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SINGULAR RIEMANNIAN FOLIATIONS WITH SECTIONS

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Abstract. A singular foliation on a complete riemannian manifold is said to be riemannian if every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets. In this paper, we study singular riemannian foliations with sections. A section is a totally geodesic complete immersed submanifold that meets each leaf orthogonally and whose dimension is the codimension of the regular leaves.

We prove here that the restriction of the foliation to a slice of a leaf is diffeomorphic to an isoparametric foliation on an open set of an euclidean space. This result provides local information about the singular foliation and in particular about the singular stratification of the foliation. It allows us to describe the plaques of the foliation as level sets of a transnormal map (a generalization of an isoparametric map). We also prove that the regular leaves of a singular riemannian foliation with sections are locally equifocal. We use this property to define a singular holonomy. Then we establish some results about this singular holonomy and illustrate them with a couple of examples.

1. Introduction

In this section we introduce the concept of singular riemannian foliations with sections. Then we review typical examples of such foliations and state our main results (Theorem 2.7 and Theorem 2.10), which relate these new classes of foliations to the notions of isoparametric and equifocal submanifolds.

We start by recalling the definition of a singular riemannian foliation (see the book of P. Molino [6]).

Definition 1.1. A partition $\mathcal{F}$ of a complete riemannian manifold M by connected immersed submanifolds (the *leaves*) is called *singular riemannian foliation on M* if it verifies the following conditions:

(1) $\mathcal{F}$ is *singular*, i.e., the module $\mathcal{X}_F$ of smooth vector fields on M tangent to the leaves is transitive on each leaf. In other words, for each leaf L

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and each $p \in L$, one can find vector fields $v_i \in \mathcal{X}_F$ such that $\{v_i(p)\}$ is a basis of T_pL.

(2) The partition is *transnormal*, i.e., every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Let $\mathcal{F}$ be a singular riemannian foliation on a complete riemannian manifold M. A point $p \in M$ is called *regular* if the dimension of the leaf L_p that contains p is maximal. A point is called *singular* if it is not regular. Let L be an immersed submanifold of a riemannian manifold M. A section ξ of the normal bundle $\nu(L)$ is said to be a *parallel normal field* along L if $\nabla^\nu \xi \equiv 0$, where ∇^ν is the normal connection. L is said to have globally flat normal bundle if the holonomy of the normal bundle $\nu(L)$ is trivial, i.e., if any normal vector can be extended to a globally defined parallel normal field.

Definition 1.2 (s.r.f.s.). Let $\mathcal{F}$ be a singular riemannian foliation on a complete riemannian manifold M. $\mathcal{F}$ is said to be a *singular riemannian foliation with section* (s.r.f.s. for short) if for every regular point p the set $\sigma := \exp_p(\nu L_p)$ is an immersed complete submanifold that meets each leaf orthogonally and if the regular points of σ are dense in it. The set σ is called a *section*.

Let $p \in M$ and $\mathrm{Tub}(P_p)$ be a tubular neighborhood of a plaque P_p that contains p. Then the connected component of $\exp_p(\nu P_p) \cap \mathrm{Tub}(P_p)$ that contains p is called a *slice* at p. Let us denote this slice by Σ_p. Now consider the intersection of $\mathrm{Tub}(P_p)$ with a section of the foliation. Each connected component of this set is called a *local section*. These two definitions play an important role in this work and are naturally related to each other. In fact, we show in Proposition 2.1 that *the slice at a singular point is the union of the local sections that contain this singular point.*

Typical examples of singular riemannian foliations with sections are the orbits of a polar action, parallel submanifolds of an isoparametric submanifold in a space form and parallel submanifolds of an equifocal submanifold with flat sections in a compact symmetric space, concepts that we now recall.

An isometric action of a compact Lie group G on a riemannian manifold M is called *polar* if there exists a complete immersed submanifold σ of M that meets each G-orbit orthogonally. Such a submanifold σ is called a *section*. A typical example of a polar action is a compact Lie group with a biinvariant metric that acts on itself by conjugation. In this case the maximal tori are the sections.

A submanifold of a real space form is called *isoparametric* if its normal bundle is flat and if the principal curvatures along any parallel normal vector field are constant. The history of isoparametric hypersurfaces and submanifolds and their generalizations can be found in the survey [9] of G. Thorbergsson.

Now we recall the concept of equifocal submanifolds that was introduced by C.L. Terng and G. Thorbergsson [8] as a generalization of the concept of an isoparametric submanifold.

Definition 1.3. A connected immersed submanifold L of a complete riemannian manifold M is called *equifocal* if the following holds:

(0) The normal bundle $\nu(L)$ is globally flat.
(1) For each parallel normal field ξ along L, the derivative of the map $\eta_\xi : L \to M$, defined as $\eta_\xi(x) := \exp_x(\xi)$, has constant rank.
(2) L has sections, i.e., for all $p \in L$ there exists a complete, immersed, totally geodesic submanifold σ such that $\nu_p(L) = T_p\sigma$.

A connected immersed submanifold L is called *locally equifocal* if, for each $p \in L$, there exists a neighborhood $U \subset L$ of p in L such that U is an equifocal submanifold.

We are finally ready to state our main results.

Theorem 2.7. *The regular leaves of a singular riemannian foliation with sections on a complete riemannian manifold M are locally equifocal. In the case that all the leaves are compact, the union of regular equifocal leaves is an open and dense subset of M.*

This result implies that given an equifocal leaf L we can reconstruct the singular foliation by taking all parallel submanifolds of L (see Corollary 2.9). In other words, *let L be a regular equifocal leaf and Ξ denote the set of all parallel normal fields along L. Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi\in\Xi}$.* Theorem 2.7 allows us to define a singular holonomy. Further results concerning this singular holonomy are also established (see Section 3) and illustrated with a couple of new examples (see Section 4). Theorem 2.7 is also used to prove the following result:

Theorem 2.10 (Slice Theorem). *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M and let Σ_q be the slice at a point $q \in M$. Then $\mathcal{F}$ restricted to Σ_q is diffeomorphic to an isoparametric foliation on an open set of $\mathbf{R}^n$, where n is the dimension of Σ_q.*

Owing to the slice theorem, we can view the plaques of the singular foliation, which are in a tubular neighborhood of a singular plaque P, as the product of isoparametric submanifolds and P. In particular, a better description of the singular stratification can be obtained (see Corollary 2.11).

As consequence of the slice theorem, we obtain Proposition 2.12. This result states that *the plaques of a s.r.f.s. are always level sets of a transnormal map,* whose definition is recalled below.

DEFINITION 1.4. Let M^{n+q} be a complete riemannian manifold. A smooth map $H = (h_1 \cdots h_q) : M^{n+q} \to \mathbf{R}^q$ is called *transnormal* if it satisfies:

(0) H has a regular value.

(1) For every regular value c there exists a neighborhood V of $H^{-1}(c)$ in M and smooth functions $b_{i\,j}$ on $H(V)$ such that, for every $x \in V$, $\langle \operatorname{grad} h_i(x), \operatorname{grad} h_j(x) \rangle = b_{i\,j} \circ H(x)$.

(2) There is a sufficiently small neighborhood of each regular level set such that, for every i and j, $[\operatorname{grad} h_i, \operatorname{grad} h_j]$ is a linear combination of $\operatorname{grad} h_1, \ldots, \operatorname{grad} h_q$, where the coefficients are functions of H.

