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A NOTE ON COMMUTATORS OF FRACTIONAL INTEGRALS WITH RBMO(μ) FUNCTIONS

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ABSTRACT. Let μ be a Borel measure on $\mathbb{R}^d$ which may be non-doubling. The only condition that μ must satisfy is $\mu(Q) \le c_0 l(Q)^n$ for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, for some fixed n with $0 < n \le d$. In this note we consider the commutators of fractional integrals with functions of the new BMO introduced by X. Tolsa.

1. Introduction

Let μ be a non-negative n-dimensional Borel measure on R^d, that is, a measure satisfying

$$\mu(Q) \le c_0 l(Q)^n$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \le d$. Throughout this note, all cubes we shall consider will be those with sides parallel to the coordinate axes. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$. Moreover, $Q(x, r)$ will be the cube centered at x with side length r.

The classical theory of harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^n, \mu)$ has been developed under the assumption that the underlying measure μ satisfies the doubling property, i.e., there exists a constant $c > 0$ such that $\mu(B(x, 2r)) \le c\mu(B(x, r))$ for every $x \in \mathbb{R}^n$ and $r > 0$. However, some recent results on Calderón-Zygmund operators ([4], [5], [6], [7]) and functions of bounded mean oscillation ([3], [8]) show that it should be possible to dispense with the doubling condition for most of the classical theory. The purpose of this note is to extend the main theorem in [1] to this new setting and strengthen the above point of view.

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Let us introduce some notations and definitions. Let $0 \le \beta < n$. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, we set

$$K_{Q,R}^{(\beta)} = 1 + \sum_{k=1}^{N_{Q,R}} \left[\frac{\mu(2^k Q)}{l(2^k Q)^n} \right]^{1-\beta/n},$$

where $N_{Q,R}$ is the first integer k such that $l(2^k Q) \ge l(R)$. If $\beta = 0$, then $K_{Q,R}^{(0)} = K_{Q,R}$. The latter concept was introduced by Tolsa in [8].

Given β_d (depending on d) large enough (for example, $\beta_d > 2^n$), we say that a cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \le \beta_d \mu(Q)$.

Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer ≥ 0 such that $2^N Q$ is doubling. We denote this cube by $\widetilde{Q}$.

Let $\eta > 1$ be a fixed constant. We say that $b \in L^1_{\text{loc}}(\mu)$ is in RBMO(μ) if there exists a constant c_1 such that for any cube Q

$$\frac{1}{\mu(\eta Q)} \int_Q |b - m_{\widetilde{Q}} b| d\mu \le c_1 \tag{1}$$

and

$$|m_Q b - m_R b| \le c_1 K_{Q,R} \text{ for any two doubling cubes } Q \subset R, \tag{2}$$

where $m_Q b = (1/\mu(Q)) \int_Q b d\mu$. The minimal constant c_1 is the RBMO(μ) norm of b, and it will be denoted by $\|b\|_*$. By Lemma 2.6 and Remark 2.9 in [8] one obtains equivalent norms in the space RBMO(μ) with different parameters $\eta > 1$ and $\beta_d > 2^n$.

2. Statement of the theorem and its proof

Now we can state the main result in this note.

THEOREM 1. *Let $b(x) \in$ RBMO(μ). Then the operator*

$$[b, I_\alpha](f)(x) = b(x) I_\alpha f(x) - I_\alpha(bf)(x)$$

satisfies

$$\|[b, I_\alpha](f)\|_q \le c \|b\|_* \|f\|_p,$$

where

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y),$$

$1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$ *and* $0 < \alpha < n$.

Before proving the theorem, we need another equivalent norm for RBMO(μ) and some lemmas.

Suppose that for a given function $b \in L^1_{\mathrm{loc}}(\mu)$ there exist some c_2 and a collection of numbers $\{b_Q\}_Q$ (i.e., for each cube Q there exists $b_Q \in \mathbb{R}$) such that

$$\sup_Q \frac{1}{\mu(\eta Q)} \int_Q |b - b_Q| d\mu \le c_2 \tag{3}$$

and

$$|b_Q - b_R| \le c_2 K_{Q,\,R} \text{ for any two cubes } Q \subset R. \tag{4}$$

Then we write $\|b\|_{**} = \inf c_2$, where the infimum is taken over all the constants c_2 and all the numbers $\{b_Q\}$ satisfying (3) and (4). By [8, Lemma 2.8, p. 99], for a fixed $\eta > 1$, the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are equivalent.

