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LOCALLY TRANSITIVE TOURNAMENTS AND THE CLASSIFICATION OF $(1,2)$–SYMPLECTIC METRICS ON MAXIMAL FLAG MANIFOLDS

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ABSTRACT. We give a new proof of a classification theorem of (1,2)–symplectic metrics on maximal flag manifolds proved by Cohen, Negreiros and San Martin. We use locally transitive tournaments in order to simplify the demonstration of this theorem. Finally, using a result due to Brouwer we count the number of invariant almost complex structures which admit (1,2)–symplectic metrics on a maximal flag manifold.

1. Introduction

Let $F(n)$ be the geometrical maximal flag manifold, i.e.,

$$F(n) = \{(L_1, \dots, L_n) \colon L_i \text{ is a subspace of } \mathbb{C}^n,\ \dim_{\mathbb{C}} L_i = 1,\ L_i \perp L_j\}, \tag{1}$$

endowed with an almost–complex structure J.

Burstall and Salamon [2] proved the existence of a bijective relation between almost–complex structures on $F(n)$ and tournaments with n vertices. This relation has been used to study geometric properties of $F(n)$ using properties of the tournaments; see, for example, Cohen, Negreiros and San Martin [3], [4]; Mo and Negreiros [7], [8], Negreiros and San Martin [15]; and Paredes [10], [11], [12], [13], [14].

Cohen, Negreiros and San Martin [3] proved the following result:

THEOREM 1. *The maximal flag manifold $(F(n), J)$ admits an invariant $(1,2)$–symplectic metric if and only if the associated tournament $\mathcal{T}(J)$ does not contain coned 3–cycles ((5) and (6) in Figure 1).*

This theorem was conjectured by Paredes [10], and proved in [7] for $n = 3, 4$ and in [11] for $n = 5, 6, 7$. The sufficiency of the condition that the tournament $\mathcal{T}(J)$ does not contain coned 3–cycles was studied in [12], where an affirmative answer was obtained for certain classes of tournaments. The necessity was

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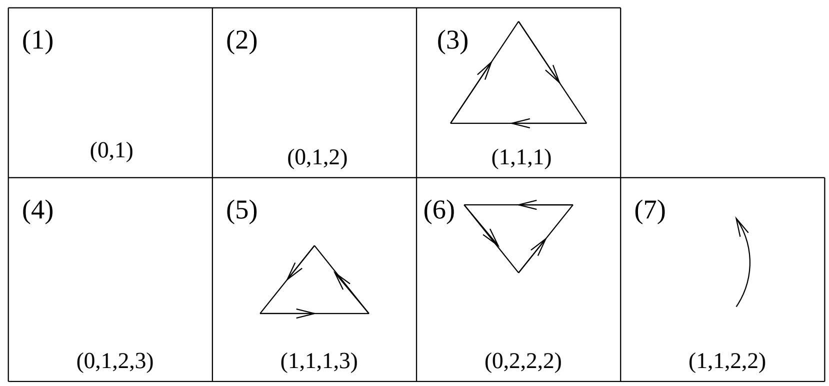

FIGURE 1. Isomorphism classes of n–tournaments for $n = 2, 3, 4$.

established in [7]. Paredes [10] stated this theorem in the following equivalent form:

THEOREM 2. *The maximal flag manifold* $(F(n), J)$ *admits an invariant* $(1, 2)$*–symplectic metric if and only if all* 4*–subtournaments of the associated tournament* $\mathcal{T}(J)$ *are isomorphic to* (4) *or* (7) *in Figure* 1.

The tournaments characterized in [12] for which $F(n)$ admits (1,2)–symplectic metrics were used to construct many examples of harmonic maps by applying a theorem due to Lichnerowicz [6].

Theorem 1 has been generalized by Negreiros and San Martin [15] to generalized flag manifolds associated to complex semi–simple Lie groups.

In the proof of Theorem 1 given by Cohen, Negreiros and San Martin [3] the concept of cone–free tournaments is used. In this note we show that the concept of cone–free tournaments is equivalent to the concept of locally transitive tournaments, and using results from Brouwer's paper [1] we rederive the classification of $(1, 2)$–symplectic metrics, obtained in [3], in a more direct way.

