4.13}$$

It follows from (4.11) that

$$\mu_{t\circ\psi_i}(U_j) > d/2 \tag{4.14}$$

for all $t \in T(B)$, $i = 1, 2$. Since $U_j \cap B_{7\eta/64}(S) = \emptyset$, we have

$$\mu_{t\circ\psi_i}(B_\eta(S)) > d/2 + \mu_{t\circ\psi_i}(B_{7\eta/64}(S)). \tag{4.15}$$

There is a $\Lambda \subset \{1, 2, \ldots, N\}$ such that $\bigcup_{i\in\Lambda} \overline{U_i} \supset S$. Suppose that Λ is smallest such subset of $\{1, 2, \ldots, N\}$. Then

$$\operatorname{supp} g_\Lambda \subset B_{7\eta/64}(S) \text{ and } \mu_{t\circ\psi_i}(B_{7\eta/64}(S)) \geq t(\psi_i(g_\Lambda)) \tag{4.16}$$

for all $t \in T(B)$ and $i = 1, 2$.

By 4.11,

$$|t \circ \psi_1(g_\Lambda) - t \circ \psi_2(g_\Lambda)| < d/8 \tag{4.17}$$

for all $t \in T(B)$. By applying (4.17), (4.16) and (4.15), it follows that

$$\begin{aligned} \mu_{t\circ\psi_1}(S) &\leq t(\psi_1(g_\Lambda)) \leq t(\psi_2(g_\Lambda)) + d/8 \\ &\leq \mu_{t\circ\psi_2}(B_{7\eta/64}(S)) + d/8 \leq \mu_{t\circ\psi_2}(B_\eta) \end{aligned} \tag{4.18}$$

for all $t \in T(B)$. Similarly,

$$\mu_{t\circ\psi_2}(S) \leq \mu_{t\circ\psi_1}(B_\eta) \tag{4.19}$$

for all $t \in T(B)$. $\square$

LEMMA 4.7. *Let X be a finite dimensional compact metric space with infinitely many points and let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$.*

Let $\varepsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $\eta > 0$ be any positive number such that

$$|f(t) - f(t')| < \varepsilon/8$$

if $\mathrm{dist}(t, t') < \eta$ *for all* $f \in \mathcal{F}$.

Let n be an integer so that $1/n < \varepsilon/4$ and let G be an open set such that $\alpha^j(G)$ are pairwise disjoint for $0 \le j \le n-1$ with the following properties:

(i) *G contains x_i, $i = 1, 2, \ldots, l$, where $\{x_1, x_2, \ldots, x_l\}$ is $\eta/2$-dense in X;*
(ii) *$\mu(\bigcup_j \alpha^j(G)) > 1 - \varepsilon/16$ for all $\mu \in T_\alpha$;*
(iii) *$\mu(\partial(G)) = 0$ for all $\mu \in T_\alpha$.*

Then there exist a (specially selected) projection $p \in A_\alpha$ with $\tau(1-p) < \varepsilon/2$ for all $\tau \in T(A_\alpha)$, a finite subset $\mathcal{G} \subset A_\alpha$ and $\delta > 0$ satisfying the following property:

If $\phi : A_\alpha \to M_k$ (with $k > ln$) is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, then there are m distinct points

$$\{y_i, i = 1, 2, \ldots, m\}$$

with $y_i \in G$, $x_i = y_i$, $i = 1, 2, \ldots, l \le m$, and $\frac{k-mn}{k} < \varepsilon/4$ such that

$$\left\| \sum_{j=0}^{n-1} \sum_{i=1}^{m} f(\alpha^j(y_i)) p_{i,j} + \sum_{i=K+1}^{N} f(z_i) p_i - \phi(p j_\alpha(f) p) \right\| < \varepsilon \tag{4.20}$$

($K = mn < N < k$) for all $f \in \mathcal{F}$, where

$$\{p_{i,j} : 1 \le i \le m, 0 \le j \le n-1\} \cup \{p_{K+1}, \ldots, p_N\}$$

is a set of mutually orthogonal rank one projections in M_k and $\{z_{K+1}, \ldots, z_N\} \subset X$.

Proof. Let $\eta_1 > 0$ such that $\eta_1 < \eta$ and

$$\mathrm{dist}(\alpha^j(x), \alpha^j(x')) < \eta/2 \tag{4.21}$$

if $\mathrm{dist}(x, x') < \eta_1$, $-n+1 \le j \le n-1$. Let $\eta_2 > 0$ be such that $\eta_2 < \eta_1$ and

$$\mathrm{dist}(\alpha^j(x), \alpha^j(x')) < \eta_1/2 \tag{4.22}$$

if $\mathrm{dist}(x, x') < \eta_2$, $j = 1, 2, \ldots, n-1$.

Since X has finite covering dimension, (X, α) has mean dimension zero (see 3.1). Let U_i be an open ball with center at x_i and radius $\eta_2/4$ such that $\mu(\overline{U_i} \setminus U_i) = 0$ for all $\mu \in T_\alpha$, $i = 1, 2, \ldots, L$.

