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VECTOR-VALUED MODULAR FORMS AND POINCARÉ SERIES

MARVIN KNOPP AND GEOFFREY MASON

ABSTRACT. We initiate a general theory of vector-valued modular forms associated to a finite-dimensional representation ρ of $SL(2,\mathbf{Z})$. We introduce vector-valued Poincaré series and Eisenstein series and a version of the Petersson inner product, and establish analogs of basic results from the classical theory of modular forms concerning these objects, at least if the weight is large enough. In particular, we show that the space of entire vector-valued modular forms of weight k associated to ρ is a finite-dimensional vector space which, for large enough k, is nonzero and spanned by Poincaré series. We show that Hecke's estimate $a_n = O(n^{k-1})$ continues to apply to the Fourier coefficients of component functions of entire vector-valued modular forms associated to ρ for large enough k.

1. Introduction

Vector-valued modular forms have been a feature of the theory of modular forms for some time, although there seems to be no systematic development of the subject in the literature. Selberg already pointed out in [S] how they could be used in estimating the growth of Fourier coefficients of scalar modular forms. Vector-valued forms arise naturally also in the work of Eichler and Zagier on Jacobi forms [EZ, Chapter II]. In a previous paper [KM1] we considered in detail a particular class of vector-valued modular forms, and in [KM2] we obtained Hecke-type estimates on the Fourier coefficients of a vector-valued modular form associated to a representation $\rho : \Gamma \to GL(p, \mathbf{C})$. Building on [KM2], the present paper develops some of the foundations of the theory of vector-valued modular forms.

Suppose we are given a multiplier system v in weight k on Γ. As we will explain in Section 2, it is both natural and convenient to restrict consideration

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to representations ρ for which the matrix

$$e^{2\pi i\kappa}\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has finite order N, where

$$e^{2\pi i\kappa} = v\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This provides the appropriate notion of level, and we shall develop a theory of vector-valued modular forms of arbitrary real weight for *any* finite-dimensional representation of Γ which satisfies this condition. It transpires that our theory works best when the weight k is "large" compared to the dimension p of the representation ρ. This phenomenon, already observed in [KM2], is related to the efficacy of certain estimates of Eichler [E]. Eichler's estimates figure prominently in [KM2], and they underlie many of the results in the present paper. Whether there is an alternative approach which would eliminate the restrictions on k is an interesting open question. In attempting to develop a general theory of vector-valued modular forms, one naturally tries to mimic as far as possible the classical theory. It turns out that a surprisingly large portion of the standard repertoire (both theorems and proofs) carries over to the more general setting, provided one makes judicious and liberal use of the Eichler estimates.

The paper is organized as follows. In Section 2 we discuss vector-valued modular forms and the idea of level. We also introduce various spaces of vector-valued modular forms, including the spaces $\mathcal{M}(k,\rho)$ and $\mathcal{S}(k,\rho)$ of entire, respectively, cusp forms of weight k associated to ρ. On the basis of the estimates in [KM2], we show that these spaces are of finite dimension. Section 3 presents the basic idea of the paper, namely the construction of vector-valued Poincaré series (for large enough k) associated to a representation ρ. Succeeding sections are devoted to developing the properties of our Poincaré series in parallel to the classical theory. In Section 4 we introduce spaces of vector-valued Eisenstein series $\mathcal{E}(k,\rho)$, and using them we show that the classical estimate $a(n) = O(n^{k-1})$ for the growth of Fourier coefficients continues to hold in the vector-valued case if k is large enough. In Section 5 we introduce a *Petersson pairing* for vector-valued modular forms. This is an analog of the classical Petersson inner product. However, instead of being defined on $\mathcal{S}(k,\rho)$, it is a pairing between spaces of vector-valued modular forms associated to a pair of representations ρ and $\rho^\vee$, the latter representation being the complex conjugate dual of the former. (These representations thus coincide precisely when ρ is *unitary.*) Though there is generally no Hilbert space available, we are able to establish that for large enough k, $\mathcal{S}(k,\rho)$ is spanned by Poincaré series and $\mathcal{E}(k,\rho)$ is the space orthogonal to $\mathcal{S}(k,\rho^\vee)$.

In Section 6 we show that the real axis is a natural boundary for the component functions of a meromorphic vector-valued modular form. Section 7 is concerned with the case that ρ is unitary. In this case we explain that it is indeed possible to eliminate the requirement that the weight is large enough, so that we obtain a theory which is in essential agreement with the classical case but which is broader in scope. In Section 8 we point out that one can associate vector-valued Dirichlet series to a vector-valued modular form, and that there are a matrix functional equation and a converse theorem in this context.

2. The spaces $M(k, \rho, \Lambda)$ of vector-valued modular forms

We generally adopt the notation of [KM2] concerning vector-valued modular forms. Recall (loc. cit.) that a vector-valued modular form (F, ρ), or simply F, of real weight k on the modular group $\Gamma = SL(2, Z)$ is a tuple $F(\tau) = (F_1(\tau), F_2(\tau), \ldots F_p(\tau))$ of functions holomorphic in the complex upper half-plane $\mathcal{H}$, together with a p-dimensional complex representation $\rho : \Gamma \longrightarrow GL(p, C)$ satisfying the following conditions:

(a) For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$(F_1(\tau), F_2(\tau), \ldots, F_p(\tau))^t|_k V(\tau) = \rho(V)(F_1(\tau), F_2(\tau), \ldots, F_p(\tau))^t \tag{1}$$

(t refers to transpose of vectors and matrices).

(b) Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = \sum_{n \geq h_j} a_n(j) q^{n/N_j} \tag{2}$$

with N_j a positive integer. (Here and below, $q = \exp(2\pi i \tau)$.) As discussed in [KM2], the slash operator $|_k V$ in (1) is defined by

$$F|_k V(\tau) = F|_k^v V(\tau) = v(V)^{-1}(c\tau + d)^{-k} F(V\tau) \tag{3}$$

with a multiplier system v with respect to Γ which we shall often omit from the notation. As usual, we assume that v satisfies the *nontriviality condition*

$$v(-I) = (-1)^{-k}. \tag{4}$$

The *level* of (F, ρ) is the least common multiple of the denominators N_j required to fulfill eqn.(2). In contrast to [KM2], we here allow the functions $F_j(\tau)$ to have poles at infinity, i.e. h_j may be negative. However, they are still required to be holomorphic in $\mathcal{H}$. The set of all vector-valued modular forms (F, ρ) of weight k is denoted by $\mathcal{F}(k, \rho, v)$. It is a complex linear space. There are subspaces $\mathcal{M}(k, \rho, v)$ and $\mathcal{S}(k, \rho, v)$ of *entire forms* and *cusp forms*, respectively, for which we require that each h_j is nonnegative, respectively, positive. These spaces are the main objects of interest in the present paper.