This definition is equivalent to saying that H has a regular value and for each regular value c there exists a neighborhood V of $H^{-1}(c)$ in M such that $H\,|_V \to H(V)$ is an integrable riemannian submersion, where the metric $(g_{i\,j})$ of $H(V)$ is the inverse matrix of $(b_{i\,j})$.

A transnormal map H is said to be an *isoparametric map* if V can be chosen to be M and $\triangle h_i = a_i \circ H$, where the functions a_i are smooth functions.

Isoparametric submanifolds in space forms and equifocal submanifolds with flat sections in simply connected symmetric spaces of compact type can always be described as regular level sets of transnormal analytic maps; see R. Palais and C.L. Terng [7] and E. Heintze, X. Liu and C. Olmos [5].

In [1] we proved that *the regular leaves of an analytic transnormal map on an analytic complete manifold are equifocal submanifolds and leaves of a singular riemannian foliation with sections.* Hence, Proposition 2.12 is a local converse of this result.

This paper is organized as follows. In Section 2 we prove some propositions about singular riemannian foliations with sections (s.r.f.s. for short), Theorem 2.7 and Theorem 2.10. In Section 3 we introduce the concept of singular holonomy of a s.r.f.s. and establish some results about it. In Section 4 we construct singular foliations by suspension of certain homomorphisms in order to illustrate some properties of singular holonomy.

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2. Proof of the main results

PROPOSITION 2.1. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and let $q \in M$. Then:*

(a) $\Sigma_q = \bigcup_{\sigma \in \Lambda(q)} \sigma$, *where $\Lambda(q)$ is the set of all local sections that contain q.*

(b) $\Sigma_x \subset \Sigma_q$ *for all $x \in \Sigma_q$.*

(c) $T_x\Sigma_q = T_x\Sigma_x \oplus T_x(U \cap \Sigma_q)$, *where* $x \in \Sigma_q$ *and* $U \subset L_x$ *is an open set of* x *in* L_x.

Proof. (a) First we show that $\Sigma_q \supset \bigcup_{\sigma\in\Lambda(q)} \sigma$. Let σ be a local section that contains q, let p be a regular point of σ and γ be the shortest segment of the geodesic that joins q to p. Then γ is orthogonal to L_p since $\gamma \subset \sigma$ and σ is orthogonal to L_p. Since $\mathcal{F}$ is a riemannian foliation, γ is also orthogonal to L_q and hence $p \in \Sigma_q$. Since the regular points are dense in σ we conclude that $\Sigma_q \supset \sigma$.

Next we show that $\Sigma_q \subset \bigcup_{\sigma\in\Lambda(q)} \sigma$. Let $p \in \Sigma_q$ be a regular point and γ be the segment of the geodesic orthogonal to L_q that joins q to p. Since $\mathcal{F}$ is a riemannian foliation, γ is orthogonal to L_p. Therefore γ belongs to the local section σ that contains p. In particular $q \in \sigma$. In other words, each regular point $p \in \Sigma_q$ belongs to a local section σ that contains q.

Finally let $z \in \Sigma_q$ be a singular point, σ a local section that contains z, and p a regular point of σ. Since the slice is defined on a tubular neighborhood of a plaque P_q, there exists a unique point $\tilde{q} \in P_q$ such that $p \in \Sigma_{\tilde{q}}$. As we have shown above, $\tilde{q} \in \sigma$. Now it follows from the first part of the proof that $z \in \sigma \subset \Sigma_{\tilde{q}}$ Since $z \in \Sigma_q$, we conclude that $\tilde{q} = q$.

(b) Let $x \in \Sigma_q$ and $\sigma \subset \Sigma_x$ be a local section. It follows from the proof of item (a) that $\sigma \subset \Sigma_q$ and $q \in \sigma$. Since Σ_x is a union of local sections that contain x, $\Sigma_x \subset \Sigma_q$.

(c) Since the foliation $\mathcal{F}$ is singular, we have

$$\begin{aligned}\dim T_x\Sigma_x + \dim T_x(U \cap \Sigma_q) + \dim L_q &= \dim M \\ &= \dim T_x\Sigma_q + \dim L_q.\end{aligned}$$

Item (b) and the above equation imply item (c). □

REMARK 2.2. In [6, p. 209] Molino showed that given a singular riemannian foliation $\mathcal{F}$ it is possible to change the metric so that the restriction of $\mathcal{F}$ to a slice is a singular riemannian foliation with respect to the new metric. The new metric respects the distance between the leaves. As we will see below, the metric need not be changed if the singular riemannian foliation has sections.

COROLLARY 2.3. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and Σ a slice. Then $\mathcal{F} \cap \Sigma$ is a s.r.f.s. on Σ with the induced metric of M.*

Proof. Let γ be a segment of the geodesic that is orthogonal to L_x, where $x \in \Sigma$. Since $\gamma \subset \Sigma_x$, it follows from item (b) of Proposition 2.1 that $\gamma \subset \Sigma$. Since $\mathcal{F}$ is riemannian, γ is orthogonal to the leaves of $\mathcal{F} \cap \Sigma$. Therefore $\mathcal{F} \cap \Sigma$ is a singular riemannian foliation.

Now let σ be a local section that contains x. Then it follows from item (a) of Proposition 2.1 that $\sigma \subset \Sigma$ and hence σ is a local section of $\mathcal{F} \cap \Sigma$. Therefore $\mathcal{F} \cap \Sigma$ is a s.r.f.s. $\square$

PROPOSITION 2.4. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and γ a geodesic orthogonal to the leaves of $\mathcal{F}$. Then the singular points of γ are either all the points of γ or isolated points of γ.*

Proof. Since the set of regular points on γ is open, we can suppose that $q = \gamma(0)$ is a singular point and that $\gamma(t)$ is a regular point for $-\delta < t < 0$. We will show that there exists $\epsilon > 0$ such that $\gamma(t)$ is also a regular point for $0 < t < \epsilon$.

First we note that we can choose $t_0 < 0$ such that q is a focal point of $L_{\gamma(t_0)}$. To see this let $\mathrm{Tub}(P_q)$ be a tubular neighborhood of a plaque P_q and let $t_0 < 0$ be such that $\gamma(t_0) \in \mathrm{Tub}(P_q)$. Since $L_{\gamma(t_0)}$ is a regular leaf and q is a singular point, it follows from item (c) of Proposition 2.1 that $L_{\gamma(t_0)} \cap \Sigma_q$ is not empty. Then we can join this submanifold to q with geodesics that belong to Σ_q. Since $\mathcal{F}$ is a riemannian foliation, these geodesics are also orthogonal to $L_{\gamma(t_0)} \cap \Sigma_q$. This implies that q is a focal point.

Since focal points are isolated along γ, we can choose $\epsilon > 0$ such that $\gamma(t)$ is not a focal point of $P_{\gamma(t_0)}$ along γ for $0 < t < \epsilon$.

Suppose that there exists $0 < t_1 < \epsilon$ such that $x = \gamma(t_1)$ is a singular point. Let σ be a local section that contains $\gamma(t_0)$. Let U be an open set of $\nu_x L$ such that $\widetilde{\Sigma}_x := \exp_x(U)$ contains $\gamma(t_0)$ and is contained in a convex neighborhood of x. We note that $\widetilde{\Sigma}_x$ is not contained in a tubular neighborhood of P_x and hence is not a slice.