Now we apply 4.5 with open subsets $\{U_i : 1 \le i \le L\}$ and $\{\alpha^j(G) : 0 \le j \le n-1\}$. Let $\delta_1 > 0$. By 4.5 for $\frac{\varepsilon}{8(n+1)}$ and $\mathcal{F}$, with sufficiently large $\mathcal{G}$ and sufficiently small δ, we may assume that k is sufficiently large and

$$\left\| \phi \circ (pj_\alpha(f)p) - \sum_{i=1}^{N} f(z_i)p_i \right\| < \min\{\varepsilon/8, \delta_1\}, \tag{4.23}$$

where $p \in A_\alpha$ is a specially selected projection with $\tau(1-p) < \varepsilon/8$ for all $\tau \in T(A_\alpha)$, where $\frac{k-N}{k} < \varepsilon/8$ and where $\{z_1, \dots, z_N\}$ is a set of distinct points of X. By applying 4.5 (with finitely many open U_i's and $\alpha^j(G)$'s in place of G_i), and using (ii) above, we may also assume that there are at least m distinct points $\{y_{i,j} : i = 1, 2, \dots, m\}$ of $\{z_1, z_2, \dots, z_N\}$ in each of $\alpha^j(G)$ (for some $1 \le J \le L$), $j = 0, 1, \dots, n-1$, such that

$$\frac{1}{n} \ge \frac{m}{k} > \frac{1}{n} - \frac{\varepsilon}{4n}. \tag{4.24}$$

Furthermore, we may assume that $m > L$ and $y_{0,i} \in U_i$ $i = 1, 2, \dots, l$. Put $\Psi(f) = \sum_{i=1}^{N} f(z_i)p_i$ for $f \in C(X)$. With sufficiently small δ_1 and sufficiently large $\mathcal{G}$, by 4.6, we may also assume that

$$\mu_{tr \circ \Psi}(S) \le \mu_{tr \circ \Psi \circ (\alpha^{-j})^*}(S_{\eta_2/2}) \text{ and } \mu_{tr \circ \Psi \circ (\alpha^{-j})^*}(S) \le \mu_{tr \circ \Psi}(S_{\eta_2/2}) \tag{4.25}$$

for any closed subset $S \subset X$, where $(\alpha^{-j})^*(f) = f \circ \alpha^{-j}$, $j = 1, 2, \dots, n-1$ and where $S_{\eta_2/2} = \{x \in X : \mathrm{dist}(x, S) < \eta_2/2\}$.

Thus, by the choice of η_2, for any $y_{s(i),j}$, $i = 1, 2, \dots, M$ with $1 \le M \le m$, there exist $\xi_1', \xi_2', \dots, \xi_M' \in \{x \in X : \mathrm{dist}(x, \{y_{1,0}, y_{2,0}, \dots, y_{m,0}\}) < \eta_1/2\}$ such that

$$\mathrm{dist}(y_{s(i),j}, \alpha^j(\xi_i')) < \eta_2/2, \;\; i = 1, 2, \dots, M.$$

Then, by the choice of η_1, there are $\xi_1, \xi_2, \dots, \xi_M \in \{y_{1,0}, y_{2,0}, \dots, y_{m,0}\}$ such that

$$\mathrm{dist}(y_{s(i),j}, \alpha^j(\xi_i)) < \eta/2, \;\; i = 1, 2, \dots, M.$$

Similarly, for any $\xi_1', \xi_2', \dots, \xi_M' \in \{y_{1,0}, y_{2,0}, \dots, y_{m,0}\}$, there exist $y'_{s(i),j}$, $i = 1, 2, \dots, M$, such that

$$\mathrm{dist}(\alpha^j(\xi_i'), y'_{s(i),j}) < \eta/2 \;\; i = 1, 2, \dots, M.$$

It follows from the "marriage lemma" ([7]) (see also 2.1 of [23]) that there is a permutation $\sigma_j : \{1, 2, \dots, m\} \to \{1, 2, \dots, m\}$ such that

$$\mathrm{dist}(y_{i,j}, \alpha^j(y_{\sigma_j(i),0})) < \eta,$$

$j = 1, 2, \dots, n-1$. By the choice of η and by replacing $\varepsilon/8$ by $\varepsilon/4$ in (4.23), we may assume that $y_{i,j} = \alpha^j(y_{i,0})$ and $y_{i,0} = x_i$ for $1 \le i \le l$. Let $y_i = y_{1,i}$, $i = 1, 2, \dots, m$. Put $K = mn$.

Thus, from above, with sufficiently large $\mathcal{G}$ and sufficiently small δ, we may also assume that,

$$\left\| \sum_{i=1}^{N} f(z_i)p_i - \left[\sum_{j=0}^{n-1} \sum_{i=1}^{m} f(\alpha^j(y_i))p_{i,j} + \sum_{i=K+1}^{N} f(z_i)p_i \right] \right\| < \varepsilon/2 \tag{4.26}$$

for all $f \in \mathcal{F}$. Then (4.20) follows from (4.23) and (4.26). Moreover, by (4.24) and (4),

$$\frac{K}{k} = \frac{nm}{k} > n(\frac{1}{n} - \frac{\varepsilon}{4n}) = 1 - \varepsilon/4,$$

as desired. □

PROPOSITION 4.8. *Let A and B be two unital separable C^*-algebras with $TR(A) = TR(B) = 0$. Suppose that $\lambda : Aff(T(A)) \to Aff(T(B))$ is a unital order affine isomorphism. Then there are finite dimensional C^*-algebras F_n, a sequence of unital contractive completely positive linear maps $\phi_n : B \to F_n$, and a sequence of unital contractive completely positive linear maps $\psi_n : A \to F_n$, satisfying the following properties:*

(1) *For all $a, b \in A$,*

$$\lim_{n\to\infty} \|\phi_n(a)\phi_n(b) - \phi_n(ab)\| = 0,$$

and for all $x, y \in B$

$$\lim_{n\to\infty} \|\psi_n(x)\psi_n(y) - \psi_n(xy)\| = 0;$$

(2) *there is an affine continuous map $\Delta_n : T(B) \to T(F_n)$ such that, for each $b \in B$,*

$$|\Delta_n(\tau)(\phi_n(b)) - \tau(b)| \to 0 \tag{4.27}$$

uniformly on $T(B)$;

(3) *for each $a \in A$,*

$$|\lambda(\hat{a})(\tau) - \Delta_n(\tau) \circ \psi_n(a)| \to 0 \tag{4.28}$$

uniformly on $T(B)$.