Recall that two representations ρ, ρ' of Γ are called *similar* in case there is an invertible matrix U such that

$$U\rho(V)U^{-1} = \rho'(V) \tag{5}$$

for all $V \in \Gamma$. In this case it is evident that the assignment

$$F^t \mapsto UF^t$$

maps a vector-valued modular form (F, ρ) to a vector-valued modular form (FU^t, ρ') and induces linear isomorphisms $\mathcal{F}(k,\rho) \overset{\cong}{\to} \mathcal{F}(k,\rho')$, $\mathcal{M}(k,\rho) \overset{\cong}{\to} \mathcal{M}(k,\rho')$, and $\mathcal{S}(k,\rho) \overset{\cong}{\to} \mathcal{S}(k,\rho')$ of the corresponding spaces of vector-valued modular forms. Of course the image of $F = (F_1(\tau), \ldots, F_p(\tau))$ has component functions which are linear combinations of the $F_j(\tau)$, and if one is interested in particular properties of some component $F_j(\tau)$ then it may be inappropriate to use these isomorphisms. For many purposes, however (e.g., estimating growth rates of Fourier coefficients), it is useful to do so.

We use $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ for the standard generators of Γ, and define $\kappa \in \mathbf{R}$ by

$$v(T) = e^{2\pi i\kappa},\ 0 \le \kappa < 1. \tag{6}$$

Before making a start on the main results of the paper, we consider some elementary consequences of the definition of vector-valued modular form. A related discussion can be found in Section 2 of [KM2]. Let $G_1(\tau), \ldots, G_r(\tau)$ be a basis for the linear space spanned by the component functions $F_1(\tau), \ldots, F_p(\tau)$, and set $F = (F_1(\tau), \ldots, F_p(\tau)), G = (G_1(\tau), \ldots, G_r(\tau), 0, \ldots, 0)$. Then there is an invertible $p \times p$ matrix U such that

$$UF^t = G^t.$$

Let ρ' be the representation of Γ defined by (5). Then (G, ρ') is a vector-valued modular form and there are representations α, γ of Γ of dimension r and $p - r$ respectively such that

$$\rho'(V) = \begin{pmatrix} \alpha(V) & \beta(V) \\ 0 & \gamma(V) \end{pmatrix} \tag{7}$$

for $V \in \Gamma$. If $\tilde{G} = (G_1(\tau), \ldots, G_r(\tau))$ then $(\tilde{G}, \alpha)$ is also a vector-valued modular form, and it enjoys the additional property that its component functions are linearly independent. Each $G_j(\tau)$ has a Fourier expansion of the form

$$G_j(\tau) = \sum_n c_n(j) q^{n/N},$$

and from (1) and (2) applied to $(\tilde{G}, \alpha)$ we see that

$$\tilde{G}(\tau)^t = \tilde{G}(\tau + N)^t = e^{2\pi i N\kappa}\alpha(T^N)\tilde{G}(\tau)^t = (e^{2\pi i\kappa}\alpha(T))^N \tilde{G}(\tau)^t. \tag{8}$$

Thanks to the linear independence of the $G_j(\tau)$, (8) implies that the matrix $e^{2\pi i\kappa}\alpha(T)$ is of finite order dividing N. Thus it can be diagonalized, and we may assume that $\alpha(T)$ is diagonal.

We get no such information concerning the second representation γ of Γ which intervenes in (7), nor can we expect any. Indeed, suppose that $(\tilde{G}, \alpha)$ is a vector-valued modular form and γ is *any* finite-dimensional representation of Γ. Consider the direct sum $\alpha \oplus \gamma$ of the representations, where an element $V \in \Gamma$ is represented by the block matrix

$$\rho'(V) = \begin{pmatrix} \alpha(V) & 0 \\ 0 & \gamma(V) \end{pmatrix}. \tag{9}$$

With $G = (\tilde{G}, 0, \ldots, 0)$ as above it is obvious that (G, ρ') is a vector-valued modular form. On the other hand, the representation γ is not contributing any information about our vector-valued modular form—essentially, it corresponds to the zero form. In this example the representation ρ' is reducible, whereas in the general situation reflected in (7) it may not be. Nevertheless, the principle that γ is of relatively little significance is unchanged.

The upshot of this discussion is: there is no serious loss of generality if one assumes from the outset that $\rho(T)$ is diagonalizable, and that $e^{2\pi i\kappa}\rho(T)$ has finite order N. *We will always make this assumption throughout the rest of the paper.* In this situation we set

$$e^{2\pi i\kappa}\rho(T) = \begin{pmatrix} e^{2\pi i m_1} & & & & \\ & e^{2\pi i m_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{2\pi i m_p} \end{pmatrix} \tag{10}$$

and normalize the exponents $m_j \in \mathbf{Q}$ so that

$$0 < m_j \leq 1. \tag{11}$$

This leads to the Fourier expansion

$$F_j(\tau) = q^{m_j} \sum_{n \geq h_j} a_n(j) q^n \tag{12}$$

for some integer h_j. Here we have redefined h_j and $a_n(j)$ compared to their occurrence in (2). In the sequel, we always take h_j to be the *least* integer n for which $a_n(j)$ is nonzero in (12) if $F_j(\tau) \neq 0$, and $h_j = 0$ if $F_j(\tau) = 0$.

For $\Lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbf{Z}^p$, write $\Lambda \geq 0$ if $\lambda_j \geq 0$ for each j. Then $(\mathbf{Z}^p, \geq)$ is a partially ordered set where $\Lambda_1 \geq \Lambda_2$ if, and only if, $\Lambda_1 - \Lambda_2 \geq 0$. Let $\mathbf{O}, \mathbf{1}$ be the zero vector and all 1's vector, respectively. As long as (10) holds, each vector-valued modular form F defines an element $\Lambda(F) \in \mathbf{Z}^p$, namely $\Lambda(F) = (h_1, \ldots, h_p)$ where h_j is as defined above.

It is convenient to isolate those indices j for which the jth component $F_j(\tau)$ of a vector-valued modular form $F(\tau)$ has an *integral* q-expansion. These are the indices satisfying $m_j = 1$, as we see from (12). We set

$$E = \{j \mid 1 \le j \le p,\ m_j = 1\}, \tag{13}$$

$$\mathbf{E} = (j_1, \dots, j_p) \text{ with } j_i = \begin{cases} -1, & i \in E \\ 0, & i \notin E. \end{cases} \tag{14}$$

For $\Lambda \in \mathbf{Z}^p$ we define

$$\mathcal{M}(k, \rho, v, \Lambda) = \{F \in \mathcal{F}(k, \rho, v) \mid \Lambda(F) \ge \Lambda\}.$$

With this notation we have, for example,

$$\mathcal{M}(k, \rho) = \mathcal{M}(k, \rho, \mathbf{E}),\ \mathcal{S}(k, \rho) = \mathcal{M}(k, \rho, \mathbf{0}).$$

LEMMA 2.1. *Let $\Delta(\tau)$ be the usual discriminant cusp-form of weight* 12 *on* Γ*. Multiplication by* Δ *induces a linear isomorphism*

$$\mathcal{M}(k, \rho, \Lambda) \xrightarrow{\cong} \mathcal{M}(k + 12, \rho, \Lambda + \mathbf{1}).$$

Proof. Let $F = (F_1(\tau), \dots, F_p(\tau)) \in \mathcal{M}(k, \rho, \Lambda)$. The isomorphism in question is rather obviously induced by the map

$$F \mapsto F\Delta = (F_1(\tau)\Delta(\tau), \dots, F_p(\tau)\Delta(\tau)).$$

That $F\Delta \in \mathcal{M}(k + 12, \rho, \Lambda + \mathbf{1})$ follows easily from (1) and well-known properties of the discriminant, namely $\Delta(V\tau) = (c\tau + d)^{12}\Delta(\tau), V \in \Gamma$, and $\Delta(\tau) = q + O(q^2)$ is holomorphic in $\mathcal{H}$. The inverse map is induced by $F \mapsto F\Delta^{-1}$, where we use the fact that Δ does not vanish in $\mathcal{H}$ to ensure that $F\Delta^{-1}$ is holomorphic in $\mathcal{H}$ whenever F is. $\square$