We have

$$\sigma \subset \widetilde{\Sigma}_x.$$

Since x is a singular point, it follows that

$$\dim \sigma < \dim \widetilde{\Sigma}_x.$$

The above equations imply that $\dim P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x > 0$. Hence we can find geodesics in $\widetilde{\Sigma}_x$ that join x to the submanifold $P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x$. Since the foliation is riemannian, these geodesics are also orthogonal to $P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x$ and hence x is a focal point of this submanifold. This contradicts our choice of ϵ and completes the proof. $\square$

In what follows we will need the following result due to Heintze, Liu and Olmos.

PROPOSITION 2.5 (Heintze, Liu and Olmos [5]). *Let M be a complete riemannian manifold, L be an immersed submanifold of M with globally flat normal bundle and ξ be a normal parallel field along L. Suppose that $\sigma_x :=$*

$\exp_x(\nu_x L)$ *is a totally geodesic complete submanifold for all* $x \in L$*, i.e.,* L *has sections. Then:*

(1) $d\eta_\xi(v)$ *is orthogonal to* σ_x *at* $\eta_\xi(x)$ *for all* $v \in T_x L$.
(2) *Suppose that* p *is not a critical point of the map* η_ξ*. Then there exists a neighborhood* U *of* p *in* L *such that* $\eta_\xi(U)$ *is an embedded submanifold, which meets* σ_x *orthogonally and has globally flat normal bundle. In addition, a parallel normal field along* U *transported to* $\eta_\xi(U)$ *by parallel translation along the geodesics* $\exp(t\xi)$ *is a parallel normal field along* $\eta_\xi(U)$.

Let $\exp^\perp$ denote the restriction of exp to $\nu(L)$. We recall that for each $w \in T_{\xi_0}\nu(L)$ there exist unique elements $w_{\mathsf{t}} \in T_{\xi_0}\nu(L)$ (the tangential vector) and $w_{\mathsf{n}} \in T_{\xi_0}\nu(L)$ (the normal vector) such that:

(1) $w = w_{\mathsf{t}} + w_{\mathsf{n}}$.
(2) $d\Pi(w_{\mathsf{n}}) = 0$, where $\Pi : \nu(L) \to L$ is the natural projection.
(3) $w_{\mathsf{t}} = \xi'(0)$, where $\xi(t)$ is the normal parallel field with $\xi(0) = \xi_0$.

We also recall that $z = \exp^\perp(\xi_0)$ is a focal point with multiplicity k along $\exp^\perp(t\,\xi_0)$ if and only if $\dim\ker d\exp^\perp_{\xi_0} = k$. We call z a focal point of L of *tangential type* if $\ker d\exp^\perp_{\xi_0}$ consists only of tangential vectors.

Corollary 2.6. *Let* L *be a submanifold defined as above,* $p \in L$ *and* $\xi_0 \in \nu_p L$*. Suppose that the point* $z = \exp_p(\xi_0)$ *is a focal point of* L *along* $\exp_p(t\,\xi_0)$ *that belongs to a normal neighborhood of* p*. Then* z *is a focal point of tangential type.*

Proof. If z is a focal point, then there exists $w \in T_{\xi_0}\nu(L)$ such that $\|d\exp^\perp_{\xi_0}(w)\| = 0$. It follows from the above proposition that

$$\langle d\exp^\perp_{\xi_0} w_{\mathsf{n}}, d\exp^\perp_{\xi_0} w_{\mathsf{t}}\rangle_{\exp^\perp(\xi_0)} = 0$$

and hence $\|d\exp^\perp_{\xi_0}(w_{\mathsf{n}})\| = 0$. Since z belongs to a normal neighborhood, w_{n} must be zero. We conclude that $w = w_{\mathsf{t}}$. □

Now we can prove one of our main results.

Theorem 2.7. *Let* $\mathcal{F}$ *be a singular riemannian foliation with sections on a complete riemannian manifold* M*. Then the regular leaves are locally equifocal. In the case that all the leaves are compact, the union of regular equifocal leaves is an open and dense subset of* M.

To prove this result, we need the following lemma.

Lemma 2.8. *Let* $\mathrm{Tub}(P_q)$ *be a tubular neighborhood of a plaque* P_q, $x_0 \in \mathrm{Tub}(P_q)$*, and* $\xi \in \nu P_{x_0}$ *such that* $\exp_{x_0}(\xi) = q$*. We also suppose that* q *is the only singular point on the segment of the geodesic* $\exp_{x_0}(t\,\xi) \cap \mathrm{Tub}(P_q)$*. Then we can find a neighborhood* U *of* x_0 *in* P_{x_0} *with the following properties:*

(1) *νU is globally flat and we can define the parallel normal field ξ on U.*
(2) *There exists a number $\epsilon > 0$ such that, for each $x \in U$, $\gamma_x \subset \mathrm{Tub}(P_q)$, where $\gamma_x(t) := \exp_x(t\,\xi)$ and $t \in [-\epsilon, 1+\epsilon]$.*
(3) *The regular points of the foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$ are not critical values of the maps $\eta_{t\,\xi}|_U$.*
(4) *$\eta_{t\,\xi}(U) \subset L_{\gamma_{x_0}(t)}$.*
(5) *$\eta_{t\,\xi} : U \to \eta_{t\,\xi}(U)$ is a local diffeomorphism for $t \neq 1$.*
(6) *$\dim \operatorname{rank} D\eta_\xi$ is constant on U.*

Proof. Item (1) follows from the fact that $\mathcal{F}$ has sections. The proof of (2) follows by standard arguments.

(3) Let $p = \eta_{r\,\xi}(x_1)$ be a regular point of the foliation and suppose that x_1 is a critical point of the map $\eta_{r\,\xi}|_U$. Then there exists a Jacobi field $J(t)$ along the geodesic γ_{x_1} such that $J(r) = 0$. In particular, there exists a smooth curve $\beta(t) \subset P_{x_0}$ such that $J(t) = \frac{\partial}{\partial s} \exp_{\beta(s)}(t\,\xi)$ and $\beta(0) = x_1$.

Since focal points are isolated along $\gamma_{x_1}(t)$, there exists a regular point of the foliation $\tilde{p} = \gamma_{x_1}(\tilde{r})$ that is not a focal point of P_{x_1} along γ_{x_1}. It follows from Proposition 2.5 that there exists a neighborhood V of x_1 in P_{x_0} such that the embedded submanifold $\eta_{\tilde{r}\,\xi}(V)$ is orthogonal to the sections that it meets. Hence $\eta_{\tilde{r}\,\xi}(V)$ is tangent to the plaques near $P_{\tilde{p}}$. Since $\eta_{\tilde{r}\,\xi}(V)$ has the dimension of the regular leaves, $\eta_{\tilde{r}\,\xi}(V)$ is an open subset of $P_{\tilde{p}}$.

Since $\tilde{p}$ can be chosen arbitrarily close to p, we can suppose that p and $\tilde{p}$ belong to a neighborhood W that contains only regular points of the foliation and such that $\mathcal{F}|_W$ is the pre-image of an integrable riemannian submersion $\pi : W \to B$. It follows from Proposition 2.5 that $\gamma'_{\beta(s)}(\tilde{r})$ is a parallel field along the curve $\eta_{\tilde{r}\xi} \circ \beta(s) \subset \eta_{\tilde{r}\,\xi}(V) \subset P_{\tilde{p}}$. Therefore $\gamma_{\beta(s)}(t) \cap W$ is a horizontal lift of a geodesic in B (the basis of the riemannian submersion π). This implies that $J(r) \neq 0$. This contradicts the assumption that p is a focal point and completes the proof of item (3).