In addition, using the concept of locally transitive tournaments we give an answer to the following question: How many invariant almost complex structures which admit $(1, 2)$–symplectic metrics on $F(n)$ are there? This problem has been studied in [10], where it was proved that for $n = 3$, 4, 5, 6, and 7 there exist, respectively, 2, 2, 4, 6, and 10 families of invariant almost complex structures which admit (1,2)–symplectic metrics on $F(n)$. In [14] the number of families of invariant almost complex structures which admit (1,2)–symplectic metrics on $F(n)$ was calculated for $n \leq 20$.

2. Tournaments

A tournament or n–tournament is a directed graph with n vertices or players without loops such that for each pair of vertices $x \neq y$ there is a unique oriented edge $x \to y$ or $y \to x$. If $x \to y$, then we say x wins against y.

A tournament $\mathcal{T}$ is transitive if for any three vertices x, y, z such that $x \to y$ and $y \to z$ we have $x \to z$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be tournaments with vertices $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively. A homomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $\phi : \{1, \dots, n\} \to \{1, \dots, m\}$ such that

$$s \xrightarrow{\mathcal{T}_1} t \quad \Longrightarrow \quad (\phi(s) \xrightarrow{\mathcal{T}_2} \phi(t) \qquad \text{or} \qquad \phi(s) = \phi(t)). \tag{2}$$

When ϕ is bijective we say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic.

An n–tournament determines a score vector $(s_1, \dots, s_n)$, $0 \leq s_1 \leq \cdots \leq s_n$, such that $\sum_{i=1}^n s_i = \binom{n}{2}$, whose components representing the number of games won by each player. Isomorphic tournaments have identical score vectors. Figure 1 shows the isomorphism classes of n–tournaments for $n = 2, 3, 4$, together with their score vectors. In Figure 1 we use Moon's notation in which not all of the arcs are included in the drawings. If an arc joining two vertices has not been drawn, then it is to be understood that the arc is oriented from the higher vertex to the lower vertex (see [9]).

Given a tournament $\mathcal{T}$ and a vertex $v \in \mathcal{T}$ we define the subtournaments

$$\mathcal{T}^-(v) = \{x \in \mathcal{T} : x \to v\} \quad \text{and} \quad \mathcal{T}^+(v) = \{x \in \mathcal{T} : v \to x\}, \tag{3}$$

which are called the in–neighbor and the out–neighbor of v, respectively. $\mathcal{T}$ is called locally transitive if and only if the subtournaments $\mathcal{T}^-(v)$ and $\mathcal{T}^+(v)$ are transitive for each vertex v (see [1]).

Given a tournament $\mathcal{T}$, we say that a 3–cycle formed by the vertices i, j, k of $\mathcal{T}$ is coned if there exists another vertex x such that

$$(x \to i,\ x \to j \text{ and } x \to k) \quad \text{or} \quad (i \to x,\ j \to x \text{ and } k \to x).$$

We say that the tournament $\mathcal{T}$ is cone–free if and only if it does not contain any coned 3–cycle.

Proposition 3. *A tournament $\mathcal{T}$ is locally transitive if and only if it is cone–free.*

Proof. Suppose that $\mathcal{T}$ is not cone–free. Then it contains one of the 4–tournaments (5) or (6) in Figure 1. If $\mathcal{T}$ contains (5) then the vertex that wins over the other three vertices is such that its out–neighbor is not transitive. Similarly, if $\mathcal{T}$ contains (6) then the vertex that loses to the other vertices is such that its in–neighbor is not transitive. Then $\mathcal{T}$ is not locally transitive.

Now, suppose that $\mathcal{T}$ is not locally transitive. Then there exists a vertex x such that $\mathcal{T}^-(x)$ is not transitive or $\mathcal{T}^+(x)$ is not transitive. If $\mathcal{T}^-(x)$ is not transitive then it contains a 3–cycle $i \to j \to k \to i$ and $i \to x$, $j \to x$ and $k \to x$. Hence, the tournament formed by the vertices i, j, k, x is the same as (6) in Figure 1. Thus $\mathcal{T}$ is not cone–free. $\square$

Proposition 4. *A tournament $\mathcal{T}$ is locally transitive if and only if all 4–subtournaments of $\mathcal{T}$ are locally transitive.*

Proof. If $\mathcal{T}$ is locally transitive then all subtournaments of $\mathcal{T}$ are locally transitive; in particular, all 4–subtournaments are locally transitive.