Lemma 1. *If $p > 1$ and $1/q = 1/p - \alpha/n$, $0 < \alpha < n$, then*

$$\|I_\alpha(f)\|_q \le c\|f\|_p.$$

If $p = 1$, then

$$\mu(\{x : I_\alpha(|f|)(x) > \lambda\}) \le (c/\lambda\|f\|_1)^{n/(n-\alpha)}.$$

Proof. See [2, p. 1269]. □

Lemma 2. *Let $p < r < n/\alpha$ and $1/q = 1/r - \alpha/n$. Then*

$$\|M^{(\alpha)}_{p,\,(\eta)} f\|_q \le c\|f\|_r,$$

where for $\eta > 1$ and $0 \le \beta < n/p$, $M^{(\beta)}_{p,\,(\eta)}$ is the non-centered maximal operator

$$M^{(\beta)}_{p,\,(\eta)} f(x) = \sup_{x \in Q} \left(\frac{1}{\mu(\eta Q)^{1-\beta p/n}} \int_Q |f(y)|^p d\mu(y) \right)^{1/p},$$

and when $\beta = 0$, we denote $M^{(0)}_{p,\,(\eta)}$ by $M_{p,\,(\eta)}$.

Proof. Note that for $0 \le \beta < n/p$ and $\eta > 1$, $M^{(\beta)}_{p,\,(\eta)}$ is controlled by the operator defined as

$$\widetilde{M}^{(\beta)}_{p,\,(\eta)} f(x) = \sup_{x \in \eta^{-1} Q} \left(\frac{1}{\mu(Q)^{1-\beta p/n}} \int_Q |f(y)|^p d\mu(y) \right)^{1/p}.$$

We only need to prove the lemma for $\widetilde{M}^{(\alpha)}_{p,\,(\eta)}$. We first prove that

$$\mu(\{x : \widetilde{M}^{(\alpha)}_{p,\,(\eta)} f(x) > \lambda\}) \le (c/\lambda\|f\|_p)^{np/(n-\alpha p)}.$$

Let us consider the set E defined by

$$E = \{x : \widetilde{M}^{(\alpha)}_{p,\,(\eta)} f(x) > \lambda\}.$$

By the Besicovitch covering lemma it follows that there exists a sequence of cubes Q_j, with bounded overlap, so that $E \subset \bigcup_j Q_j$ and on each Q_j we have

$$\frac{1}{\mu(Q_j)^{1-\alpha p/n}} \int_{Q_j} |f|^p d\mu \geq \lambda^p.$$

Let $q = np/(n - \alpha p)$. Then $p/q \leq 1$. Hence,

$$\mu(E)^{p/q} \leq \mu \left(\bigcup_j Q_j \right)^{p/q} \leq \sum_j \mu(Q_j)^{p/q}.$$

Now

$$\mu(Q_j)^{1-\alpha p/n} \leq \frac{1}{\lambda^p} \int_{Q_j} |f|^p d\mu,$$

and since $p/q = 1 - \alpha p/n$,

$$\sum_j \mu(Q_j)^{p/q} \leq \frac{1}{\lambda^p} \int |f|^p \left(\sum_j \chi_{Q_j} \right) d\mu.$$

Hence

$$\mu(E) \leq \frac{c}{\lambda^q} \|f\|_p^q.$$

Note now that if $p < s < n/\alpha$, then using Hölder's inequality

$$\widetilde{M}^{(\alpha)}_{p,(\eta)} f(x) \leq \widetilde{M}^{(\alpha)}_{s,(\eta)} f(x).$$

Hence by the preceding arguments we have

$$\mu(E) \leq \left(\frac{c}{\lambda} \|f\|_s \right)^{ns/(n-\alpha s)}.$$

The lemma follows by the Marcinkiewicz interpolation theorem. □

LEMMA 3. *For $K^{(\beta)}_{Q,R}$, $0 \leq \beta < n$, we have the following properties:*

1. *If $Q \subset R \subset S$ are cubes in $\mathbb{R}^d$, then $K^{(\beta)}_{Q,R} \leq K^{(\beta)}_{Q,S}$, $K^{(\beta)}_{R,S} \leq cK^{(\beta)}_{Q,S}$ and $K^{(\beta)}_{Q,S} \leq c(K^{(\beta)}_{Q,R} + K^{(\beta)}_{R,S})$.*
2. *If $Q \subset R$ have comparable sizes, then $K^{(\beta)}_{Q,R} \leq c$.*
3. *If N is a positive integer and the cubes $2Q, 2^2Q, \ldots, 2^{N-1}Q$ are non-doubling, then $K^{(\beta)}_{Q,2^NQ} \leq c$. So, $K^{(\beta)}_{Q,\tilde{Q}} \leq c$.*