(4) First we prove item (4) for $t \neq 1$. Fix a number $t_0 \neq 1$ and define $K := \{k \in U \text{ such that } \eta_{t_0\,\xi}(k) \in P_{\gamma_{x_0}(t_0)}\}$. Since $\gamma_{x_0}(t_0)$ is a regular point of the foliation, it follows from item (3) that all points of $P_{\gamma_{x_0}(t_0)}$ are regular values of the map $\eta_{t_0\,\xi}$. Hence for each $k \in K$ there exists a neighborhood V of k in U such that $\eta_{t_0\,\xi}(V)$ is an embedded submanifold. As noted in the proof of item (3), $\eta_{t_0\,\xi}(V)$ is an open set of $P_{\gamma_{x_0}(t_0)}$, because this embedded submanifold is orthogonal to the sections and has the same dimension as the plaques. Thus we conclude that K is an open set. One can prove that K is closed using standard arguments and the fact that the plaques are equidistant. Since U is connected, $K = U$.

Now we prove item (4) for $t = 1$. We define $f(x,t) := d(\eta_{t\,\xi}(x), P_q) - d(\eta_{t\,\xi}(x_0), P_q)$. As already seen, $\eta_{t\,\xi}(x)$ and $\eta_{t\,\xi}(x_0)$ belong to the same plaque for $t \neq 1$. This means that $f(x,t) = 0$ for all $t \neq 1$ and hence $f(x,1) = 0$, i.e., $\eta_\xi(x) \subset P_q$.

(5) Item (5) follows from items (3) and (4).

(6) Fix a point $x_1 \in U$. It follows from Corollary 2.6 that the focal points of U along $\gamma_x(t)$ are of tangential type. This means that $\gamma_x(t_0)$ is a focal point of U along γ_x with multiplicity k if and only if x is a critical point of $\eta_{t_0\,\xi}$ and $\dim\ker d\eta_{t_0\,\xi}(x) = k$. Furthermore, it follows from item (5) that only for $t = 1$ the map $\eta_{t\,\xi}$ may not be a diffeomorphism. Therefore we have

$$m(\gamma_x) = \dim\ker d\eta_\xi(x), \tag{2.1}$$

where $m(\gamma_x)$ denotes the number of focal points on $\gamma_x(t)$, counted with multiplicities.

On the other hand, we have

$$m(\gamma_x) \geq m(\gamma_{x_1}) \tag{2.2}$$

for x in a neighborhood of x_1 in U. Indeed one can argue like Q.M. Wang [10] to see that (2.2) follows from the Morse index theorem.

Equations (2.1) and (2.2) together with the elementary inequality $\dim\ker d\eta_\xi(x) \leq \dim\ker d\eta_\xi(x_1)$ imply that $\dim\ker d\eta_\xi$ is constant on a neighborhood of x_1 in U. Since this holds for each $x_1 \in U$, we conclude that $\dim\ker d\eta_\xi$ is constant on U. □

Proof of Theorem 2.7. Let L be a leaf of $\mathcal{F}$, U be an open set of L that has normal bundle globally flat and ξ be a parallel normal field along U. First we will prove that $\dim \operatorname{rank} d\eta_\xi|_U$ is constant, i.e., that L is locally equifocal.

Let $p \in U$. Since singular points are isolated along $\gamma_p(t) = \exp_p(t\,\xi)|_{[-\epsilon,1+\epsilon]}$ (see Proposition 2.4), we can cover this arc of geodesic with a finite number of tubular neighborhoods $\operatorname{Tub}(P_{\gamma_p(t_i)})$, where $t_0 = 0$ and $t_n = 1$.

Let $P_{\gamma_p(r_i)}$ be regular plaques that belong to $\operatorname{Tub}(P_{\gamma_p(t_{i-1})}) \cap \operatorname{Tub}(P_{\gamma_p(t_i)})$, where $t_{i-1} < r_i < t_i$. Applying Lemma 2.8 and Proposition 2.5, we can find an open set $U_0 \subset P_p$ of the plaque P_p, open sets $U_i \subset P_{\gamma_p(r_i)}$ of the plaques $P_{\gamma_p(r_i)}$ and parallel normal fields ξ_i along U_i, with the following properties:

(1) For each U_i, the parallel normal field ξ_i is tangent to the geodesics $\gamma_x(t)$, where $x \in U_0$.
(2) $\eta_{\xi_i} : U_i \to U_{i+1}$ is a local diffeomorphism for $i < n$.
(3) $\eta_\xi|_{U_0} = \eta_{\xi_n} \circ \eta_{\xi_{n-1}} \circ \cdots \circ \eta_{\xi_0}|_{U_0}$.

Because $\dim \operatorname{rank} d\eta_{\xi_i}$ is constant on U_i, it follows that $\dim d\eta_\xi$ is constant on U_0. Since this holds for each $p \in U$, $\dim d\eta_\xi$ is constant on U, i.e., L is locally equifocal.

Finally let us consider the case when the leaves of the foliation are all compact. According to Molino (Proposition 3.7, page 95, of [6]), the union of the regular leaves with trivial holonomy of a singular riemannian foliation is an open and dense set in the set of regular points. Moreover, the set of regular points is an open and dense set in M (see page 197 of [6]). Since the leaves of a s.r.f.s. having trivial holonomy are exactly the leaves that have normal bundle globally flat, the union of regular leaves that are equifocal is an open and dense set in M.

Corollary 2.9. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and L be a regular leaf of $\mathcal{F}$.*

(a) *Let $\beta(s) \subset L$ be a smooth curve and ξ a parallel normal field along $\beta(s)$. Then the curve $\exp_{\beta(s)}(\xi)$ belongs to a leaf of the foliation.*

(b) *Let L be a regular equifocal leaf and Ξ denote the set of all parallel normal fields along L. Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi \in \Xi}$.*

Proof. (a) Item (a) can be easily proved using item (4) of Lemma 2.8 and gluing together tubular neighborhoods as we have already done in the proof of Theorem 2.7.

(b) We first prove:

Claim. *Let $x_0 \in L$ and $q = \eta_\xi(x_0)$. Then there exists a neighborhood $U \subset L$ of x_0 in L such that $\eta_\xi(U) \subset P_q$ is an open set in L_q.*

To prove this claim, it is enough to suppose that $x_0 \in \mathrm{Tub}(P_q)$, for the general case can be proved by gluing together tubular neighborhoods as we have done in proof of Theorem 2.7. Now the claim follows if we note that $\eta_\xi : U \to P_q$ is a submersion whose fibers are the intersections of U with the slices of P_q. To see that each fiber is contained in $U \cap \Sigma$ one can use the fact that the rank of $d\eta_\xi$ is constant and the fact that the foliation is riemannian. To see that each fiber contains $U \cap \Sigma$ one can use item (4) of Lemma 2.8 together with item (a) of Proposition 2.1.

It follows from the above claim that $\eta_\xi(L)$ is an open set in L_q. We will see that $\eta_\xi(L)$ is also a closed set in L_q. Let $z \in L_q$ and $\{z_i\}$ be a sequence in $\eta_\xi(L)$ such that $z_i \to z$.