Proof. Let $\varepsilon > 0$, $\mathcal{F} \subset A$ and $\mathcal{G} \subset B$ be two finite subsets. To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ and $\mathcal{G}$ are in the unit balls of A and B, respectively.

Since $TR(A) = 0$, by [9], for any $\delta > 0$, there exist a projection $p \in A$ and a finite dimensional C^*-subalgebra C of A with $p = 1_C$ such that

(i) $\|pa - ap\| < \delta/8$ for all $a \in \mathcal{F}$,
(ii) $\mathrm{dist}(pap, C) < \delta/8$ for all $a \in \mathcal{F}$,
(iii) $t(1 - q) < \delta/4$ for all $t \in T(A)$.

We choose $\delta < \min\{\varepsilon/4, 1\}$. Moreover, by 2.3.5 of [11], there exists a contractive completely positive linear map $\tilde{\psi}' : pAp \to C$ such that $\tilde{\psi}(c) = c$ if $c \in C$. Define $\tilde{\psi}(a) = \tilde{\psi}'(pap)$ for all $a \in A$.

Write $C = \bigoplus_{i=1}^{k} M_{R(i)}$. Denote by e_i a minimal rank one projection in $M_{R(i)}$, $i = 1, 2, \ldots, k$. Since $TR(B) = 0$, $\rho_B(K_0(B))$ is dense in $Aff(T(B))$. So there exists a projection $p_i \in B$ such that

$$\lambda(\widehat{e_i})(\tau) - \delta/8 < \tau(p_i) < \lambda(\widehat{e_i})(\tau) \tag{4.29}$$

for all $\tau \in T(B)$, $i = 1, 2, \ldots, k$. Note

$$\sum_{i=1}^{k} R(i)[p_i] < [1_B]$$

in $K_0(B)$. Thus (since $TR(B) = 0$) we obtain a C^*-subalgebra $B_0 \subset B$ for which there exists an isomorphism $\psi_1 : C \to B_0$ so that $\psi_1(e_i) = p_{i,1}$, $i = 1, 2, \ldots, k$.

Choose $\mathcal{G}_1$ which contains $\mathcal{G}$ and $\psi_1 \circ \tilde{\psi}(\mathcal{F})$ as well as a set of generators of B_0. For any $\delta_1 > 0$, there is a projection $q \in B$ and a finite dimensional C^*-subalgebra F of B with $q = 1_F$ such that

(1) $\|qb - bq\| < \delta_1/8$ for all $b \in \mathcal{G}_1$;
(2) $\mathrm{dist}(qbq, F) < \delta_1/8$ for all $b \in \mathcal{G}_1$;
(3) $\tau(1 - q) < \delta_1/4$ for all $\tau \in T(B)$.

We may assume that $\delta_1 < \min\{\varepsilon/4, 1\}$. By 2.3.5 of [11], we may assume that there exists a contractive completely positive linear map $\phi' : qBq \to F$ such that $\phi(b) = b$ if $b \in F$. Define $\phi : B \to F$ by $\phi(b) = \phi'(qbq)$ for all $b \in B$. Then ϕ is a $\mathcal{G}_1$-$\delta_1/4$-multiplicative contractive completely positive linear map.

Furthermore, by 4.1, we may assume that there exists a homomorphism $h : B_0 \to F$ so that

$$\|h - \phi|_{B_0}\| < \varepsilon/8.$$

For each $\tau \in T(B)$ define $\Delta(\tau) = \frac{1}{\tau(q)}\tau|_F$. Since, for any $b \in B$,

$$\tau((1 - q)bq) = 0 = \tau(qb(1 - q)),$$

we have

$$|\tau(b) - \tau(qbq)| < \delta_1/4 \tag{4.30}$$

for all $\tau \in T(B)$. With $\delta_1 < 1$, for any $f \in F$,

$$|\tau(f) - \Delta(\tau)(f)| < \left(1 - \frac{1}{1 - \delta_1/4}\right)|\tau(f)| < (\delta_1/3)|\tau(f)| \tag{4.31}$$

for all $\tau \in T(B)$. By (2) above, (4.30) and (4.31), we estimate that

$$|\tau(b) - \Delta(\tau)(\phi(b))| < \delta_1/4 + \delta_1/8 + (\delta_1/3)(1 + \delta_1/8) + \delta_1/8 < \varepsilon/2 \tag{4.32}$$

for all $b \in \mathcal{G}_1$.

Define $\psi(a) = h \circ (\tilde{\psi}(a))$. Note that ψ is from A to $F \subset B$ and it is $\mathcal{F}$-ε-multiplicative. We also compute that

$$|\lambda(\hat{a})(\tau) - \Delta(\tau)(\psi(a))| < \varepsilon$$

for all $a \in \mathcal{F}$. □

5. Uniform approximate conjugacy in measure

Definition 5.1. Let X be a compact metric space and let $\alpha : X \to X$ be a minimal homeomorphism. Define $F(X)$ to be the set of those measures ν that are concentrated on finite subsets of X.