Now, suppose that $\mathcal{T}$ is not locally transitive. Then there exists a vertex v of $\mathcal{T}$ such that either $\mathcal{T}^-(v)$ or $\mathcal{T}^+(v)$ is not transitive. If $\mathcal{T}^-(v)$ is not transitive then it contains a 3–cycle. Therefore the 4–subtournament determined by the 3–cycle and v is not locally transitive. $\square$

The following result and its proof are taken from Brouwer's paper [1].

Proposition 5. *Let $\mathcal{T}$ be a locally transitive tournament and a, x vertices of $\mathcal{T}$. If $a \in \mathcal{T}^+(x)$ then $\mathcal{T}^+(a)$ is the union of a terminal interval in $\mathcal{T}^+(x)$ and an initial interval in $\mathcal{T}^-(x)$.*

Proof. Suppose that the vertices $b, c \in \mathcal{T}^-(x)$ are such that $b \to c$, $c \in \mathcal{T}^+(a)$ and $b \notin \mathcal{T}^+(a)$. Then $c, x, a \in \mathcal{T}^+(b)$, and therefore $\mathcal{T}^+(b)$ contains the 3–cycle formed by c, x, a. But this is a contradiction because $\mathcal{T}^+(b)$ is transitive and does not contain 3–cycles. $\square$

For each n–tournament $\mathcal{T}$ we can associate an $n \times n$ matrix $m(\mathcal{T}) = (a_{ij})$, called the incidence matrix of $\mathcal{T}$, defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \to j, \\ 0 & \text{if } i = j, \\ -1 & \text{if } j \to i\,. \end{cases}$$

The following definition was introduced by Cohen, Negreiros and San Martin [3]. An n–tournament $\mathcal{T}$ with vertices $\{1, \ldots, n\}$ is called stair–shaped if there are integers s, t, with $1 \leq s \leq t \leq n$, such that the following axioms are satisfied:

(1) The subtournament of $\mathcal{T}$ formed by the vertices $1, \ldots, t$ is a maximal transitive tournament.
(2) The subtournament of $\mathcal{T}$ formed by the vertices $s, \ldots, n$ is a maximal transitive tournament.
(3) If $z \to x$ and $z > x$, then $x < s$ and $t < z$.
(4) If $x' \leq x$ and $z \leq z'$, then $z \to x$ and $z > x$ implies $z' \to x'$ and $z' > x'$.

In other words, a tournament is stair–shaped if its incidence matrix is of the following type:

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & 1 & 1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \ddots & 1 & 1 & 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

The following result was proved in [3].

THEOREM 6. *A tournament is cone–free if and only if it is isomorphic to a stair–shaped tournament.*

This theorem and Proposition 3 imply that a tournament $\mathcal{T}$ is locally transitive if and only if it is isomorphic to a stair–shaped tournament.

The proof of Theorem 6 is long and complicated. We will give here a simpler proof using the following characterization of the tournaments under consideration.

Let $\mathcal{T}$ be a locally transitive tournament and choose an arbitrary vertex v of $\mathcal{T}$. Given another vertex x of $\mathcal{T}$, we say that x has type 1 if $v \to x$ and type 2 if $x \to v$. Define the weight of x, $w(x)$, as the number of vertices y such that $y \to x$ and y has the same type as x. Then reorder the vertices in the following way: vertex v, vertices of type 1 in increasing order of weight and vertices of type 2 in increasing order of weight. Using Proposition 5 it is easy to see that this method produces the incidence matrix of $\mathcal{T}$ in stair–shaped form.

For example, consider the 8–tournament corresponding to the incidence matrix

$$\begin{pmatrix} 0 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 0 \end{pmatrix},$$

and suppose that the vertices of this tournament are labelled so that vertex i corresponds to row i. We choose vertex 1 in order to apply the method above. Then we have the following types of vertices:

type 1:	2, 3, 5, 6, 8
type 2:	4, 7

The corresponding weights are $w(1) = 0, w(2) = 2, w(3) = 4, w(5) = 1$, $w(6) = 3, w(8) = 0, w(4) = 1$ and $w(7) = 0$. Reordering the vertices using our method we obtain the stair–shaped matrix of the tournament

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Clearly, this method to obtain the stair–shaped form of the incidence matrix is simpler than the method used by Cohen, Negreiros and San Martin, which was described in the definition of stair–shaped tournaments. In fact, our method yields the largest transitive subtournament contained in the tournament, and we call the matrix obtained by this method maximal stair–shaped.