First suppose that L_q is a regular leaf. If follows from Proposition 2.5 that there exists a parallel normal field $\hat{\xi}$ along $\eta_\xi(L)$ such that $\hat{\xi}_{\eta_\xi(x)}$ is tangent to the geodesic $\exp_x(t\,\xi)$. Since the normal bundle of P_z is globally flat, we can extend $\hat{\xi}$ along P_z. Item (4) of Lemma 2.8 implies that $\eta_{-\hat{\xi}} : P_z \to L$. By construction $\eta_\xi \circ \eta_{-\hat{\xi}}(z_i) = z_i$. Therefore $\eta_\xi \circ \eta_{-\hat{\xi}}(z) = z$. This means that $z \in \eta_\xi(L)$.

Finally, suppose that L_q is a singular leaf. There exists $x_i \in L$ such that $z_i = \eta_\xi(x_i) \in P_z$. We can find a number $s < 1$ such that $y_i = \eta_{s\,\xi}(x_i)$ is a regular point. Since y_i is a regular point, the plaque P_{y_i} is an open set of $\eta_{s\,\xi}(L)$ as we have proved above. There exists a parallel normal field $\hat{\xi}$ along P_{y_i} such that $\eta_{\hat{\xi}} \circ \eta_{s\,\xi} = \eta_\xi$.

It follows from item (4) of Lemma 2.8 that $\eta_{\hat{\xi}}(P_{y_i}) \subset P_z$. On the other hand, since the foliation is singular, the plaque P_{y_i} intercepts the slice Σ_z. These two facts imply that $z \in \eta_{\hat{\xi}}(P_{y_i})$. Therefore $z \in \eta_\xi(L)$. □

Let $\mathcal{F}$ (respectively $\widetilde{\mathcal{F}}$) be a foliation on a manifold M^n (respectively $\widetilde{M^n}$) and $\varphi : M \to \widetilde{M}$ be a diffeomorphism. We say that φ is a *diffeomorphism between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$* if each leaf L of $\mathcal{F}$ is diffeomorphic to a leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$.

THEOREM 2.10 (Slice Theorem). *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M and Σ_q be a slice at a point $q \in M$. Then $\mathcal{F}$ restricted to Σ_q is diffeomorphic to an isoparametric foliation on an open set of $\mathbf{R}^n$, where n is the dimension of Σ_q.*

Proof. According to H. Boualem (see Proposition 1.2.3 and Lemma 1.2.4 of [3]), we have:

(1) The map $\exp^{-1}$ is a diffeomorphism between the foliation $\mathcal{F}|_{\Sigma_q}$ and a singular riemannian foliation with sections $\widetilde{F}$ on an open set of the inner product space $(T_q\Sigma_q, \langle , \rangle_q)$, where $\langle , \rangle_q$ denotes the metric of T_qM.
(2) The sections of the singular foliation $\widetilde{F}$ are the vector subspaces $\exp^{-1}(\sigma)$, where σ is a local section of $\mathcal{F}$.

Let $\langle , \rangle_0$ denote the canonical euclidean product. Then there exists a positive definite symmetric matrix A such that $\langle X, Y\rangle_q = \langle A\,X, Y\rangle_0$. The isometry $\sqrt{A} : (T_q\Sigma_q, \langle , \rangle_q) \to (\mathbf{R}^n, \langle , \rangle_0)$ is a diffeomorphism between the foliation $\widetilde{F}$ and a singular riemannian foliation with section $\widehat{F}$ on an open set of the inner product space $(\mathbf{R}^n, \langle , \rangle_0)$. Since $0 \in \mathbf{R}^n$ is a singular leaf of the foliation $\widehat{\mathcal{F}}$, the leaves of this foliation are contained in spheres in the euclidean space.

CLAIM 1. *The restriction of the foliation $\widehat{\mathcal{F}}$ to a sphere $\mathbf{S}^{n-1}(r)$ is a singular riemannian foliation with sections on $\mathbf{S}^{n-1}(r)$.*

The first step in proving this claim is to note that $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ is a singular foliation, since $\widehat{\mathcal{F}}$ is so. Next, we notice that if $\widehat{\sigma}$ is a section of $\widehat{F}$ then $\sigma_s := \widehat{\sigma} \cap \mathbf{S}^{n-1}(r)$ is a section of the foliation $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$. To conclude, it suffices to show that $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ is a transnormal system. Let γ be a geodesic of $\mathbf{S}^{n-1}(r)$ that is orthogonal to a leaf $L_{\gamma(0)}$ of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$. Since a slice of $\widehat{F}$ is a union of sections, γ is tangent to a section $\hat{\sigma}$ at the point $\gamma(0)$ and hence is tangent to a section σ_s of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ at the point $\gamma(0)$. This implies that $\gamma \subset \sigma_s$, which means that γ is orthogonal to each leaf that it meets, i.e., the partition is transnormal.

Now Theorem 2.7 guarantees that the leaves of a singular riemannian foliation with sections are locally equifocal. Therefore the leaves of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ are locally equifocal.

The next claim follows from standard calculations on space forms.

CLAIM 2. *The locally equifocal submanifolds in $\mathbf{S}^{n-1}(r)$ are isoparametric submanifolds in $\mathbf{S}^{n-1}(r)$.*

Since isoparametric submanifolds in spheres are isoparametric submanifolds in euclidean spaces (see Palais and Terng, Proposition 6.3.17 of [7]), we conclude that the regular leaves of $\widehat{\mathcal{F}}$ are isoparametric submanifolds in an open set of the euclidean space $\mathbf{R}^n$.

Finally, we note that Corollary 2.9 implies that the singular leaves of $\widehat{\mathcal{F}}$ are the focal leaves. Therefore $\widehat{\mathcal{F}}$ is an isoparametric foliation on an open set of the euclidean space. This completes the proof of the theorem. □

Corollary 2.11. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and σ be a local section contained in a slice Σ_q of dimension n. According to the slice theorem there exist an open set $U \subset \mathbf{R}^n$ and a diffeomorphism $\Psi : \Sigma_q \to U$ sending the foliation $\mathcal{F} \cap \Sigma_q$ to an isoparametric foliation $\widehat{\mathcal{F}}$ on U. Then the set of singular points of $\mathcal{F}$ contained in σ is a finite union of totally geodesic hypersurfaces that are sent by Ψ onto the focal hyperplanes of $\widehat{\mathcal{F}}$ contained in a section of this isoparametric foliation.*

We will call *singular stratification of the local section σ* this set of singular points of $\mathcal{F}$ contained in σ.

Proof. It follows from Molino [6] (page 194, Proposition 6.3) that the intersection of the singular leaves with a section is a union of totally geodesic submanifolds. Now the slice theorem implies that these totally geodesic submanifolds are, in fact, hypersurfaces diffeomorphic to focal hyperplanes. □

Proposition 2.12. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and $q \in M$. Then there exist a tubular neighborhood* $\mathrm{Tub}(P_q)$, *an open set $W \subset \mathbf{R}^k$ and a transnormal map $H : \mathrm{Tub}(P_q) \to W$ such that the pre-images of H are leaves of the singular foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$. The leaf $H^{-1}(c)$ is regular if and only if c is a regular value.*

Proof. We start by recalling a result that can be found in the book of Palais and Terng.