Fix a finite set of points $x_1, x_2, \dots, x_k \in X$ and k positive affine continuous functions $a_1, a_2, \dots, a_k \in Aff(T(A_\alpha))$ with $\sum_{i=1}^k a_i = 1$. One can define an affine continuous map $\Delta : T_\alpha \to F(X)$ by

$$\int f d\Delta(\mu) = \sum_{i=1}^{k} a_i(\tau_\mu) f(x_i) \tag{5.1}$$

for all $f \in C(X)$. To simplify notation, we also use Δ for the induced affine continuous map from $T(A_\alpha)$ to $F(X)$.

Definition 5.2. Let X be a compact metric space and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. We say that α and β are *approximately conjugate uniformly in measure* if there is a sequence of open subsets $\{O_n\}$, with each O_n being $1/n$-dense in X, and a sequence of Borel isomorphisms $\{\gamma_n\}$ on X, with the following properties:

(1) For each $\sigma > 0$,

$$\mu(\{x \in X : \mathrm{dist}(\gamma_n^{-1}\alpha\gamma_n(x), \beta(x)) \geq \sigma\}) \to 0, \tag{5.2}$$

$$\mu(\{x \in X : \mathrm{dist}(\alpha\gamma_n(x), \gamma_n\beta(x)) \geq \sigma\}) \to 0, \tag{5.3}$$

and

$$\nu(\{x \in X : \mathrm{dist}(\gamma_n\beta\gamma_n^{-1}(x), \alpha(x)) \geq \sigma\}) \to 0, \tag{5.4}$$

$$\nu(\{x \in X : \mathrm{dist}(\beta\gamma_n^{-1}(x), \gamma_n^{-1}\alpha(x)) \geq \sigma\}) \to 0, \tag{5.5}$$

uniformly on T_β and T_α, respectively.

(2) $\gamma_n(O_n)$ is a $\frac{1}{n}$-dense open subset, γ_n is continuous on O_n and γ_n^{-1} is continuous on $\gamma_n(O_n)$.

(3) There exists an affine continuous map $\Delta_n : T_\beta \to F(X)$ such that $\int f \circ \gamma_n d\Delta_n(\mu)$ converges uniformly on T_β for all $f \in C(X)$, which defines an affine homeomorphism $r : T_\beta \to T_\alpha$ and

$$\left| \int f d\mu - \int f d\Delta_n(\mu) \right| \to 0 \tag{5.6}$$

uniformly on T_β for all $f \in C(X)$, and there exists an affine continuous map $\tilde{\Delta}_n : T_\alpha \to F(X)$ such that $\int f \circ \gamma_n^{-1} d\tilde{\Delta}_n(\nu)$ converges uniformly on T_α for all $f \in C(X)$, which defines the affine homeomorphism $r^{-1} : T_\alpha \to T_\beta$, and

$$\left| \int f d\mu - \int f d\tilde{\Delta}_n(\mu) \right| \to 0 \tag{5.7}$$

uniformly on T_α for all $f \in C(X)$.

REMARK 5.3. In general, one should not expect that $\{\gamma_n\}$ converges in any suitable sense. Nevertheless, it is important that $\{\gamma_n\}$ carries some consistent information. Note that Borel equivalences (or even homeomorphisms) do not preserve measures. For a sequence of homeomorphisms $\{\gamma_n\}$ from X onto X, even if each γ_n does not map positive measure sets to sets with zero measure, it could still happen that, for example, $\mu(\gamma_n(E)) \to 0$ for some Borel set E with $\mu(E) > 0$. Therefore one should regard (3) as a crucial part of the definition.

It should be noted that the relation of approximate conjugacy uniformly in measure is a rather weak relation. Given an affine homeomorphism $r : T_\alpha \to T_\beta$, Theorem 5.6 provides a sequence of maps $\{\gamma_n\}$ which induces the map r in the sense of (3) in 5.2 and $\gamma_n^{-1}\alpha\gamma_n(x)$ converges to β and $\gamma_n\beta\gamma_n^{-1}$ converges to α in measure uniformly on T_β and T_α, respectively. It is interesting to see that there exists a sequence $\{\gamma_n\}$ which induces r.

For convenience, we list two known facts below.

LEMMA 5.4. *Let X be a compact metric space and $\alpha : X \to X$ be a minimal homeomorphism. Then, for any $x, y \in X$ and any two open neighborhoods $N(x)$ and $N(y)$ of x and y, there exist a neighborhood $O(x) \subset N(x)$, an open subset $O \subset N(y)$, and a homeomorphism α' from $O(x)$ onto O.*

Proof. This follows from the minimality immediately. In fact, for any $\varepsilon > 0$, there exists $n \geq 1$, such that

$$\mathrm{dist}(\alpha^n(x), y) < \varepsilon/2.$$

Since α^n is continuous, there exists $\delta > 0$ such that

$$\alpha^n(\{\xi \in X : \mathrm{dist}(x, \xi) < \delta\}) \subset \{\xi \in X : \mathrm{dist}(y, \xi) < \varepsilon\}.$$

This means that the homeomorphism α^n maps $\{x \in X : \mathrm{dist}(x, \xi) < \delta\}$ into the neighborhood $\{\xi \in X : \mathrm{dist}(y, \xi) < \varepsilon\}$. □

LEMMA 5.5. *Two second countable locally compact Hausdorff spaces are Borel equivalent if they have the same cardinality ($\leq 2^{\aleph_0}$) .*

See 4.6.13 of [22] for a proof of 5.5.

We remind the reader that when X is a finite dimensional compact metric space and α is minimal, (X, α) has mean dimension zero ([19]).