Concerning the direct sum of representations (9), the following is straightforward:

LEMMA 2.2. *The map $F \oplus F' \mapsto (F, F')$ induces a linear isomorphism*

$$\mathcal{M}(k, \rho \oplus \rho', \Lambda_1 \oplus \Lambda_2) \cong \mathcal{M}(k, \rho, \Lambda_1) \oplus \mathcal{M}(k, \rho', \Lambda_2). \quad \square$$

LEMMA 2.3. *Let* Hol_k *denote the vector space of holomorphic functions in* $\mathcal{H}$*, regarded as a right* Γ*-module with respect to the group action* (3)*. Then* S^2 *acts as the identity.*

Proof. This is nothing more than the nontriviality condition (4). From (3) with $V = S^2$ we get

$$f|_k S^2(\tau) = v(S^2)^{-1}(-1)^{-k} f(\tau) = f(\tau),$$

and the lemma follows. $\square$

Now consider a representation ρ of Γ, with A the underlying linear space. Since $S^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then we may write

$$A = A^+ \oplus A^-,$$

where $A^{\pm}$ are the ± 1-eigenspaces for $\rho(S^2)$. Moreover since S^2 is in the center of Γ then both subspaces are Γ-submodules of A and correspond to representations $\rho^{\pm}$. So ρ is similar to $\rho^+ \oplus \rho^-$. However, if $F \in \mathcal{M}(k, \rho^-, d)$ then $F|_k S^2 = -F$ by definition, and therefore $F = 0$ by Lemma 2.3. So $\mathcal{M}(k, \rho^-, \Lambda) = 0$ for all triples (k, ρ, Λ) and from Lemma 2.2 we see that there is a natural identification $\mathcal{M}(k, \rho, \Lambda) \cong \mathcal{M}(k, \rho^+, \Lambda)$.

Let us call a representation ρ of Γ *normal* in case it satisfies the two conditions (a) $\rho(T)$ is diagonal, and (b) $\rho(S^2) = \mathrm{Id}$. Our discussion shows that when dealing with vector-valued modular forms there is little loss of generality in assuming that the corresponding representation is normal. *We shall make this assumption throughout the rest of the paper.*

Given a representation ρ of Γ, a certain (nonnegative) constant α, which depends only on ρ, was introduced in [KM2]. This constant will occur throughout the discussion to follow. We will have more to say about it in due course; here we simply record a result involving α (loc. cit., Lemma 4.1) that we will soon need.

LEMMA 2.4. *$\mathcal{M}(k, \rho) = 0$ whenever $k < -2\alpha$.*

Our first main result is the following.

THEOREM 2.5. *Each space $\mathcal{M}(k, \rho, \Lambda)$ has finite dimension. In particular, setting $\delta = p(1 + \alpha/6)$ we have*

$$\dim \mathcal{M}(k, \rho) \leq \frac{pk}{12} + \delta. \tag{15}$$

Proof. Let l be the integer satisfying

$$-2\alpha - 12 \leq k - 12l < -2\alpha. \tag{16}$$

By Lemma 2.1 we have $\mathcal{M}(k, \rho, \Lambda) \cong \mathcal{M}(k - 12l, \rho, \Lambda - l\mathbf{1})$. Moreover the latter space contains no nonzero entire vector-valued modular forms thanks to Lemma 2.4 and the second inequality in (16).

Now a typical vector-valued modular form $F \in \mathcal{M}(k - 12l, \rho, \Lambda - l\mathbf{1})$ has component functions $F_1(\tau), \ldots, F_p(\tau)$ with q-expansions (2). In particular, there are only a finite number of nonzero Fourier coefficients of the $F_j(\tau)$'s which correspond to negative powers of q. Mapping F onto any one such Fourier coefficient defines a linear map $\mathcal{M}(k - 12l, \rho, \Lambda - l\mathbf{1}) \to \mathbf{C}$, and the intersection of the kernels of these maps is trivial, being the subspace of entire vector-valued modular forms. Since the intersection of the kernels has finite codimension, the first assertion of the theorem is established.

Now let us specialize to the case $\Lambda = \mathbf{E}$ (cf. (13). In this case there are no more than pl nonzero Fourier coefficients corresponding to negative powers of q in the Fourier expansions of the component functions of a vector-valued modular form $F \in \mathcal{M}(k - 12l, \rho, \mathbf{E} - l\mathbf{1})$. Using the first inequality in (16) yields

$$\dim \mathcal{M}(k, \rho) = \dim \mathcal{M}(k - 12l, \rho, \mathbf{E} - l\mathbf{1}) \leq pl \leq \frac{pk}{12} + \delta,$$

as claimed. This completes the proof of the theorem. □

Once we have developed the theory of Poincaré series in the next section, we shall be able to give a lower bound for the dimension of $\mathcal{M}(k, \rho)$ for $k > 2+2\alpha$. Of course, it is of interest to determine the *precise* dimension when possible. If $p = 1$ then $\mathcal{M}(k, \rho)$ is a space of classical entire modular forms on Γ of weight k. Then $\alpha = 0$, leading to $\dim \mathcal{M}(k, \rho) \leq 1 + \frac{k}{12}$. We can say a little bit more about the general case.

Suppose that $\chi : \Gamma \to \mathbf{C}$ is a character and ρ a representation of Γ. We denote the corresponding space of *semiinvariants* by ρ^{χ}. Thus,

$$\rho^{\chi} = \{u \in \mathbf{C}^p \mid \rho(V)u = \chi(V)u, \text{all } V \in \Gamma\}. \tag{17}$$

Lemma 2.6. $\rho^{\overline{v}}$ *is the space of* constant *vector-valued modular forms in* $\mathcal{M}(0, \rho, v)$. *(Recall that v is a character of Γ when $k = 0$.)*

Proof. From (1) we see that a vector of scalars $F = (F_1, \dots, F_p) \in \mathbf{C}^p$ lies in $M(0, \rho, v)$ if, and only if, $\rho(V)F^t = v(V)^{-1}F^t$ for all $V \in \Gamma$. □

Taking $k = 0$ in (15) together with the last lemma yields

$$\dim \rho^{\overline{v}} \leq \dim \mathcal{M}(0, \rho) \leq \delta.$$

Remark. Based on the classical case, one might think that we always have $\rho^{\overline{v}} = \mathcal{M}(0, \rho)$. That is, an entire vector-valued modular form of weight 0 is necessarily constant. However, this is *false* for a general representation ρ. Counterexamples are constructed in [KM1]. On the other hand, it seems likely that one generally has $\dim \mathcal{M}(0, \rho) \leq p$.

3. Poincaré series

In this section we begin to develop a theory of Poincaré series for vector-valued modular forms that runs parallel to the classical theory ([L], [R]). In particular, we will see that for *any* finite-dimensional representation $\rho : \Gamma \to GL(p, \mathbf{C})$, the space $M(k, \rho)$ is nonzero for all large enough weights k.