Proof. The properties (1) and (2) are easy to check. Let us prove (3). Note that $\beta_d > 2^n$. For $k = 1, \ldots, N-1$, we have $\mu(2^{k+1}Q) > \beta_d \mu(2^k Q)$. Thus

$$\mu(2^k Q) < \frac{\mu(2^N Q)}{\beta_d^{N-k}}$$

for $k = 1, \ldots, N-1$. Therefore

$$\begin{aligned} K^{(\beta)}_{Q,\,2^N Q} &\le 1 + \sum_{k=1}^{N-1} \left[\frac{\mu(2^N Q)}{\beta_d^{N-k} l(2^k Q)^n} \right]^{1-\beta/n} + \left[\frac{\mu(2^N Q)}{l(2^N Q)^n} \right]^{1-\beta/n} \\ &\le 1 + c_0^{1-\beta/n} + \left[\frac{\mu(2^N Q)}{l(2^N Q)^n} \right]^{1-\beta/n} \sum_{k=1}^{N-1} \left[\frac{1}{\beta_d^{N-k} 2^{(k-N)n}} \right]^{1-\beta/n} \\ &\le 1 + c_0^{1-\beta/n} + c_0^{1-\beta/n} \sum_{k=1}^{\infty} (2^n/\beta_d)^{k(1-\beta/n)} \le c. \qquad \square \end{aligned}$$

In [8], Tolsa defined a sharp maximal operator $M^{\#} f(x)$ such that

$$f \in \mathrm{RBMO}(\mu) \Longleftrightarrow M^{\#} f \in L^\infty(\mu).$$

In order to prove the theorem, we need to introduce a variant of this sharp maximal operator $M^{\#,\,(\beta)} f(x)$ such that $M^{\#} f(x) = M^{\#,\,(0)} f(x)$. We define

$$M^{\#,\,(\beta)} f(x) = \sup_{x \in Q} \frac{1}{\mu((3/2)Q)} \int_Q |f - m_{\widetilde{Q}} f| d\mu + \sup_{\substack{x \in Q \subset R \\ Q,\, R \,\text{doubling}}} \frac{|m_Q f - m_R f|}{K^{(\beta)}_{Q,\,R}}.$$

We also consider the non-centered doubling maximal operator N, defined by

$$N f(x) = \sup_{\substack{x \in Q \\ Q \,\text{doubling}}} \frac{1}{\mu(Q)} \int_Q |f| d\mu.$$

By Remark 2.3 of [8], for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \frac{1}{\mu(Q_k)} \int_{Q_k} b(y) d\mu(y) = b(x).$$

So, $|f(x)| \le N f(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Moreover, it is easy to show that N is of weak type $(1, 1)$ and bounded on $L^p(\mu)$, $p \in (1, \infty]$.

LEMMA 4. *Let $f \in L^1_{\mathrm{loc}}(\mu)$ with $\int f d\mu = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$, if $\inf(1, Nf) \in L^p(\mu)$, then for $0 \le \beta < n$ we have*

$$\|N f\|_{L^p(\mu)} \le c \|M^{\#,\,(\beta)} f\|_{L^p(\mu)}.$$

When $\beta = 0$, this is Theorem 6.2 of [8]. With minor changes in the proof one can obtain the present lemma. We omit the proof here for brevity.

LEMMA 5. *For $0 \le \beta < n$ there exists a constant P_β (large enough) depending on c_0, n and β such that if $Q_1 \subset Q_2 \subset \cdots \subset Q_m$ are concentric* *cubes with* $K^{(\beta)}_{Q_i,Q_{i+1}} > P_\beta$ *for* $i = 1, 2, \dots, m-1$*, then*

$$\sum_{i=1}^{m-1} K^{(\beta)}_{Q_i,Q_{i+1}} \le c_3 K^{(\beta)}_{Q_1,Q_m},$$

where c_3 *depends only on* c_0*,* n *and* β*.*

Proof. Let Q'_i be a cube concentric with Q_i such that $l(Q_i) \le l(Q'_i) < 2l(Q_i)$ with $l(Q'_i) = 2^k l(Q_1)$ for some $k \ge 0$. Then

$$c_4^{-1} K^{(\beta)}_{Q_i,\,Q_{i+1}} \le K^{(\beta)}_{Q'_i,\,Q'_{i+1}} \le c_4 K^{(\beta)}_{Q_i,\,Q_{i+1}},$$

for all i with c_4 depending on c_0, n and β.