Lemma 2.13 (Theorem 6.4.4, page 129, of [7]). *Let N be a rank k isoparametric submanifold in $\boldsymbol{R}^n$, whose associated Coxeter group is denoted by W. Consider a point q in N along with the affine normal plane $\nu_q = q + \nu(N)$ at q. Finally let $u_1, \dots, u_k$ be a set of generators of the W-invariant polynomials on ν_q. Then $u = (u_1, \dots, u_k)$ extends uniquely to an isoparametric polynomial map $g : \mathbf{R}^n \to \boldsymbol{R}^k$ having N as a regular level set. Moreover,*

(1) *each regular set is connected,*
(2) *the focal set of N is the set of critical points of g,*
(3) $\nu_q \cap N = W \cdot q$,
(4) $g(\boldsymbol{R}^n) = u(\nu_q)$,
(5) *for $x \in \nu_q$, $g(x)$ is a regular value if and only if x is W-regular,*

(6) *$\nu(N)$ is globally flat.*

The above result implies that the leaves of the isoparametric foliation, which has N as a leaf, can be described as pre-images of a map g. Note that this still holds if N is not a full isoparametric submanifold of $\mathbf{R}^n$.

Now we define $\widetilde{H} : \Sigma_q \to \mathbf{R}^k$ as $\widetilde{H} := g \circ \Psi$, where $\Psi : \Sigma_q \to \mathbf{R}^n$ is the diffeomorphism, given by the slice theorem, that sends $\mathcal{F}|_{\Sigma_q}$ to an isoparametric foliation on an open set W of $\mathbf{R}^n$.

Since $\mathcal{F}$ is a singular foliation, there exists a projection $\Pi : \mathrm{Tub}(P_q) \to \Sigma_q$ such that $\Pi(P) = P \cap \Sigma_q$ for each plaque P.

Finally we define $H := \widetilde{H} \circ \Pi$. Then the pre-images of H are leaves of the foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$.

The claim below, which can be found in Molino [6, page 77], implies that H is a transnormal map.

Claim. *Let U be a simple neighborhood of a riemannian foliation (with section) and let $H : U \to \widetilde{U} \subset \boldsymbol{R}^k$ be a smooth map such that the sets $H^{-1}(c)$ are leaves of $\mathcal{F}|_U$. Then we can choose a metric for $\widetilde{U}$ such that $H : U \to \widetilde{U}$ is a (integrable) riemannian submersion.* $\square$

3. Singular holonomy

The slice theorem gives us a description of the plaques of a singular riemannian foliation with sections. However, it does not guarantee that different plaques belong to different leaves. To obtain this kind of information, we must extend the concept of holonomy to describe not only what happens near a regular leaf, but also what happens in a neighborhood of a singular leaf.

In this section, we will introduce the concept of singular holonomy and establish some of its properties.

Proposition 3.1. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M, L_p a regular leaf, σ a local section and $\beta(s) \subset L_p$ a smooth curve, where $p = \beta(0)$ and $\beta(1)$ belong to σ. Let $[\beta]$ denote the homotopy class of β. Then there exists an isometry $\varphi_{[\beta]} : U \to W$, where the source U and the target W contain σ, with the following properties:*

(1) *$\varphi_{[\beta]}(x) \in L_x$ for each $x \in \sigma$.*
(2) *$d\varphi_{[\beta]}\xi(0) = \xi(1)$, where $\xi(s)$ is a parallel normal field along $\beta(s)$.*

Proof. Since σ is a local section, for each $x \in \sigma$ there exists a unique $\xi \in T_p\sigma$ such that $\exp_p(\xi) = x$. Let $\xi(t)$ be the parallel transport of ξ along β and define $\varphi_\beta(x) := \exp_{\beta(1)}(\xi(1))$. It is easy to see that φ_β is a bijection. It follows from Corollary 2.9 that $\exp_\beta(\xi) \subset F_x$, and this proves a part of item (1). Since φ_β is an extension of the holonomy map, $d\varphi_\beta\xi(0) = \xi(1)$, and this proves a part of item (2). The fact that φ_β is an extension of the holonomy map implies that the restriction of φ_β to a small neighborhood of σ depends only on the homotopy class of β. Since isometries are determined by the image of a point and the derivative at this point, it is enough to prove that φ_β is an isometry in order to see that φ_β depends only on the homotopy class of β. To see that φ_β is an isometry it is enough to prove the following claim.

Claim. *Given a point $x_0 \in \sigma$ there exists an open set $V \subset \sigma$ of x_0 in σ such that $d(x_1, x_0) = d(\varphi_\beta(x_1), \varphi_\beta(x_0))$, for each $x_1 \in V$.*

To prove this claim, let $\xi_0(s)$ and $\xi_1(s)$ be normal parallel fields along $\beta(s)$ such that $x_j = \exp_p(\xi_j(0))$ for $j = 0, 1$. Define $\alpha_j(s) = \exp_{\beta(s)}(\xi_j(s))$ for $j = 0, 1$. Since $\varphi_\beta(x_j) = \alpha_j(1)$ the claim follows from the equation

$$d(\alpha_0(s), \alpha_1(s)) = d(\alpha_0(0), \alpha_1(0)),$$

which in turn is a consequence of the following facts:

(1) $\alpha_j(s) \in L_{x_j}$.
(2) Singular riemannian foliations are locally equidistant.
(3) $\alpha_0(s)$ and $\alpha_1(s)$ are always in the same local section. □

Definition 3.2. The pseudogroup of isometries generated by the isometries constructed above is called *pseudogroup of singular holonomy of the local section* σ. Let $\mathrm{Holsing}(\sigma)$ denote this pseudogroup.

Proposition 3.3. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and σ a local section. Then the reflections in the hypersurfaces of the singular stratification of the local section σ leave $\mathcal{F} \cap \sigma$ invariant. Moreover, these reflections are elements of* $\mathrm{Holsing}(\sigma)$.

Proof. The proposition is known to hold if the singular foliation is an isoparametric foliation on an euclidean space. In what follows we will use this fact and the slice theorem to construct the desired reflections.

Let S be a complete totally geodesic hypersurface of the singular stratification of the local section σ and Σ be a slice of a point of S. Note that σ is contained in Σ. It follows from the slice theorem that there exists a diffeomorphism $\Psi : \Sigma \to V \subset \mathbf{R}^n$ that sends $\mathcal{F} \cap \Sigma$ to an isoparametric foliation $\widetilde{F}$ on an open set V of $\mathbf{R}^n$. Consider a regular point $p \in \sigma$ and set $\tilde{L} := \Psi(L_p \cap \Sigma)$ and $\tilde{\sigma} := \Psi(\sigma)$. We note that $\tilde{\sigma}$ is a local section of the isoparametric foliation $\widetilde{F}$.

It follows from Corollary 2.11 and from the theory of isoparametric submanifolds [7] that $\tilde{S} := \Psi(S)$ is a focal hyperplane associated to a curvature distribution E. Consider a path $\beta \subset \Sigma \cap \mathcal{F}$, with $\beta(0) = p$ and $\beta(1) \in \sigma$, such that $\tilde{\beta} := \Psi \circ \beta$ is tangent to the distribution E. Finally, let $z \in S$, $\xi \in T_p\sigma$ be such that $\exp_p(\xi) = z$ and let $\xi(s)$ denote the parallel transport of ξ along β.

Claim. $\exp_{\beta(s)}(\xi) = z$.