Fix a normal representation ρ of Γ of dimension p and multiplier system v in weight k. Let $e_j = (0, \dots, 1, \dots, 0)^t$ denote the jth standard basis (column) vector in $\mathbf{C}^p$, so that e_j has entry 1 in the jth position and 0 elsewhere.

Definition (Poincaré series). Fix integers ν and r with $1 \leq r \leq p$. The Poincaré series $P(\tau)$ is defined as

$$P(\tau; \rho, k, v, \nu, r) = \frac{1}{2} \sum_{M} \frac{\exp[2\pi i(\nu + m_r)M\tau]}{v(M)(c\tau + d)^k} \rho(M)^{-1} e_r, \tag{18}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ ranges over a set of coset representatives for $< T > \backslash \Gamma$ and m_r is as in (10), (11).

Of course (18) defines a *column* vector of functions rather than a row vector. We are going to show that the transpose $P(\tau)^t$ is a vector-valued modular form of weight k. To begin with, suppose that we replace M by $T^n M = \left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right)$ in (18). The corresponding term in the sum then acquires an additional factor

$$\frac{e^{2\pi i n m_r}}{v(T^n)}\rho(T^n)^{-1}_{rr} = \frac{e^{2\pi i n m_r}}{e^{2\pi i n\kappa}}e^{-2\pi i n(m_r-\kappa)} = 1.$$

This shows that (18) is well-defined. As a result we may, and later shall, use the set of coset representatives denoted by $\mathcal{M}$ in [KM2] (cf. [K1], [L]).

Next we show *formally* (that is, assuming absolute convergence, so that we can rearrange terms at will) that $P(\tau)$ is invariant with respect to the action $|_k$ of Γ. To this end, let $V = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ and write

$$M' = MV = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix}.$$

Then

$$\begin{aligned}
P|_k^v V(\tau) &= \frac{1}{2}\sum_M \frac{\exp[2\pi i(\nu+m_r)MV\tau]}{v(M)v(V)(cV\tau+d)^k(\gamma\tau+\delta)^k}\rho(M)^{-1}e_r \\
&= \frac{1}{2}\sum_M \frac{\exp[2\pi i(\nu+m_r)M'\tau]}{v(MV)(c'\tau+d')^k}\rho(M)^{-1}e_r \\
&= \frac{1}{2}\sum_{M'} \frac{\exp[2\pi i(\nu+m_r)M'\tau]}{v(M')(c'\tau+d')^k}\rho(M'V^{-1})^{-1}e_r \\
&= \frac{1}{2}\rho(V)\sum_{M'} \frac{\exp[2\pi i(\nu+m_r)M'\tau]}{v(M')(c'\tau+d')^k}\rho(M')^{-1}e_r \\
&= \rho(V)P(\tau),
\end{aligned}$$

where we have rearranged terms and used independence of choice of coset representatives. This establishes formal invariance of $P(\tau)$ under the action of Γ.

PROPOSITION 3.1. *If $k > 2+2\alpha$ then the component functions $P_j(\tau)$ of $P(\tau;\rho,k,v,\nu,r), 1 \le j \le p$, converge absolutely-uniformly on compact subsets of $\mathcal{H}$. In particular, each $P_j(\tau)$ is holomorphic in $\mathcal{H}$.*

Proof. Clearly

$$P_j(\tau) = \frac{1}{2}\sum_M \exp[2\pi i(\nu+m_r)M\tau]v(M)^{-1}(c\tau+d)^{-k}\rho(M^{-1})_{jr}, \tag{19}$$

and we have to estimate the exponential term and the matrix element. The former estimate is standard. Setting $\tau = x + iy$, use (11) to see that

$$|\exp[2\pi i(\nu + m_r)M\tau]| \leq \exp(2\pi|\nu|/y), \tag{20}$$

for $c \neq 0$. Note that $c = 0$ corresponds to the terms $M = \pm I$ in (19). As for the expression $\rho(M^{-1})_{jr}$, we estimate it using results from [KM2]. Namely, if $p = \dim\rho$ then display (9) of (loc. cit.) shows that

$$|\rho(M^{-1})_{jr}| \leq p^{-1}(pK_1)^{L(M)},$$

where $L(M)$ is the *Eichler length* of M (and satisfies $L(M) = L(M^{-1})$), and where K_1 is a positive constant depending only on ρ. By (7) (loc. cit.), the previous inequality may be put in the form

$$|\rho(M^{-1})_{jr}| \leq p^{-1}(pK_1)^{n_2}\mu(M)^{\alpha}. \tag{21}$$

Here, α is the constant that was introduced prior to Lemma 2.4. Indeed,

$$\alpha = n_1 \log(pK_1), \tag{22}$$

$$\mu(M) = a^2 + b^2 + c^2 + d^2 \tag{23}$$

and n_1, n_2 are universal constants. Now we specialize to the set of coset representatives $M \in \mathcal{M}$. For such M we have (loc. cit.)

$$\mu(M) \leq K_3(c^2 + d^2), \tag{24}$$

and from Lemma 4 of [K1] and display (13) of [KM2] we see that $c^2 + d^2 \leq |c\tau + d|^2(1 + |\tau|^2)/y^2$. So for $M \in \mathcal{M}$, (21) shows that

$$|\rho(M^{-1})_{jr}| \leq K_4|c\tau + d|^{2\alpha}(1 + |\tau|^2)^{\alpha}/y^{2\alpha} \tag{25}$$

for a constant K_4 depending only on ρ. Combine (25) and (20) to see that for $c \neq 0$, the absolute value of the summand in (19) is bounded above by

$$K_4|c\tau + d|^{-(k-2\alpha)}(1 + |\tau|^2)^{\alpha}/y^{2\alpha}.$$

Now a standard argument shows that if $k - 2\alpha > 2$ then (19) converges absolutely uniformly on compact subsets of $\mathcal{H}$, and the proposition is proved. $\square$

It remains for us to consider the q-expansions of the component functions $P_j(\tau)$. From (19) we have

$$P_j(\tau) = \frac{1}{2}\sum_{c=-\infty}^{\infty}\sum_{\substack{d=-\infty \\ (c,d)=1}}^{\infty} \exp[2\pi i(\nu + m_r)M\tau]v(M)^{-1}(c\tau + d)^{-k}\rho(M^{-1})_{jr}.$$

By assumption ρ is a normal representation, in particular $\rho(\pm M) = \rho(M)$. It follows from this and (4) that M and $-M$ make equal contributions to the right side of the previous equation. Noting that the terms with $c = 0$ correspond to $M = \pm I$ we obtain

$$P_j(\tau) = \delta_{jr} \exp[2\pi i(\nu + m_r)\tau] \\ + \sum_{c=1}^{\infty} \sum_{\substack{d=-\infty \\ (c,d)=1}}^{\infty} \exp[2\pi i(\nu + m_r)M\tau] \upsilon(M)^{-1}(c\tau + d)^{-k}\rho(M^{-1})_{jr}.$$