THEOREM 5.6. *Let X be a finite dimensional compact metric space with infinitely many points and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Suppose that $\rho(K_0(A_\alpha))$ is dense in $Aff(T_\alpha))$ and $\rho(K_0(A_\beta))$ is dense in $K_0(A_\beta)$. Then the following are equivalent:*

(1) *There is an affine homeomorphism $r : T_\beta \to T_\alpha$.*
(2) *α and β are approximately conjugate uniformly in measure.*

Proof. It suffices to prove "(1)⇒(2)".

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Fix $\eta_0 > 0$ such that

$$|f(x) - f(x')| < \varepsilon/8$$

if $\text{dist}(x, x') < \eta_0$.

Choose an integer $n > 0$ such that $1/n < \varepsilon/8$. Choose $\eta_1 > 0$ such that

$$\text{dist}(\alpha^j(x), \alpha^j(y)) < \eta_0/2 \text{ and } \text{dist}(\beta^j(x), \beta^j(y)) < \eta_0/2 \tag{5.8}$$

if $\text{dist}(x, y) < \eta_1$, $j = 1, 2, \ldots, n-1$.

Let $\eta = \min\{\varepsilon/4, \eta_1/4, \eta_0/4\}$.

By 3.4, one obtains an open subset G that satisfies the following properties:

(i) G contains $\bigcup_{i=1}^{l}\{x \in X : \text{dist}(x, x_i) < d\}$ for some $d > 0$, where $\{x_1, x_2, \ldots, x_t\}$ is $\eta/6$-dense;
(ii) $\alpha^j(G)$ are pairwise disjoint for $0 \le j \le n-1$;
(iii) $\mu(X \setminus \bigcup_{j=0}^{n-1} \alpha^j(G)) < \varepsilon/8$ for all $\mu \in T_\alpha$;
(iv) $\mu(\partial(G)) = 0$ for all $\mu \in T_\alpha$.

Similarly, let Ω be an open subset that satisfies the following properties:

(i') Ω contains at least one open ball of ξ_i, where $\{\xi_1, \xi_2, \ldots, \xi_t\}$ is $\eta/2$-dense in X;
(ii') $\beta^j(\Omega)$ are pairwise disjoint for $0 \le j \le n-1$;
(iii') $\mu(X \setminus \bigcup_{j=0}^{n-1} \beta^j(\Omega)) > 1 - \varepsilon/8$ for all $\mu \in T_\beta$;
(iv') $\mu(\partial(\Omega)) = 0$ for all $\mu \in T_\alpha$.

Note that we can use the same number t for the number of points in $\{x_1, x_2, \ldots, x_t\}$ and in $\{\xi_1, \xi_2, \ldots, \xi_t\}$. When we apply 3.4 to obtain Ω, we use the $\eta/6$-dense set $\{x_1, x_2, \ldots, x_t\}$ to obtain the $\eta/2$-dense set $\{\xi_1, \xi_2, \ldots, \xi_t\}$.

Suppose that $O(x_i)$ are open balls of x_i so that $O(x_i) \subset G$ and $O(\xi_i)$ are open balls of ξ_i so that $O(\xi_i) \subset \Omega$. Since (X, α) has mean dimension zero, let $\{O_1, O_2, \ldots, O_L\}$ be a finite set of pairwise disjoint open subsets of X such that each O_i has diameter less than $\eta_1/2$, $X = \bigcup_{i=1}^{L} \overline{O_i}$ and $\mu(\overline{O_i} \setminus O_i) = 0$ for all $\mu \in T_\alpha$. We may assume that $O(x_i) \subset O_{i'} \cap G$ for some i', by choosing a smaller open ball of x_i if necessary. Further, by considering a suitable open ball of x_i with universal null boundary, we may simply assume that $O_i = O(x_i)$, $i = 1, 2, \ldots, t$ and $L > t$.

Let $\{U_1, U_2, \dots, U_{L_1}\}$ be a finite set of pairwise disjoint open subsets of X such that each U_i has diameter less than $\eta_1/2$, $X = \bigcup_{i=1}^{L_1} \overline{U_i}$ and $\nu(\overline{U_i} \setminus U_i) = 0$ for all $\nu \in T_\beta$. We may also assume that $O(\xi_i) = U_i$, $i = 1, 2, \dots, t$ and $t < L_1$.

Let $p \in A_\alpha$ and $q \in A_\beta$ be the specially selected projections as given by 4.7 with

$$\tau(1-p) < \varepsilon/16 \text{ and } \theta(1-q) < \varepsilon/16 \tag{5.9}$$

for all $\tau \in T(A_\alpha)$ and $\theta \in T(A_\beta)$ for $\varepsilon/4$, $\mathcal{F}$, η, n and G above and $\varepsilon/4$, $\mathcal{F}$, η, n and Ω above.

Let $\mathcal{G}_1 \subset A_\beta$ be a finite subset (in place of $\mathcal{G}$) and $\delta > 0$ as given by 4.7 for the above $\varepsilon/4$, $\mathcal{F}$, n, η and Ω. Let $\mathcal{G}_2 \subset A_\alpha$ be a finite subset and $\delta_1 > 0$ as given by 4.7 for the above $\varepsilon/4$, $\mathcal{F}$, n η and G.