To prove this claim, we recall that $\tilde{S} \subset \tilde{\sigma}_{\tilde{\beta}}$, where $\tilde{\sigma}_{\tilde{\beta}}$ is a local section of $\tilde{\mathcal{F}}$ that contains $\tilde{\beta(s)}$ (see Theorem 6.2.9 of [7]). Therefore $S \subset \sigma_{\beta(s)}$. On the other hand, it follows from Corollary 2.9 that $\exp_{\beta(s)}(\xi) \subset P_z$. Hence $\exp_{\beta(s)}(\xi) \subset P_z \cap S$. Now the claim follows from the fact that $P_z \cap S = \{z\}$.

The claim implies that the isometry $\varphi_{[\beta]}$ leaves the points of S fixed. Therefore $\varphi_{[\beta]}$ is a reflection in a totally geodesic hypersurface. Since $\varphi_{[\beta]}(x) \in L_x$, these reflections leave $\mathcal{F} \cap \sigma$ invariant. $\square$

Proposition 3.4. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M. Suppose that the leaves are compact and that the holonomy of each regular leaf is trivial. Let σ be a local section and Ω be a connected component of the set of regular points in σ. Then:*

(a) Ω *is convex.*

(b) *If an isometry $\varphi_{[\beta]}$, as in Proposition* 3.1*, leaves Ω invariant, then it is the identity.*

(c) Holsing(σ) *is generated by the reflections in the hypersurfaces of the singular stratification of the local section.*

Proof. (a) Let p and q be points in Ω and γ the shortest segment of the geodesic that joins p to q. It follows from Proposition 3.3 that there exists a curve $\beta \subset \overline{\Omega}$ that joins p to q such that the length of β is equal to the length of γ. Since Ω is contained in a normal neighborhood, β coincides with γ and hence $\gamma \subset \overline{\Omega}$. Since the walls of the singular stratification are totally geodesic, γ cannot be tangent to them and hence $\gamma \subset \Omega$.

(b) Let p a point of Ω. Since the leaves are compact, L_p intercepts Ω only a finite number of times. Hence, there exists a number n_0 such that $\varphi_{[\beta]}^{n_0}(p) = p$. Let $K := \{\varphi_{[\beta]}^{i}(p)\}_{0 \leq i < n_0} \subset \Omega$.

Lemma 3.5. *There exists a unique $x \in \Omega$ such that the closed ball $\overline{B_r(x)} \supset K$ has minimal radius.*

Proof. The proof is standard, so we only sketch the main steps.

Claim 1. *There exists a closed ball $\overline{B_r(x)} \supset K$ with minimal radius. The center x belongs to Ω.*

Let $B_l(y)$ be a ball that contains K and define $Q := \overline{B_l(y) \cap \Omega}$. Since Ω is convex, Q is a compact convex set that contains K.

First, note that if $B_{r_0}(x_0) \supset K$ then there exists $\tilde{x}_0 \in Q$ such that $B_{r_0}(\tilde{x}_0) \supset K$. Indeed, if x_0 is not contained in Q then we can increase a radius ϵ until the ball $B_\epsilon(x_0)$ has a first contact point $\tilde{x}_0$ with Q. Since Q is convex, it is not difficulty to see that $B_{r_0}(\tilde{x}_0) \supset K$.

Now define

$$r := \inf\{\tilde{r} \mid \text{ there exists } \tilde{x} \text{ such that } B_{\tilde{r}}(\tilde{x}) \supset K\}.$$

Hence we can find sequences $\{r_n\}$ and $\{x_n\}$ such that $r_n \to r$ and $B_{r_n}(x_n) \supset K$. As we have noted above, we can suppose that $x_n \in Q$. Since Q is compact, there exists a subsequence of $\{x_n\}$ that converges to a point $x \in Q$. It is easy to see that $\overline{B_r(x)} \supset K$.

Finally, note that x belongs to $\Omega \cap Q$. Suppose that x belongs to $\partial\Omega \cap Q$. Let γ be a segment of the geodesic that contains x and is orthogonal to a wall P of $\partial\Omega$. Since the points of K do not belong to P, it is not difficulty to see that there exists a point $\tilde{x}$ in γ and a radius $\tilde{r} < r$ such that $\overline{B_{\tilde{r}}(\tilde{x})} \supset K$. This is a contradiction.

Claim 2. *A closed ball $\overline{B_r(x)} \supset K$ with minimal radius is unique.*

To prove this claim suppose that there exist two closed balls $\overline{B_r(x_1)}$ and $\overline{B_r(x_2)}$ that contain K and have minimal radius r. Let x_3 be the midpoint of the segment that joins x_1 to x_2. Then it is possible to find a radius $\tilde{r} < r$, such that $\overline{B_{\tilde{r}}(x_3)} \supset (\overline{B_r(x_1)} \cap \overline{B_r(x_2)})$ and this is a contradiction. □

Now we return to the proof of item (b) of the proposition.

Since $\varphi_{[\beta]}$ leaves K invariant, $K = \varphi_{[\beta]}(K) \subset \overline{B_r(\varphi_{[\beta]}(x))}$. Since $\overline{B_r(\varphi_{[\beta]}(x))}$ is the closed ball with minimal radius that contains K, $\varphi_{[\beta]}$ fixes the point $x \in \Omega$. On the other hand, since the holonomy of regular leaves is trivial, $d_x\varphi_{[\beta]}$ is the identity. Since $\varphi_{[\beta]}$ is an isometry, it is the identity.

(c) Let $\varphi_{[\beta]} \in \mathrm{Holsing}(\sigma)$. We can compose $\varphi_{[\beta]}$ with reflections R_i's in the walls of the singular stratification so that $R_1 \circ \cdots \circ R_k \circ \varphi_{[\beta]}$ leaves Ω invariant. Then it results from item (a) that $R_1 \circ \cdots \circ R_k \circ \varphi_{[\beta]}$ is the identity. Therefore we conclude that $\mathrm{Holsing}(\sigma)$ is generated by the reflections in the hypersurfaces of the singular stratification. □

Corollary 3.6. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M. Suppose that the leaves are compact and the holonomy of regular leaves is trivial. Consider a tubular neighborhood* $\mathrm{Tub}(L_q)$ *of a leaf L_q and let L_p be a regular leaf contained in* $\mathrm{Tub}(L_q)$. *Finally denote by $\Pi : \mathrm{Tub}(L_q) \to L_q$ the orthogonal projection. Then L_p is the total space of a fiber bundle with a projection Π, a basis L_q and a fiber diffeomorphic to an isoparametric submanifold of an euclidean space.*

Proof. First we recall that $\Pi : L_p \to L_q$ is a submersion because the foliation is singular.

Claim. $\Pi^{-1}(c) = \Sigma_c \cap L_p$ *has only one connected component.*

To prove this claim suppose that $\tilde{L}_x, \tilde{L}_y \subset \Sigma \cap L_p$ are two disjoint leaves of $\Sigma \cap \mathcal{F}$. We can suppose that both x and y belong to the same local section. Since $x, y \in L_p$, there exists $\varphi_{[\beta]} \in \mathrm{Holsing}(\sigma)$ such that $\varphi_{[\beta]}(x) = y$. The above corollary implies that $\varphi_{[\beta]}$ is a composition of reflections in the hypersurfaces of the singular stratification and hence that $y = \varphi_{[\beta]}(x)$ belongs to $\tilde{L}_x$. Therefore $\tilde{L}_x, \tilde{L}_y$ must be the same leaf.