From this expression we can derive formulas for the Fourier coefficients of $P_j(\tau)$ by the standard method of invoking the Lipschitz summation formula [KR, Theorem 1]. (See, e.g., [L, pp. 295–299] or [R, pp. 155–164].) The explicit expressions that arise will be familiar to those who have studied the Fourier coefficients of classical (i.e. scalar) modular forms. To state these we introduce the Bessel functions of the first kind [Wa]:

$$J_n(z) = \sum_{t=0}^{\infty} \frac{(-1)^t (z/2)^{n+2t}}{t!\Gamma(n+t+1)}, \tag{26}$$

$$I_n(z) = \sum_{t=0}^{\infty} \frac{(z/2)^{n+2t}}{t!\Gamma(n+t+1)}. \tag{27}$$

We omit the proof, stating the result as

Theorem 3.2. *Let ρ be a normal representation of Γ of dimension p. For integers ν, r with $1 \le r \le p$, let $P(\tau) = P(\tau; \rho, k, \upsilon, \nu, r)$ be the Poincaré series* (18). *As long as $k > 2 + 2\alpha$, $P(\tau) \in \mathcal{F}(k, \rho)$ is a vector-valued modular form of weight k. Setting $\sigma_r = \nu + m_r$ we have*

$$P_j(\tau) = \delta_{j,r} q^{\sigma_r} + q^{m_j} \sum_{n=0}^{\infty} c_n(j; k, \rho, \upsilon, \nu, r) q^n, \tag{28}$$

where the Fourier coefficients $c_n(j; k, \rho, \upsilon, \nu, r)$ are as follows and exactly one of the following holds:

Case (a)*: $\sigma_r = \nu + m_r > 0$.*

$$P_j(\tau) = \delta_{jr} q^{\sigma_r} + \sum_{n=0}^{\infty} c_n(j; k, \rho, \upsilon, \nu, r) q^{n+m_j},$$

where

$$c_n(j; k, \rho, \upsilon, \nu, r) = 2\pi i^{-k} \sum_{c=1}^{\infty} c^{-1} A_c^{(j,r)}(n; \rho, \upsilon, \nu) \left(\frac{(n+m_j)}{\sigma_r}\right)^{(k-1)/2} \\ J_{k-1}\left(\frac{4\pi}{c}\sqrt{\sigma_r(n+m_j)}\right),$$

and

$$(29)\quad A_c^{(j,r)}(n;\rho,v,\nu) = \sum_{\substack{-d'=0\\(c,d')=1}}^{c-1} v(M_{c,d'})^{-1}\rho(M_{c,d'}^{-1})_{jr}\exp\left\{\frac{2\pi i}{c}[\sigma_r a+(n+m_j)d']\right\}.$$

In this case $P(\tau)$ is a cusp form, possibly equal to 0.

Case (b)*: $\sigma_r = 0$, i.e. $m_r = 1 = -\nu$.*

$$P_j(\tau) = \delta_{jr} + \sum_{n=0}^{\infty} c_n(j;k,\rho,v,r)q^{n+m_j},$$

where

$$(30)\quad c_n(j;k,\rho,v,r) = \frac{(-2\pi i)^k}{\Gamma(k)}\sum_{c=1}^{\infty} c^{-k}A_c^{(j,r)}(n;\rho,v,-1)(n+m_j)^{k-1},$$

and

$$A_c^{(j,r)}(n;\rho,v,-1) = \sum_{\substack{-d'=0\\(c,d')=1}}^{c-1} v(M_{c,d'})^{-1}\rho(M_{c,d'}^{-1})_{jr}\exp\left(\frac{2\pi i}{c}(n+m_j)d'\right).$$

In this case $P(\tau)$ is entire and $P_r(\tau) = 1+\dots$ is the unique component of $P(\tau)$ which does not vanish at $i\infty$.

Case (c)*: $\sigma_r = \nu + m_r < 0$.*

$$P_j(\tau) = \delta_{j,r}q^{\sigma_r} + \sum_{n=0}^{\infty} c_n(j;k,\rho,v,\nu,r)q^{n+m_j},$$

where

$$(31)\quad c_n(j;k,\rho,v,\nu,r) = 2\pi i^{-k}\sum_{c=1}^{\infty} c^{-1}A_c^{(j,r)}(n;\rho,v,\nu)\left(\frac{(n+m_j)}{-\sigma_r}\right)^{(k-1)/2} I_{k-1}\left(\frac{4\pi}{c}\sqrt{-\sigma_r(n+m_j)}\right).$$

In this case $P_r(\tau) = q^{\sigma_r}+\dots$ has a pole at $i\infty$, and all other components of $P(\tau)$ are entire. □

Remark. The expression $A_c^{(j,r)}(n;\rho,v,\nu)$ (29) may be considered to be a *generalized Kloosterman sum*, an analog of the classical Kloosterman sum.

4. Eisenstein series and estimates for Fourier coefficients

We continue with the notation of the previous section, and start with the

DEFINITION (Eisenstein series). Let ρ be a normal representation of Γ and let $k > 2+2\alpha$. Let $\mathcal{E}(k,\rho,v)$ be the space spanned by the Poincaré series $P(\tau;\rho,k,v,-1,r)$ where r ranges over the indices in E (cf. (13)) and $\nu=-1$. Elements of $\mathcal{E}(k,\rho)$ are called (vector-valued) *Eisenstein series.*

REMARK. We will give an *intrinsic* characterization of the space of Eisenstein series in the next section.

THEOREM 4.1. *Let ρ be a normal representation of Γ and let $k > 2+2\alpha$. Then the following holds:*

$$\mathcal{M}(k,\rho)=\mathcal{E}(k,\rho)\oplus\mathcal{S}(k,\rho). \tag{32}$$

Moreover

$$\dim\mathcal{E}(k,\rho)=|E|.$$

Proof. As long as $r\in E$, i.e. $m_r=1$, we may take $\nu=-1$ and, using Theorem 3.2, obtain an entire Poincaré series $P(\tau;\rho,k,v,-1,r)\in\mathcal{E}(k,\rho)$. Furthermore, as r varies the corresponding series are clearly linearly independent. The decomposition (32) is then an immediate consequence of Theorem 3.2, and the present theorem is proved. □

THEOREM 4.2. *Let the assumptions be as in Theorem 4.1. Then*

$$\dim\mathcal{S}(k,\rho)\geq\frac{pk}{12}-\delta-\frac{p}{6},$$

where δ is defined in Theorem 2.5.

Proof. Let l be the integer satisfying

$$k-12l>2+2\alpha\geq k-12l-12. \tag{33}$$

Note that we have $l\geq 0$ as a consequence of the second inequality together with the assumption that $k>2+2\alpha$. By Lemma 2.1 we have $\mathcal{M}(k,\rho)=\mathcal{M}(k,\rho,\mathbf{E})\cong\mathcal{M}(k-12l,\rho,\mathbf{E}-l\mathbf{1})$. We may apply Theorem 3.2 to the latter space because $k-12l>2+2\alpha$. It tells us that the subspace spanned by those Poincaré series occurring in Theorem 3.2 which are not entire is exactly pl-dimensional. In addition, there is the space of Eisenstein series $\mathcal{E}(k-12l,\rho)\subseteq\mathcal{M}(k-12l,\rho,\mathbf{E}-l\mathbf{1})$ of dimension $|E|$ (cf. Theorem 4.1). Hence,

$$\begin{aligned}\dim\mathcal{M}(k,\rho)&=\dim\mathcal{M}(k-12l,\rho,\mathbf{E}-l\mathbf{1})\\&\geq pl+|E|\geq\frac{pk}{12}-\delta+|E|-\frac{p}{6}\end{aligned}$$

thanks to the second inequality in (33). Since $\dim \mathcal{S}(k,\rho) = \dim \mathcal{M}(k,\rho) - |E|$ by Theorem 4.1, the present theorem follows. □