Let $r^\natural : Aff(T(A_\alpha)) \to Aff(T(A_\beta))$ be the affine isomorphism induced by r. It follows from 4.8 (and (5.9)) that, with sufficiently large $\mathcal{G}_1$ and sufficiently small δ, there is a finite dimensional C^*-algebra B_0, a unital $\mathcal{G}_1$-δ-multiplicative contractive completely positive linear map $\phi : A_\beta \to B_0$, a $\mathcal{G}_2$-δ-multiplicative contractive completely positive linear map $\psi : A_\alpha \to B_0$, and an affine continuous map $\Delta_0 : T(A_\beta) \to T(B_0)$, such that

(1) for all $\tau \in T(A_\beta)$ and $f \in \mathcal{F}$,

$$|\Delta_0(\tau) \circ \phi(q j_\beta(f) q) - \tau \circ j_\beta(f)| < \varepsilon/8; \tag{5.10}$$

(2) for all $\tau \in T(A_\beta)$ and $f \in \mathcal{F}$,

$$|r^\natural(\widehat{j_\alpha(f)})(\tau) - \Delta_0(\tau) \circ \psi(p j_\alpha(f) p)| < \varepsilon/8. \tag{5.11}$$

Write $B_0 = \oplus_{s=1}^{k_0} M_{R(s)}$ and let $\pi_s : B_0 \to M_{R(s)}$ be the canonical projection map. By applying 4.7, for each s, there are integers $K(s) = m_s n$ and $K'(s) = m_s' n$ with $m_s = \sum_{i=1}^{L} m_s(i)$ and $m_s' = \sum_{i'=1}^{L_1} m_s'(i')$, and points $y_{i,l}(s) \in O_i \cap G$, $l = 1, 2, \dots, m_s(i)$, $i = 1, 2, \dots, L$, $Y_{i',l'}(s) \in U_i \cap \Omega$, $l' = 1, 2, \dots, m_s'(i')$, $i' = 1, 2, \dots, L_1$, such that

$$\begin{aligned} \Bigg\| \sum_{i,l,j} f(\alpha^j(y_{i,l}(s))) p_{s,i,l,j} \\ + \sum_{i=K(s)+1}^{N(s)} f(z_i) p_{s,i} - \pi_s \circ \psi \circ (p j_\alpha(f) p) \Bigg\| < \varepsilon/4 \end{aligned} \tag{5.12}$$

for all $f \in \mathcal{F}$ and

$$\left\| \sum_{i',l',j} f(\beta^j(Y_{i,l}(s)))q_{s,i',l',j} + \sum_{i'=K'(s)+1}^{N'(s)} f(z_i')q_{s,i'} - \pi_s \circ \phi \circ (qj_\beta(f)q) \right\| < \varepsilon/4 \tag{5.13}$$

for all $f \in \mathcal{F}$, where

$$\{p_{s,i,l,j} : i,l,j\} \cup \{p_{s,i} : i > N(s)\} \text{ and } \{q_{s,i',l',j} : i',l',j\} \cup \{q_{s,i'} : i' > N'(s)\}$$

are sets of mutually orthogonal rank one projections in $M_{R(s)}$ and $z_i, z_{i'} \in X$. In addition, by 4.7, we may assume that $y_{i,1}(1) = x_i$ and $Y_{i,1}(1) = \xi_i$, $i = 1, 2, \dots, t$.

Furthermore,

$$\frac{R(s) - K(s)}{R(s)} < \varepsilon/4 \text{ and } \frac{R(s) - K'(s)}{R(s)} < \varepsilon/4 \tag{5.14}$$

for $s = 1, 2, \dots, k_0$. Without loss of generality, since X has no isolated points, we may assume that $\{y_{i,l}(s) : i,l,s\}$ and $\{Y_{i',l'})(s) : i',l',s\}$ are two sets of distinct points. If $m_s' > m_s$, we will move $m_s' - m_s$ many points of $Y_{i',l'}(s)$ to the set $\{z_i' : i'\}$. If, on the other hand, $m_s > m_s'$, we will move $m_s - m_s'$ many points to $\{z_i : i\}$. So, either way, we may assume that $m_s = m_s'$ and $K(s) = K'(s)$. Note that we still have $\frac{R(s)-K'(s)}{R(s)} < \varepsilon/4$.

By replacing ϕ by ad $u \circ \phi$, for a suitable unitary in B_0, we may assume that

$$\{p_{s,i,l,j} : 1 \le i \le L, 1 \le l \le m_s(i), 0 \le j \le n-1\} = \{q_{s,i',l',j}\}.$$

Since now we assume that $m_s = m_s'$, we define, for each s, $\tilde{\gamma}(Y_{i',l'}(s))$ to be a one-to-one bijection between $\{Y_{i'l'}(s) : i',l',s\}$ and $\{y_{i,l}(s) : i,l,s\}$. We may also assume that $\tilde{\gamma}(Y_{i,1}(1)) = y_{i,1}(1)$, $i = 1, 2, \dots, t$.

To construct the desired map γ, we divide $O_i \cap G$ into $\sum_{s=1}^{k_0} m_s(i)$ pairwise disjoint sets $B_{s,i,l}$ as follows: Choose $d(s,i,l) > 0$ so that the open balls $B(y_{i,l}(s), d(s,i,l))$ are mutually disjoint. If $(s,i,l) \neq (1,i,2)$, define $B_{s,i,l} = B(y_{i,l}(s), d(s,i,l))$. Define $B_{1,i,2} = (O_i \cap G) \setminus (\bigcap_{(s,i,l)\neq(1,i,2)} B_{s,i,l})$. Similarly, we then divide $U_{i'} \cap \Omega$ into $\sum_{s=1}^{k_0} m_s'(i')$ pairwise disjoint subsets $C_{s,i',l'}$ which is either an open neighborhood of $Y_{i',l}(s)$ or a closed subset which contains an open neighborhood of $Y_{i',l}(s)$.