Now the proposition follows from the slice theorem and from the following theorem of Ehresmann [4]:

Let $\Pi : L \to K$ be a submersion, where L and K are compact manifolds. Suppose that $\Pi^{-1}(c)$ has only one connected component for each value c. Then the pre-images are pairwise diffeomorphic and $\Pi : L \to K$ is the projection of a fiber bundle with total space L, basis K and fiber $\Pi^{-1}(c)$. □

PROPOSITION 3.7. *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and let σ be a local section. Consider a point $p \in \sigma$. Then*

$$\overline{\text{Holsing}(\sigma)} \cdot p = \overline{L_p \cap \sigma}.$$

In other words, the closure of $L_p \cap \sigma$ is an orbit of complete closed pseudogroup of local isometries. In particular, $\overline{L_p \cap \sigma}$ is a closed submanifold.

Proof. This result follows directly from results of E. Salem about pseudogroups of isometries (see Appendix D in [6]).

Using arguments of Salem (see Proposition 2.6 in [6]) one can prove that $\overline{\text{Holsing}(\sigma)}$ is complete and closed for the C^1 topology. It follows from Theorem 3.1 in [6] that a complete closed pseudogroup of isometry is a Lie pseudogroup. It also follows from E. Salem that an orbit of this Lie pseudogroup is a closed submanifold (see Corollary 3.3 in [6]). Therefore $\overline{\text{Holsing}(\sigma)} \cdot p$ is a closed submanifold. Now it is easy to see that $\overline{\text{Holsing}(\sigma)} \cdot p \supset \overline{\text{Holsing}(\sigma) \cdot p}$. It is also easy to see that $\overline{\text{Holsing}(\sigma)} \cdot p \subset \overline{\text{Holsing}(\sigma) \cdot p}$. To finish the proof we have only to recall that $\text{Holsing}(\sigma) \cdot p = L_p \cap \sigma$. □

4. Examples

In this section we illustrate some properties of the singular holonomy by constructing singular riemannian foliations with sections by means of suspension of homomorphisms.

We start by recalling the method of suspension. For further details see, for example, the book of Molino [6, pages 28,29,96,97].

Let B and T be riemannian manifolds with dimensions p and n, respectively, and let $\rho : \pi_1(B, b_0) \to \text{Iso}(T)$ be a homomorphism from the fundamental group of B to the group of isometries of T. Let $\hat{P} : \hat{B} \to B$ be the projection of the universal cover of B into B. Then we can define an action of $\pi_1(B, b_0)$ on $\widetilde{M} := \hat{B} \times T$ by

$$[\alpha] \cdot (\hat{b}, t) := ([\alpha] \cdot \hat{b}, \rho(\alpha^{-1}) \cdot t),$$

where $[\alpha] \cdot \hat{b}$ denotes the deck transformation associated to $[\alpha]$ applied to a point $\hat{b} \in \hat{B}$.

We denote the set of orbits of this action by M and the canonical projection by $\Pi : \widetilde{M} \to M$. One can show that M is a manifold. Indeed, given a simple open neighborhood $U_j \subset B$, we can construct the following bijection:

$$\begin{array}{rrcl} \Psi_j : & \Pi(\hat{P}^{-1}(U_j) \times T) & \to & U_j \times T \\ & \Pi(\hat{b}, t) & \to & (\hat{P}(\hat{b}) \times t). \end{array}$$

If $U_i \cap U_j \neq \emptyset$ and connected, we see that

$$\Psi_i \cap \Psi_j^{-1}(b, t) = (b, \rho([\alpha]^{-1})t)$$

for a fixed $[\alpha]$. So there exists a unique manifold structure on M for which Ψ_j are local diffeomorphisms. We define a map P by

$$\begin{array}{rlcl} P : & M & \to & B \\ & \Pi(\hat{b}, t) & \to & \hat{P}(\hat{b}) \end{array}$$

It follows that M is a total space of a fiber bundle, which has P as the projection over the basis B. Moreover, the fiber of this bundle is T and the structural group is given by the image of ρ.

Finally, we define $\mathcal{F} := \{\Pi(\hat{B}, t)\}$, i.e., the projection of the trivial foliation defined as the product of $\hat{B}$ with each t. One can show that this is a foliation transverse to the fibers of the fiber bundle. In addition, this foliation is a riemannian foliation whose transverse metric coincides with the metric of T.

Example 4.1. In what follows we construct a singular riemannian foliation with sections such that the intersection of a local section with the closure of a regular leaf is an orbit of an action of a subgroup of isometries of the local section. This illustrates Proposition 3.7.

Let T denote the product $\mathbf{R}^2 \times \mathbf{S}^1$ and $\hat{\mathcal{F}}_0$ the singular foliation of codimension 2 on T whose leaves are the product of a point in $\mathbf{S}^1$ with a circle in $\mathbf{R}^2$ centered at $(0, 0)$. It is easy to see that the foliation $\hat{\mathcal{F}}_0$ is a singular riemannian foliation with sections and these sections are cylinders. Let B be the circle $\mathbf{S}^1$ and q be an irrational number. Then we define the homomorphism ρ as

$$\begin{array}{rlcl} \rho : & \pi_1(B, b_0) & \to & \mathrm{Iso}(T) \\ & n & \to & ((x, s) \to (x, \exp(i\, n\, q) \cdot s)). \end{array}$$

Finally we set $\mathcal{F} := \Pi(\hat{B} \times \hat{\mathcal{F}}_0)$. It turns out that $\mathcal{F}$ is a singular riemannian foliation with sections such that the intersection of each section with the closure of a regular leaf is an orbit of an isometric action on the section. Indeed one can regard this action as a translation along the meridians of a cylinder, which is a section of the foliation.

Example 4.2. Next we construct a singular riemannian foliation with sections such that Holsing(σ) has an element that cannot be generated by the reflections in the hypersurfaces of the singular stratification.

Let T be a compact Lie group (e.g., $T = SU(3)$) and consider a manifold B such that $\pi_1(B) = \mathbf{Z}_2$ (e.g., $B = SO(n)$). We define a homomorphism ρ as follows:

$$\begin{array}{llll} \rho : & \pi_1(B, b_0) & \rightarrow & \mathrm{Iso}(T) \\ & 0 & \rightarrow & (t \rightarrow t) \\ & 1 & \rightarrow & (t \rightarrow t^{-1}). \end{array}$$

Let us consider the action of T on itself by conjugation, i.e., $t \cdot g := t\, g\, t^{-1}$. The orbits of this action are leaves of a singular riemannian foliation whose sections are tori. Denote this singular foliation by $\hat{F}_0$. Clearly $(T \cdot g)^{-1} = T \cdot g^{-1}$. This assures us that $\mathcal{F} := \Pi(\hat{B} \times \hat{\mathcal{F}}_0)$ is a singular foliation on M. Next let M be endowed with a metric inducing on the fibers the original metric of T. Then $\mathcal{F}$ turns out to be a singular riemannian foliation whose sections are contained in the fibers. These sections are tori.

Now it can be checked that the leaves of $\mathcal{F}$ intersect a Weyl chamber of each torus in more than one point. In fact, given a point x_1 belonging to a Weyl chamber, we can reflect it in the walls of the singular stratification to obtain another point x_2 belonging to another Weyl chamber and such that x_2^{-1} belongs to the same Weyl chamber of x_1. Since inverse points belong to the same leaf, x_2^{-1} belongs to the same leaf that contains x_1.

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