In [KM2] we obtained estimates

$$c_n(j) = \begin{cases} O(n^{k+2\alpha}), & F \text{ entire} \\ O(n^{k/2+\alpha}), & F \text{ cuspidal} \end{cases}$$

for the Fourier coefficients $c_n(j)$ of the jth component of an entire vector-valued modular form $F(\tau)$. We are going to use the Poincaré series to show that the *classical* estimate continues to hold for entire vector-valued modular forms of large enough weight. Precisely, we will prove

THEOREM 4.3. *Suppose that $F(\tau) \in \mathcal{M}(k,\rho)$ is an entire vector-valued modular form of weight $k > 2 + 2\alpha$. Then $c_n(j) = O(n^{k-1})$ for $n \to \infty$.*

Proof. Note that $k/2 + \alpha < k - 1$. So if $F(\tau)$ is cuspidal then the desired estimate is no better than that already obtained in [KM2]. We may therefore assume that $F(\tau)$ is not cuspidal. By Theorem 4.1 we may, and shall, further assume that $F(\tau)$ is an Eisenstein series $P(\tau;\rho,k,v,-1,r)$.

Next we need the 'trivial' estimate for our generalized Kloosterman sum $A_c^{(j,r)}(n;\rho,v,\nu)$.

PROPOSITION 4.4. *We have*

$$|A_c^{(j,r)}(n;\rho,v,\nu)| \le Kc^{1+2\alpha}$$

for a constant K depending only on ρ. □

We omit the proof, a straightforward application of (10) and (21)–(24).

Theorem 4.3 follows immediately from Proposition 4.4. □

We can also get an *asymptotic estimate* for the Fourier coefficients $c_n(j;k,\rho,v,\nu,r)$ of the corresponding Poincaré series in the case that there is a pole, i.e. $\sigma_r < 0$. The result is as follows:

THEOREM 4.5. *Suppose that $k > 2 + 2\alpha$. Then we have*

$$c_n(j;k,\rho,v,\nu,r) \\ \sim \frac{A_1^{(j,r)}(n;\rho,v,\nu)}{2\sqrt{2\pi}|\sigma_r|^{k-3/4}}(n+m_j)^{k-5/4}\exp\left(4\pi\sqrt{|\sigma_r|(n+m_j)}\right).$$

REMARKS. (i) $A_1^{(j,r)}(n;\rho,v,\nu)$ is bounded by Proposition 4.4.
(ii) The estimate is *independent* of α.

The proof of this estimate is standard, an easy deduction from the following two elementary inequalities:

$$I_{k-1}(x) \le x^{k-2}\sinh x \text{ for } x \ge 0, k \ge 2$$

(cf. (27)),

$$\sinh x \leq \frac{x\ \sinh B}{B}, 0 \leq x \leq B,$$

and the asymptotic estimate [Wa, p. 203]

$$I_{k-1}(x) \sim \frac{e^x}{\sqrt{2\pi x}} \text{ for } x \to \infty. \qquad \square$$

5. The Petersson pairing

In this section we show how to define a Petersson scalar product for vector-valued modular forms, and draw some conclusions. Given a representation $\rho : \Gamma \to GL(p, \mathbf{C})$, there is a second representation $\rho^\vee$ defined by

$$\rho^\vee(V) = \overline{\rho(V^{-1})}^t, V \in \Gamma.$$

Note that $\rho^{\vee\vee}(V) = \rho(V)$ and that ρ is *unitary* precisely when $\rho(V) = \rho^\vee(V)$ for all V. Furthermore if ρ is a normal representation then $\rho(T) = \rho^\vee(T)$ and $\rho(S^2) = \rho^\vee(S^2)$. In particular, ρ is normal if, and only if, $\rho^\vee$ is normal.

For row vectors $F = (F_1, \dots, F_p), G = (G_1, \dots, G_p)$ we write the usual dot product as $F \cdot G = FG^t = F_1G_1 + \cdots + F_pG_p$. Now suppose that $F \in \mathcal{M}(k, \rho, v)$ and $G \in \mathcal{S}(k, \rho^\vee, v)$. We define

$$< F, G >= \int_{\mathcal{R}} F(\tau) \cdot \overline{G(\tau)} y^k \frac{dxdy}{y^2}, \tag{34}$$

where $\mathcal{R}$ is any fundamental region for Γ.

Lemma 5.1. *The integral* (34) *is* independent *of the choice of fundamental region.*

Proof. Note that if $V \in \Gamma$ then

$$\begin{aligned}
(F|_k V(\tau)) \cdot \overline{(G|_k V(\tau))} &= (\rho(V) F(\tau)^t)^t \ \overline{\rho^\vee(V) G(\tau)^t} \\
&= F(\tau)\rho(V)^t \overline{\overline{\rho(V^{-1})^t} G(\tau)^t} \\
&= F(\tau)\rho(V)^t \rho(V^{-1})^t \overline{G(\tau)}^t \\
&= F(\tau)\overline{G(\tau)}^t,
\end{aligned}$$

so that

$$(F(V(\tau)) \cdot (\overline{G(V(\tau))}) = F(\tau)\overline{G(\tau)}^t |c\tau + d|^{2k}.$$

The lemma is a standard consequence of the last equality. $\square$

Lemma 5.2. *The integral* (34) *is absolutely convergent.*

Proof. By Lemma 5.1 we may assume that $\mathcal{R}$ is the standard fundamental region for Γ bounded by the lines $\mathrm{Re}(\tau) = \pm 1/2$ and an arc of the unit circle. We have

$$< F, G >= \sum_{i=1}^{p} \int_{\mathcal{R}} F_i(\tau)\overline{G_i(\tau)} y^k \frac{dxdy}{y^2}$$

and by hypothesis each product $F_i(\tau)\overline{G_i(\tau)}$ is cuspidal in the sense that its q-expansion vanishes at $i\infty$. By a standard argument, each of the integrals in the previous expression is absolutely convergent, and the lemma follows. □

Remark. Let $F(\tau)$ and $G(\tau)$ be as above. It is clear that the previous two lemmas continue to hold as long as $\sum_i F_i(\tau)\overline{G_i(\tau)}$ vanishes at $i\infty$.

As a result of the previous two lemmas, the map $(F(\tau), G(\tau)) \mapsto < F, G >$ defines a pairing

$$\mathcal{M}(k,\rho) \ \times \ \mathcal{S}(k,\rho^\vee) \to \mathbf{C} \tag{35}$$

which is linear in the first argument and conjugate-linear in the second. We refer to it as the *Petersson pairing.* As expected, Poincaré series behave well with respect to the Petersson pairing:

Theorem 5.3. *Let ρ be a normal representation with $P(\tau) = P(\tau; \rho, k, v, \nu, r)$ an entire Poincaré series* (18) *of weight $k > 2 + 2\alpha$ (thus, $\sigma_r \geq 0$). Let $G(\tau) \in \mathcal{S}(k, \rho^\vee, v)$ be a cusp form whose rth component has Fourier expansion*

$$G_r(\tau) = q^{m_r} \sum_{n=0}^{\infty} b_n(r) q^n.$$

Then

$$< P(\tau)^t, G(\tau) >= \begin{cases} \overline{b_\nu(r)} \ \frac{\Gamma(k-1)}{(4\pi\sigma_r)^{k-1}}, & \sigma_r > 0 \\ 0, & \sigma_r = 0 \end{cases} . \tag{36}$$

The proof of Theorem 5.3 follows along lines familiar from the classical case, and will be omitted. (See [L, pp. 286–288] or [R, pp. 149–151].) □

With regard to the Petersson pairing (35), for a subspace U of $\mathcal{M}(k,\rho)$ or $\mathcal{S}(k,\rho^\vee)$ we let $U^\perp$ denote the subspace of $\mathcal{S}(k,\rho^\vee)$, resp. $\mathcal{M}(k,\rho)$, which is *orthogonal* to U.