3. Almost complex structures and metrics on $F(n)$

The natural action of the unitary group $U(n)$ on $F(n)$ is transitive and provides the following representation of $F(n)$ as a homogeneous space:

$$F(n) = \frac{U(n)}{T} = \frac{U(n)}{U(1) \times \cdots \times U(1)}, \tag{4}$$

where $T = U(1) \times \cdots \times U(1)$ is a maximal torus of $U(n)$. Using this representation it is easy to see that the tangent space of $F(n)$ at the origin is the subspace $\mathfrak{p} \subset \mathfrak{u}(n)$ of zero–diagonal matrices, where $\mathfrak{u}(n)$ is the Lie algebra of skew–symetric matrices. Let E_{ij} be the canonical basis matrix. Then $\mathfrak{p} = \oplus_{i \neq j} \mathfrak{p}_{ij}$, where $\mathfrak{p}_{ij} = (\mathbb{C}E_{ij} + \mathbb{C}E_{ji}) \cap \mathfrak{u}(n)$.

Since the action of $U(n)$ on $F(n)$ is transitive, to define any tensor on $F(n)$ it is sufficient to specify it on $\mathfrak{p}$. An invariant almost complex structure on $F(n)$ is determined by a linear map $J\colon \mathfrak{p} \to \mathfrak{p}$ satisfying $J^2 = -I$ that commutes with the adjoint representation of the torus T on $\mathfrak{p}$. Then $J(\mathfrak{p}_{ij}) = \mathfrak{p}_{ji}$, for all $i \neq j$. Thus we have $J(A) = A'$, $A = (a_{ij})$, $A' = (a'_{ij})$, with $a'_{ij} = \varepsilon_{ij}\sqrt{-1}a_{ij}$, such that $\varepsilon_{ij} = \pm 1$ and $\varepsilon_{ij} = -\varepsilon_{ji}$. In other words, an invariant almost complex structure is completely determined by a skew–symetric matrix (ε_{ij}) with off–diagonal entries in $\{\pm 1\}$. This establishes a one–to–one correspondence between invariant almost complex structures on

$F(n)$ and n–tournaments through the incidence matrix. Given an almost complex structure J, we denote by $\mathcal{T}(J)$ the associated tournament.

The complexification V of $\mathfrak{p}$ is the subspace of complex matrices with zero–diagonal entries. It decomposes as $V = \oplus_{i\neq j} V_{ij}$, where $V_{ij} = \mathrm{span}_{\mathbb{C}}\{E_{ij}\}$. An almost complex structure J extends to a $\mathbb{C}$–linear operator on V, which we also denote by J. The eigenvalues of this operator are $\pm\sqrt{-1}$ and the corresponding eigenspaces are

$$V^{10} = \bigoplus\{V_{ij} : \varepsilon_{ij} = 1\}, \qquad V^{01} = \bigoplus\{V_{ij} : \varepsilon_{ij} = -1\},$$

for $\sqrt{-1}$ and $-\sqrt{-1}$, respectively, where (ε_{ij}) is the incidence matrix of $\mathcal{T}(J)$.

A $U(n)$–invariant Riemannian metric on $F(n)$ is completely determined by an inner product $(\cdot,\cdot)$ in $\mathfrak{p}$ invariant under T. In order to describe these inner products we start with the Cartan–Killing form on $\mathfrak{p}$,

$$\langle X, Y\rangle = \mathrm{tr}(\mathrm{adj}(X)\circ Y)\,, \qquad X, Y \in \mathfrak{p}\,, \tag{5}$$

which is an inner product invariant under T. Any invariant metric on $F(n)$ has the form

$$ds^2_\Lambda(X,Y) = -\left\langle \Lambda(X), Y\right\rangle\,, \tag{6}$$

where $\Lambda\colon \mathfrak{p}\to\mathfrak{p}$ is positive–definite with respect to $\langle\cdot,\cdot\rangle$. Furthermore, $ds^2_\Lambda(\cdot,\cdot)$ is invariant under T if and only if the elements of the standard basis $\sqrt{-1}(E_{ij}+E_{ji})$, $E_{ij}-E_{ji}$ are eigenvectors of Λ. Then, $\Lambda(E_{ij}) = \lambda_{ij}E_{ij}$ with $\lambda_{ij} > 0$ and $\lambda_{ij} = \lambda_{ji}$. It is easy to see that these metrics are Hermitian with respect to J.