Note that, since every point in X is condensed, $B_{s,i,l}$ and $C_{s,i',l'}$ are second countable locally compact Hausdorff spaces with cardinality $2^{\aleph_0}$. By 5.5, they are all Borel equivalent.

Define a Borel equivalence $\gamma : X \to X$ as follows:

By 5.4, there is an open neighborhood $Z(i,1,s)$ of $Y_{i,1}(s)$ in $C_{s,i,1}$ (for $1 \le i \le t$) and a open subset $\tilde{Z}(i,1,s)$ of $B_{s,i,1}$ which are homeomorphic. In particular, the closure of a smaller open neighborhood of $Y_{i,1}(s)$ is homeomorphic to the closure of an open subset of $\tilde{Z}(i,1,s)$. Thus, by taking a sufficiently small such neighborhood and by applying 5.5, one obtains a Borel equivalence γ from $C_{s,i,1}$ onto $B_{s,i,1}$ which maps a non-empty neighborhood $Z(i,1,s)$ of $Y_{i,1}(s)$ to an open subset of a neighborhood of $y_{i,1}(s)$ homeomorphically for $1 \le i \le t$.

For the rest of $C_{s,i,l}$ ($l > 1$ or $l = 1$, but $i > t$), we define γ to be a Borel equivalence from $C_{s,i,l}$ to $B_{s,i',l'}$ if $\tilde{\gamma}(Y_{i,l}(s)) = y_{i,l}(s)$.

We define γ on $\beta^j(C_{s,i',l'})$ to be $\alpha^j \circ \gamma \circ \beta^{-j}$, $j = 1, 2, \dots, n-2$.

Since $X \setminus \bigcup_{j=0}^{n-2} \alpha^j(G)$ (which is a compact subset of X which contains $\alpha^{n-1}(G)$) and $X \setminus \bigcup_{j=0}^{n-2} \alpha^j(\Omega_j)$ (which is a compact subset of X which contains $\alpha^{n-1}(\Omega)$) are Borel equivalent, we obtain a Borel equivalence γ of X which is bi-continuous on $O = \bigcup_{i',s} Z(i',1,s)$. Note that γ maps $\bigcup_{j=0}^{n-2} \beta^j(\Omega)$ onto $\bigcup_{j=0}^{n-2} \alpha^j(G)$. We also have $\gamma(Z(i,1,s)) \subset \tilde{Z}(i,1,s)$. Since $\bigcup_{i=1}^{L} O_i$ and $\bigcup_{i'=1}^{L_1} U_i$ have diameter less than $\eta/2$, by the construction, we see that O and $\gamma(O)$ are η -dense in X.

Moreover, on each $\beta^j(C_{s,i'l'})$ with $0 \le j \le n-2$,

$$\mathrm{dist}(\gamma^{-1}\alpha\gamma(x), \beta(x)) < \eta \text{ and } \mathrm{dist}(\alpha\gamma(x), \gamma\beta(x)) < \eta. \tag{5.15}$$

We also have, on each $\alpha^j(B_{i,l,s})$ with $0 \le j \le n-2$,

$$\mathrm{dist}(\gamma\beta\gamma^{-1}(x), \alpha(x)) < \eta \text{ and } \mathrm{dist}(\beta\gamma^{-1}(x), \gamma^{-1}\alpha(x)) < \eta. \tag{5.16}$$

Since

$$\nu(\beta^{n-1}(\Omega)) < 1/n < \varepsilon/8 \text{ and } \mu(\alpha^{n-1}(G)) < 1/n < \varepsilon/8 \tag{5.17}$$

for all β-invariant probability measures ν and α-invariant probability measures μ, we conclude that

$$\nu(\{x \in X : \mathrm{dist}(\gamma^{-1}\alpha\gamma(x), \beta(x)) > \eta\}) < \varepsilon/4 \text{ and} \tag{5.18}$$

$$\mu(\{x \in X : \mathrm{dist}(\gamma\beta\gamma^{-1}(x), \alpha(x)) > \eta\}) < \varepsilon/4 \tag{5.19}$$

for all β-invariant probability measures ν and α-invariant probability measures μ.

To complete the proof, it remains to check (3) of 5.2. to this end, we note that, by (5.12), (5.13) and (5.14),

$$\left| \sum_{s,i,l,j} f(\alpha^j(y_{i,l}(s)))\Delta_0(\tau)(p_{s,i,l,j}) - \Delta_0(\tau)(\psi \circ (pj_\alpha(f)p)) \right| < \varepsilon/2 \tag{5.20}$$

and

$$\left| \sum_{s,i',l',j'} f(\beta^j(Y_{i,l}(s)))\Delta_0(\tau)(q_{s,i',l',j}) - \Delta_0(\tau)(\phi \circ (qj_\beta(f)q)) \right| < \varepsilon/2 \tag{5.21}$$

for all $f \in \mathcal{F}$ and $\tau \in T(A_\beta)$. Note also that, for each s, $\Delta_0(\tau)(p_{s,i,l,j}) = \Delta_0(\tau)(q_{s,i',l',j}) = \frac{c_\tau}{R(s)}$ for all i, i', l, l', j and for some non-negative constant c_τ.

We also estimate that, for each s,

$$\Big| \sum_{i',l',0\le j\le n-2} f\circ\gamma(\beta^j(Y_{i',l'}(s)))\frac{c_\tau}{R(s)} - \sum_{i,l,0\le j\le n-2} f(\alpha^j(y_{i,l}(s)))\frac{c_\tau}{R(s)}\Big| < \varepsilon/8 \tag{5.22}$$

and

$$\Big| \sum_{i,l,o\le j\le n-2} f\circ\gamma^{-1}(\alpha^j(y_{i,l}(s)))\frac{c_\tau}{R(s)} - \sum_{i,l,0\le j\le n-2} f(\beta^j(Y_{i',l'}(s)))\frac{c_\tau}{R(s)}\Big| < \varepsilon/8 \tag{5.23}$$

for all $f \in \mathcal{F}$ and $\tau \in T_\beta$.