Theorem 5.4. *Suppose that ρ is a normal representation and $k > 2+2\alpha$. Then the following hold:*

(a) *$S(k,\rho)$ is* spanned *by Poincaré series.*

(b) *The Petersson pairing induces a* perfect *pairing*

$$\mathcal{S}(k,\rho) \;\times\; \mathcal{S}(k,\rho^\vee) \to \mathbf{C}, \tag{37}$$

i.e. $\mathcal{S}(k,\rho)^\perp = \mathcal{S}(k,\rho) \cap S(k,\rho^\vee)^\perp = 0$. *Furthermore,* $\dim \mathcal{S}(k,\rho) = \dim \mathcal{S}(k,\rho^\vee)$.

Proof. Note first that by Theorem 2.5, the spaces $\mathcal{S}(k,\rho)$ and $\mathcal{S}(k,\rho^\vee)$ have finite dimension. Moreover, we may assume without loss that $\dim \mathcal{S}(k,\rho) \leq \dim \mathcal{S}(k,\rho^\vee)$. If not, simply reverse the rôles of ρ and $\rho^\vee$.

Let $\mathcal{P} \subseteq \mathcal{S}(k,\rho)$ be the subspace spanned by the $P(\tau;\rho,k,v,\nu,r)$ for which $\sigma_r > 0$ (the Poincaré series which are cuspidal), and let $G(\tau) \in \mathcal{P}^\perp$. From Theorem 5.3 it follows that for each r, all Fourier coefficients of the rth component function $G_r(\tau)$ vanish. So we must have $G(\tau) = 0$, and this shows that in fact $\mathcal{P}^\perp = 0$. It is a consequence of elementary linear algebra together with the existence of the pairing (37) and the vanishing of $\mathcal{P}^\perp$ that

$$\dim \mathcal{S}(k,\rho^\vee) \leq \dim \mathcal{P} \leq \dim \mathcal{S}(k,\rho).$$

So in fact the inequalities are both equalities, and we have $\mathcal{S}(k,\rho) = \mathcal{P} \cong \mathcal{S}(k,\rho^\vee)$. In particular, (a) holds. Furthermore we have $\mathcal{S}(k,\rho)^\perp = 0$ since $\mathcal{S}(k,\rho)^\perp \subseteq \mathcal{P}^\perp = 0$. Reversing the rôles of ρ and $\rho^\vee$ similarly shows that $\mathcal{S}(k,\rho) \cap S(k,\rho^\vee)^\perp = 0$. This completes the proof of the theorem. □

Remark. Quantitative improvements on part (a) of the theorem are possible. We present one in the next section.

Theorem 5.5. *Suppose that ρ is a normal representation and $k > 2+2\alpha$. Then*

$$\mathcal{S}(k,\rho^\vee)^\perp = \mathcal{E}(k,\rho).$$

Proof. $\mathcal{E}(k,\rho)$ is spanned by those Poincaré series $P(\tau;\rho,k,v,\nu,r)$ with $m_r = -\nu = 1$ (cf. Theorem 3.2), and we know that these lie in $\mathcal{S}(k,\rho^\vee)^\perp$ by Theorem 5.3. Certainly then, we have $\mathcal{E}(k,\rho) \subseteq \mathcal{S}(k,\rho^\vee)^\perp$. That this containment is one of equality now follows from Theorems 4.1 and 5.4(b). This completes the proof of the theorem. □

6. The natural boundary

Lemma 6.1. *Suppose that $\varphi(\tau)$ is defined in $\mathcal{H}$, k is a real number, and that one of the following further conditions holds:*

(a) $k \geq 0, \varphi(\tau)$ *is continuous at a rational point* $a/c, (a,c) = 1, c \neq 0$, $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, *and* $\varphi|_k V(\tau)$ *is periodic.*

(b) $k < 0, \varphi(\tau)$ *is holomorphic in a region containing both* $\mathcal{H}$ *and a nonempty interval* I *of the real axis, and* $\varphi|_k V(\tau)$ *is periodic for each* $V \in \Gamma$.

Then φ is identically zero if $k \neq 0$ and constant if $k = 0$.

Proof. Let $N = N_V$ be a positive integral period for $\varphi|_k V(\tau)$ whenever this function is in fact periodic. Then we have for all n that

$$\frac{(c\tau+d)^k}{(c(\tau+nN)+d)^k}\varphi\left(\frac{a+b/(\tau+nN)}{c+d/(\tau+nN)}\right) = \varphi(V\tau). \tag{38}$$

If $k > 0$, so that (a) holds, letting $n \to \infty$ in (38) shows that $\varphi(V\tau) = 0$ for all $\tau \in \mathcal{H}$. Similarly if $k = 0$, letting $n \to \infty$ shows that $\varphi(a/c) = \varphi(V\tau)$ for all $\tau \in \mathcal{H}$. Finally, if $k < 0$, writing (38) in the form

$$\varphi\left(\frac{a+b/(\tau+nN)}{c+d/(\tau+nN)}\right) = \frac{(c\tau+d)^{-k}}{(c(\tau+nN)+d)^{-k}}\varphi(V\tau)$$

shows that $\varphi(a/c) = 0$ for all rational a/c in the interval I . Because (b) holds in this case, we again conclude that $\varphi(\tau) \equiv 0$. This completes the proof of the lemma. □

Lemma 6.2. *Let $F(\tau)$ be a (meromorphic) vector-valued modular form in $\mathcal{F}(k, \rho)$, and assume that for some index j, the Fourier expansion of the jth component $F_j(\tau)$ reduces to a* finite *sum. Then one of the following holds:*

(a) $F_j(\tau) = 0$.
(b) $k = 0$ *and* $F_j(\tau)$ *is a constant.*

Proof. Because $F_j(\tau)$ has a finite q-expansion, it satisfies the conditions imposed upon $\varphi(\tau)$ in Lemma 6.1. The present lemma follows immediately. □

Remark. The last two lemmas show that the real axis is a *natural boundary* for each nonconstant component of a (meromorphic) vector-valued modular form.