The metric $ds^2_\Lambda(X,Y)$ admits a natural extension to a symetric bilinear form on the complexification V of $\mathfrak{p}$, which is also denoted by $ds^2_\Lambda(X,Y)$. Here the two–dimensional real eigenspace $\mathfrak{p}_{ij}$ of Λ, whose basis is $\sqrt{-1}(E_{ij}+E_{ji})$, $E_{ij}-E_{ji}$, extends to a complex space having basis E_{ij}.

We define the Kähler form by

$$\Omega_{J,\Lambda} = ds^2_\Lambda(X, J(Y)) = -\left\langle \Lambda(X), J(Y)\right\rangle = -\mathrm{tr}\left(\mathrm{adj}(\Lambda(X))\circ J(Y)\right), \tag{7}$$

for each $X, Y\in\mathfrak{p}$. $\Omega_{J,\Lambda}$ is a differential 2–form, so its exterior differential $d\Omega_{J,\Lambda}$ is a differential 3–form and

$$d\Omega_{J,\Lambda} = (d\Omega_{J,\Lambda})^{(3,0)} + (d\Omega_{J,\Lambda})^{(2,1)} + (d\Omega_{J,\Lambda})^{(1,2)} + (d\Omega_{J,\Lambda})^{(0,3)}\,, \tag{8}$$

where $(d\Omega_{J,\Lambda})^{(i,j)}$ is a differential form of type (i,j). It can be verified that $\big((d\Omega_{J,\Lambda})^{(3,0)}\big)^* = -(d\Omega_{J,\Lambda})^{(0,3)}$ and $\big((d\Omega_{J,\Lambda})^{(2,1)}\big)^* = -(d\Omega_{J,\Lambda})^{(1,2)}$.

$(F(n), J, ds^2_\Lambda)$ is said to be almost Kähler if $d\Omega_{J,\Lambda} = 0$ and it is called Kähler if furthermore J is integrable. $(F(n), J, ds^2_\Lambda)$ is said to be $(1,2)$–symplectic if $(d\Omega_{J,\Lambda})^{(1,2)} = 0$.

The following proposition was proved in [3] and was also studied in [10].

Proposition 7. *$(F(n), J, ds^2_\Lambda)$ is $(1,2)$–symplectic if and only if for all transitive 3–subtournaments $\{i, j, k\}$ of $\mathcal{T}(J)$ we have*

$$\lambda_{ik} = \lambda_{ij} + \lambda_{jk},$$

where $i \to j \to k$.

4. Results

In this section we present a new proof of the classification, due to Cohen, Negreiros and San Martin (see [3, Theorem 1.1]), of the $(1,2)$–symplectic metrics on $F(n)$. Using properties of locally transitive tournaments we obtain this classification in a more direct way.

Theorem 8. *Let $(F(n), J, ds^2_\Lambda)$ be the maximal flag manifold. The metric ds^2_Λ is $(1,2)$–symplectic if and only if the associated tournament $\mathcal{T}(J)$ is locally transitive.*

Proof. The necessity of the condition for $\mathcal{T}(J)$ to be a cone–free tournament or a locally transitive tournament was proved in [7]. In fact, there it was shown that if the metric ds^2_Λ is $(1,2)$–symplectic then the associated tournament $\mathcal{T}(J)$ is cone–free.