Define $\Delta : T_\beta \to F(X)$ by

$$\int f d(\Delta(\mu)) = \sum_{s=1}^{k_0} \sum_{i',l',0\le j\le n-2} f(\beta^j(Y_{i'.l'}(s)))\frac{c_\tau}{R(s)}$$

for all $\mu \in T_\beta$ (where $\mu = \mu_\tau$) and all $f \in C(X)$. Note that $\int f d(r(\mu)) = r^\natural(\widehat{j_\alpha(f)})(\tau)$ $(\mu = \mu_\tau)$. Combining (5.9), (5.10), (5.11), (5.14), (5.20) and (5.22), we have

$$\Big|\int f d\mu - \int f d(\Delta(\mu))\Big| < \varepsilon, \tag{5.24}$$

$$\Big|\int f\circ\gamma d(\Delta(\mu)) - \int f d(r(\mu))\Big| < \varepsilon \tag{5.25}$$

for all β-invariant probability measures μ and all $f \in \mathcal{F}$.

Define $\tilde{\Delta} : T_\alpha \to F(X)$ by $\tilde{\Delta}(\nu) = \Delta(r_\natural^{-1}(\nu))$ for $\nu \in T_\alpha$. Then we have, by (5.9), (5.10), (5.11), (5.14), (5.21) and (5.23),

$$\Big|\int f d\tilde{\Delta}(\nu) - \int f d\nu\Big| < \varepsilon, \tag{5.26}$$

$$\Big|\int f\circ\gamma^{-1} d\tilde{\Delta}(\nu) - \int f d(r^{-1}(\nu))\Big| < \varepsilon \tag{5.27}$$

for all α-invariant probability measures ν and all $f \in \mathcal{F}$. $\square$

6. Concluding remarks

6.1. Let X be a compact metric space and T be a convex subset of probability Borel measures. Suppose that $\Gamma_n, \Gamma : X \to X$ are Borel maps and $\Gamma_n \to \Lambda_n$ in measure uniformly on T. Then a uniform Egorov theorem holds. Put

$$S_{m,k} = \{x \in X : \mathrm{dist}(\Gamma_m, \Gamma(x)) \geq 1/k\}, \tag{6.1}$$

$k = 1, 2, \ldots,$ and $m = 1, 2, \ldots$ Let $\delta > 0$. For each $k > 0$, there exists an integer $n(k)$ such that

$$\mu(S_{n(k),k}) < \frac{\delta}{2^k} \tag{6.2}$$

for all $\mu \in T$, if $n \geq n(k)$. Put

$$E = \bigcap_{k=1}^{\infty} \bigcap_{m=n(k)}^{\infty} \{x \in X : \mathrm{dist}(\Gamma_m(x), \Gamma(x)) < 1/k\}. \tag{6.3}$$

Then Γ_n converges to Γ uniformly on E. Furthermore,

$$\mu(X \setminus E) \leq \mu\left(\bigcup_{k=1}^{\infty} S_{n(k),k}\right) \leq \sum_{k=1}^{\infty} \mu(S_{n(k),k}) < \delta \tag{6.4}$$

for all $\mu \in T$. Thus, in Theorem 5.6, for any $\delta > 0$, there exists a Borel subset $E \subset X$ with $\mu(X \setminus E) < \delta$ for all $\mu \in T_\beta$ such that $\gamma_n^{-1}\alpha\gamma_n$ converges to β uniformly on E. Moreover, there exists a Borel subset $E' \subset X$ with $\mu(X \setminus E') < \delta$ such that $\gamma_n\beta\gamma_n^{-1}$ converges to α uniformly on E'. A similar measure theoretical argument, by taking a subsequence of $\{\gamma_n\}$, shows that there exist Borel measurable subsets $F_\alpha, F_\beta \subset X$ such that $\gamma_n^{-1}\alpha\gamma_n$ converges to β on F_β and $\gamma_n\beta\gamma_n^{-1}$ converges to α on F_α and $X \setminus F_\beta$ and $X \setminus F_\beta$ are universally null, i.e., $\mu(X \setminus F_\beta) = 0$ for all $\mu \in T_\beta$ and $\nu(X \setminus F_\alpha) = 0$ for all $\nu \in T_\alpha$.

6.2. Suppose that X is the Cantor set and suppose that $\alpha, \beta : X \to X$ are two minimal homeomorphisms. Then in Theorem 3.4 G can be chosen to be clopen. Since a non-empty clopen subset of the Cantor set can be divided into m non-empty clopen subsets for any integer $m > 0$, in the proof of 5.6, $B_{i,l,s}$ and $C_{i',l',s}$ can be chosen to be also non-empty clopen subsets of X. They all are homeomorphic. It is then easy to see that the map γ in the proof can be made a homeomorphism. In other words, we have the following corollary:

Corollary 6.1. *Let X be the Cantor set and let $\alpha, \beta : X \to X$ be minimal homeomorphisms. Then α and β are approximately conjugate uniformly in measure if and only if there is an affine homeomorphism $r\colon T_\alpha \to T_\beta$. Moreover, when α and β are approximately conjugate uniformly in measure, the conjugating maps γ_n can be chosen to be homeomorphisms.*

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