Lemma 6.3. *Suppose that ρ is a normal representation, and that $k > 2 + 2\alpha$. Then any* cofinite *set of cuspidal Poincaré series spans $\mathcal{S}(k, \rho)$.*

Proof. Let $\mathcal{P}$ denote the set of cuspidal Poincaré series (18) (i.e. those with $\sigma_r > 0$), and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be a cofinite subset—that is, $\mathcal{P}_0$ contains all but finitely many elements of $\mathcal{P}$. The assertion to be proved is that $\mathcal{P}_0$ spans $\mathcal{S}(k, \rho)$. If not, by Theorem 5.4(b) we can find $0 \neq G(\tau) \in \mathcal{P}_0^{\perp} \cap \mathcal{S}(k, \rho^{\vee})$. Theorem 5.3 then informs us that all but a finite number of Fourier coefficients of each of the component functions of $G(\tau)$ vanish. Since $G(\tau) \neq 0$, this can happen only if $k = 0$ by Lemma 6.2, a contradiction which completes the proof. □

7. The unitary case

It turns out that if the representation ρ is *unitary*, so that $\rho : \Gamma \to U(p, \mathbf{C})$, then the results obtained in previous sections may be considerably strengthened. This is due to the observation that in this case, the constant α may be taken to be 0. This assertion follows from the analysis in Section 3 of [KM2]. Namely, in the unitary case each matrix entry satisfies

$$|\rho_{jm}(V)| \leq 1.$$

Then (with notation as in [KM2]), equation (11) (loc. cit.) may be replaced by

$$g_j(Vz) < K_2|cz + d|^{k-2\sigma} v^{\delta\sigma}$$

in which the constant α no longer appears. The rest of the argument in (loc. cit.) depends only on this inequality, so that, for example, we obtain the estimate

$$a_n(j) = O(n^k) \text{ as } n \to \infty \tag{39}$$

for the Fourier coefficients $a_n(j)$ of the jth component of a unitary, entire vector-valued modular form. From this, it follows that Lemma 2.4 and Theorem 2.5 continue to hold with $\alpha = 0$ in the unitary case. In particular,

$$\dim \mathcal{M}(k, \rho) \leq p\left(1 + \frac{k}{12}\right). \tag{40}$$

For similar reasons, our construction of Poincaré series applies for all weights $k > 2$ in the unitary case, and the main results of Sections 4 and 5 also hold for $k > 2$ in this case. (We have already noted that in the unitary case, the space of cusp forms $\mathcal{S}(k, \rho)$ is a Hilbert space with respect to the Petersson pairing.)

We turn now to a discussion of some unitary representations of Γ. Our intention is not to discuss systematically what is certainly an intricate subject, but rather to illustrate that Γ has a large number of finite-dimensional unitary representations which are not of the type classically considered in the context of modular forms, and to which our previous comments apply. In order to simplify the discussion a bit, we will consider only representations ρ for which $\rho(T)$ has finite order N (cf. the discussion in Section 2). Since the modular group $PSL(2, Z)$ is the free product of the group of order 2 and the group of order 3, we are concerned with representations of the group

$$\tilde{\Gamma}(N) =< x, y \,|x^2 = y^3 = (xy)^N = 1 > . \tag{41}$$

Let $\Delta(N)$ be the normal closure of $\pm T^N$ in Γ, that is,

$$\Delta(N) =< \pm V T^N V^{-1} \mid V \in \Gamma > .$$

Then $\tilde{\Gamma}(N) \cong \Gamma/\Delta(N)$, and there is a short exact sequence

$$1 \to \pm\Gamma(N)/\Delta(N) \to \tilde{\Gamma}(N) \to PSL(2, Z/NZ) \to 1. \tag{42}$$

Here, $\Gamma(N)$ is the usual principal congruence subgroup of level N in Γ, consisting of matrices which are congruent (mod N) to the identity matrix I, and $\pm\Gamma(N)$ is obtained by adjoining $-I$ to $\Gamma(N)$. In [Wo], Wohlfahrt showed that there is an isomorphism

$$\pm\Gamma(N)/\Delta(N) \cong \pi_1(\Gamma(N)\backslash\mathcal{H}^*)$$

where the group on the right hand side is the fundamental group of the compact Riemann surface resulting from the action of $\Gamma(N)$ on the upper half-plane. Thus, if $g = g(N)$ is the genus of the Riemann surface in question, we have

$$\pm\,\Gamma(N)/\Delta(N) \cong < a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 > . \tag{43}$$

Let us denote this group by Π_g. Let F_g be the free group of rank g, and let G be any subgroup of $U(p, \mathbf{C})$ which is generated by g elements. Then there are homomorphisms

$$\Pi_g \xrightarrow{f} F_g \xrightarrow{h} G \xrightarrow{i} U(p, \mathbf{C}). \tag{44}$$

Here, i is the canonical injection, h the natural surjection in which the free generators of F_g map onto the g generators of G, and f is the surjection with kernel generated by the conjugates of the elements $a_1b_1^{-1}, \ldots, a_gb_g^{-1}$. The composition of the maps in (44) thus defines a unitary representation of Π_g whose image is an *arbitrary* subgroup of $U(p, \mathbf{C})$ generated by g elements.

Finally, given a group G, a subgroup K of finite index, and a unitary representation $\sigma : K \to U(H)$ of K in a finite-dimensional Hilbert space H, one knows that the induced representation

$$\mathrm{Ind}_K^G(\sigma) : G \to \mathbf{C}G \otimes_{\mathbf{C}K} H$$

is also unitary (and finite-dimensional). Indeed, suppose that $<,>: H \otimes H \to \mathbf{C}$ is a nondegenerate K-invariant Hermitian form on H, and $g_1, \ldots g_t$ a set of coset representatives of K in G. Then the formula

$$(g_i \otimes u, g_j \otimes v) = \delta_{ij} < u, v >,\ u, v \in H$$

defines a nondegenerate G-invariant Hermitian form on the induced module. (For more details on induced representations, see [M].) Applying this formalism in the case that $G = \tilde{\Gamma}(N), K = \Pi_g$ and σ is the composition (44) yields an enormous number of unitary representations of $\tilde{\Gamma}(N)$.

It is classical [R] that

$$g = g(N) = 1 + \frac{N^2}{24}(N-6)\prod_{p|N}(1 - 1/p^2)$$

for $N \geq 3$. Moreover $g = 0$ if, and only if, $N \leq 5$. Thus it is only for level $N \leq 5$ that Π_g is trivial and the group $\tilde{\Gamma}(N)$ is finite (cf. [K2]). So if $N \leq 5$, a vector-valued modular form is essentially a classical modular form (together with its Fourier expansions at the various cusps). The next case is level 6, where $g = 1$ and $\Pi_g = Z \oplus Z$. This is a special case of the more general situation, considered in [KM1], in which the representation ρ of Γ is induced from a 1-dimensional representation of $\Gamma(N)$.

8. The vector-valued Dirichlet series

As in the classical case we can associate Dirichlet series to an entire vector-valued modular form of weight k (with the usual proviso that $k > 2 + 2\alpha$), more precisely a vector of Dirichlet series $\Phi(s)$. There is a functional equation relating $\Phi(s)$ and $\Phi(k - s)$; in particular, at the center $s = k/2$ of the critical strip $\Phi(k/2)^t$ is an *eigenvector* for $\rho(S)$ with eigenvalue $i^k v(S)$. The proof, in case $F(\tau) = (F_1(\tau), \ldots, F_p(\tau)) \in \mathcal{S}(k, \rho)$ is a cusp-form associated with a normal representation ρ, follows Hecke and Berndt (cf. [H], [R]).

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Marvin Knopp, Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Geoffrey mason, Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

E-mail address: gem@cats.ucsc.edu