To prove the sufficiency we suppose that $\mathcal{T}(J)$ is a locally transitive tournament. Given a vertex s in $\mathcal{T}(J)$, the subtournaments $\mathcal{T}(J)^-(s)$ and $\mathcal{T}(J)^+(s)$ are transitive. We can enumerate the vertices of $\mathcal{T}(J)$ so that

$$\mathcal{T}(J)^+(s) = \{1, 2, \ldots, s-1\} \quad \text{and} \quad \mathcal{T}(J)^-(s) = \{s+1, s+2, \ldots, n\}.$$

To determine $(d\Omega_{J,\Lambda})^{(1,2)}$, we need to calculate $d\Omega(X, Y, Z)$, with $X, Y \in V^{10}$ and $Z \in V^{01}$ (see [5]). Suppose that $X = E_{ij}$, $Y = E_{jk}$ and $Z = E_{ki}$, with $i < j < k$. Then we have two cases:

1. The subtournament determined by i, j, k is contained in $\mathcal{T}(J)^+(s)$ or $\mathcal{T}(J)^-(s)$. In this case we have $i < j < k \leq s$ or $s \leq i < j < k$.
2. The subtournament determined by i, j, k is not contained in $\mathcal{T}(J)^+(s)$ or $\mathcal{T}(J)^-(s)$. In this case we can suppose $i < j < s < k$.

We will prove that there exist $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that for $i < j$,

(a) $\lambda_{ij} = \sum_{k=1}^{j-1} \lambda_k$, if $\varepsilon_{ij} = 1$;
(b) $\lambda_{ij} = \lambda_n + \sum_{k=1}^{i-1} \lambda_k + \sum_{k=j}^{n-1} \lambda_k$ otherwise; or, equivalently, $\lambda_{ij} = q - \sum_{k=i}^{j-1} \lambda_k$, where $q = \sum_{k=1}^{n} \lambda_k$.

We apply our method to put the incidence matrix in the maximal stair–shaped form. It is easy to see that Case 1 above implies (a), because in this case the 3–subtournament determined by i, j, k is transitive. Then we choose $\lambda_k = \lambda_{k(k+1)}$, for $k = 1, \ldots, n-1$.

In order to prove that Case 2 implies (b), we use induction over the number t of -1's on the top right side of the incidence matrix of $\mathcal{T}(J)$. We count these

-1's beginning with the first -1 in the first row toward the right side of the incidence matrix and following in zigzag form.

If the incidence matrix has only one -1, then it is in position $1n$ and choosing $\lambda_n = \lambda_{1n}$ we have $\lambda_{1n} = q - \sum_{k=1}^{n-1} \lambda_k$.

By the maximal stair–shaped form of the incidence matrix we can suppose that the m–th entry -1 is in a position jn, for some $s < j < n$. That is, $\varepsilon_{jn} = -1$. Then $\varepsilon_{(j-1)n} = -1$. This implies that the 3–subtournament formed by the vertices $j-1$, j and n is transitive. Then $\varepsilon_{(j-1)j} = 1$ and $\lambda_{jn} = \lambda_{(j-1)n} + \lambda_{(j-1)j}$. Now, we apply the induction hypothesis, that is, we suppose that the theorem is true for all $t < m$. Then,

$$\begin{aligned}
\lambda_{jn} &= \lambda_{(j-1)n} + \lambda_{(j-1)j} \\
&= q - \sum_{k=j-1}^{n-1} \lambda_k + \lambda_{j-1} \\
&= q - \lambda_{j-1} - \lambda_j - \cdots - \lambda_{n-1} + \lambda_{j-1} \\
&= q - \lambda_j - \cdots - \lambda_{n-1} \\
&= q - \sum_{k=j}^{n-1} \lambda_k . \qquad \square
\end{aligned}$$

The main theorem in Brouwer's paper [1] says that the number of locally transitive n–tournaments, up to permutations, is

$$\sum_{d|n} \left(\frac{2^{d-1}}{d} \ \mathrm{odd}\left(\frac{n}{d}\right) \sum_{e|\frac{n}{d}} \frac{\mu(e)}{e} \right) , \tag{9}$$

where μ is the Möbius function and $\mathrm{odd}(i)$ is equal to 1 if i is odd, and 0 if i is even. Theorem 8 implies that (9) is also the number of invariant almost complex structures which admit (1,2)–symplectic metrics on $F(n)$. Using the Euler function ϕ we derive the following simpler formula for (9).

Theorem 9. *The number of invariant almost complex structures which admit* $(1,2)$*–symplectic metrics, up to permutations, on* $F(n)$ *is*

$$\frac{1}{n} \sum_{q|n} 2^{\frac{n}{q}-1} \ \mathrm{odd}(q) \ \phi(q) \, . \tag{10}$$

Proof. In equation (9), we set $S = \sum_{e|\frac{n}{d}} \frac{\mu(e)}{e}$. If we write $q = n/d$ and $r = n/de$, then we obtain

$$S = \sum_{q/r|q} \frac{\mu(q/r)}{q/r} = \frac{1}{q} \sum_{q/r|q} r\mu(q/r) = \frac{1}{q} \sum_{r|q} r\mu(q/r). \tag{11}$$

Choose $f(n) = n$. It is known that $f(n) = \sum_{d|n} \phi(d)$. By Möbius inversion we get

$$\phi(n) = \sum_{d|n} d\mu(n/d).$$

With this, (11) can be rewritten as

$$S = \frac{\phi(q)}{q}.$$

Hence, (9) can be written in the following way:

$$\begin{aligned}
\sum_{d|n} \left(\frac{2^{d-1}}{d} \operatorname{odd}\left(\frac{n}{d}\right) \sum_{e|\frac{n}{d}} \frac{\mu(e)}{e} \right) &= \sum_{d|n} \frac{2^{d-1}}{d} \operatorname{odd}\left(\frac{n}{d}\right) \; \frac{\phi(n/d)}{n/d} \\
&= \frac{1}{n} \sum_{d|n} 2^{d-1} \operatorname{odd}\left(\frac{n}{d}\right) \; \phi\left(\frac{n}{d}\right) \\
&= \frac{1}{n} \sum_{q|n} 2^{\frac{n}{q}-1} \operatorname{odd}(q) \; \phi(q). \qquad \square
\end{aligned}$$

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References

[1] A. E. Brouwer, *The enumeration of locally transitive tournaments*, Afdeling Zuivere Wiskunde [Department of Pure Mathematics], vol. 138, Mathematisch Centrum, Amsterdam, 1980. MR 585334 (81j:05068)

[2] F. E. Burstall and S. M. Salamon, *Tournaments, flags, and harmonic maps*, Math. Ann. **277** (1987), 249–265. MR 886422 (88g:58039)

[3] N. Cohen, C. J. C. Negreiros, and L. A. B. San Martin, $(1,2)$-*symplectic metrics, flag manifolds and tournaments*, Bull. London Math. Soc. **34** (2002), 641–649. MR 1924350 (2003h:53090)

[4] ———, *A rank-three condition for invariant* $(1,2)$-*symplectic almost Hermitian structures on flag manifolds*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), 49–73. MR 1934283 (2004b:53070)

[5] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969. MR 0238225 (38 #6501)

[6] A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69), Academic Press, London, 1968/1969, pp. 341–402. MR 0262993 (41 #7598)

[7] X. Mo and C. J. C. Negreiros, $(1,2)$-*symplectic structures on flag manifolds*, Tohoku Math. J. (2) **52** (2000), 271–282. MR 1756098 (2001c:53106)

[8] ———, *Tournaments and geometry of full flag manifolds*, XI Brazilian Topology Meeting (Rio Claro, 1998), World Sci. Publishing, River Edge, NJ, 2000, pp. 183–191. MR 1835701 (2002f:53114)

[9] J. W. Moon, *Topics on tournaments*, Holt, Rinehart and Winston, New York, 1968. MR 0256919 (41 #1574)
[10] M. Paredes, *Aspectos da geometria complexa das variedades bandeira*, Doctoral Thesis, Universidade Estadual de Campinas, Brasil, 2000.
[11] ——, *Some results on the geometry of full flag manifolds and harmonic maps*, Rev. Colombiana Mat. **34** (2000), 57–89. MR 1905207 (2003g:53113)
[12] ——, *Families of* $(1,2)$*-symplectic metrics on full flag manifolds*, Int. J. Math. Math. Sci. **29** (2002), 651–664. MR 1902836 (2003m:53080)
[13] ——. *Torneos y estructuras parablicas sobre variedades bandera maximales*, Revista Integración, Vol. 17, No. 1, 1999.
[14] M. Paredes, P. Gonzlez y B. McKay, *Sobre un tipo especial de torneos y una clase de mtricas sobre variedades bandera*, Proceedings of the First Iberoamerican Conference of Computational Mathematics, Sociedad Colombiana de Matemticas, Bogot, 2001.
[15] L. A. B. San Martin and C. J. C. Negreiros, *Invariant almost Hermitian structures on flag manifolds*, Adv. Math. **178** (2003), 277-310. MR 1994221

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