Observe also that if we take P_β so that $c_4^{-1} P_\beta \ge 2$, then $K^{(\beta)}_{Q'_i,\,Q'_{i+1}} > 2$ and so

$$K^{(\beta)}_{Q'_i,\,Q'_{i+1}} \le 2 \sum_{k=1}^{N_{Q'_i,\,Q'_{i+1}}} \left[\frac{\mu(2^k Q'_i)}{l(2^k Q'_i)^n}\right]^{1-\beta/n}.$$

Therefore

$$\sum_{i=1}^{m-1} K^{(\beta)}_{Q'_i,\,Q'_{i+1}} \le 2 \sum_{i=1}^{m-1} \sum_{k=1}^{N_{Q'_i,\,Q'_{i+1}}} \left[\frac{\mu(2^k Q'_i)}{l(2^k Q'_i)^n}\right]^{1-\beta/n}. \tag{5}$$

On the other hand, if P_β is large enough, then $Q'_i \ne Q'_{i+1}$. Indeed,

$$c_0^{1-\beta/n} N_{Q_i,\,Q_{i+1}} \ge \sum_{k=1}^{N_{Q_i,\,Q_{i+1}}} \left[\frac{\mu(2^k Q_i)}{l(2^k Q_i)^n}\right]^{1-\beta/n} \ge P_\beta - 1,$$

and so $N_{Q_i,\,Q_{i+1}} \ge (P_\beta - 1)/c_0^{1-\beta/n} > 2$, assuming P_β large enough. This implies $l(Q_{i+1}) > 2l(Q_i)$, so $Q'_i \ne Q'_{i+1}$. As a consequence, there is no overlapping in the terms $[\mu(2^k Q'_i)/l(2^k Q'_i)^n]^{1-\beta/n}$ on the right hand side of (5). Thus

$$\sum_{i=1}^{m-1} K^{(\beta)}_{Q_i,\,Q_{i+1}} \le c_4 \sum_{i=1}^{m-1} K^{(\beta)}_{Q'_i,\,Q'_{i+1}} \le 2c_4 K^{(\beta)}_{Q'_1,\,Q'_m} \le 2c_4^2 K^{(\beta)}_{Q_1,\,Q_m}. \qquad \square$$

Lemma 6. *For* $0 \le \beta < n$ *there exists a constant* P'_β *(large enough) depending on* c_0*,* n *and* β *such that if* $x \in \mathbb{R}^d$ *is a fixed point and* $\{f_Q\}_{Q \ni x}$ *is a collection of numbers such that* $|f_Q - f_R| \le K^{(\beta)}_{Q,\,R} C_x$ *for all doubling cubes* $Q \subset R$ *with* $x \in Q$ *such that* $K^{(\beta)}_{Q,\,R} \le P'_\beta$*, then*

$$|f_Q - f_R| \le c_5 K^{(\beta)}_{Q,\,R} C_x \ \text{ for all doubling cubes } Q \subset R \text{ with } x \in Q,$$

where c_5 *depends on* c_0*,* n *and* β*.*

Proof. Let $Q \subset R$ be two doubling cubes in $\mathbb{R}^d$ with $x \in Q =: Q_0$. Let Q_1 be the first cube of the form $2^k Q$, $k \geq 0$, such that $K^{(\beta)}_{Q,\,Q_1} > P_\beta$. Since $K^{(\beta)}_{Q,\,2^{-1}Q_1} \leq P_\beta$, we have $K^{(\beta)}_{Q,\,Q_1} \leq 2P_\beta + c_6$ by Lemma 3. So, for the doubling cube $\widetilde{Q}_1$, we have $K^{(\beta)}_{Q,\,\widetilde{Q}_1} \leq c_7$ with c_7 depending on P_β, n, c_0 and β.

In general, given $\widetilde{Q_i}$, we denote by Q_{i+1} the first cube of the form $2^k \widetilde{Q_i}$, $k \geq 0$, such that $K^{(\beta)}_{\widetilde{Q_i},\,Q_{i+1}} > P_\beta$. We consider the doubling cube $\widetilde{Q_{i+1}}$. We have $K^{(\beta)}_{\widetilde{Q_i},\,\widetilde{Q_{i+1}}} \leq c_7$ and $K^{(\beta)}_{\widetilde{Q_i},\,\widetilde{Q_{i+1}}} \geq K^{(\beta)}_{\widetilde{Q_i},\,Q_{i+1}} > P_\beta$. Then we obtain

$$|f_Q - f_R| \leq \sum_{i=1}^{N} |f_{\widetilde{Q_{i-1}}} - f_{\widetilde{Q_i}}| + |f_{\widetilde{Q_N}} - f_R|, \tag{6}$$

where $\widetilde{Q_N}$ is the first cube of the sequence $\{\widetilde{Q_i}\}_i$ such that $\widetilde{Q_{N+1}} \supset R$. Since $K^{(\beta)}_{\widetilde{Q_N},\,\widetilde{Q_{N+1}}} \leq c_7$, we also have $K^{(\beta)}_{\widetilde{Q_N},\,R} \leq c_7$. By (6) and Lemma 5, if we set $P'_\beta = c_7$, we get

$$\begin{aligned} |f_Q - f_R| &\leq \sum_{i=1}^{N} K^{(\beta)}_{\widetilde{Q_{i-1}},\,\widetilde{Q_i}} C_x + K^{(\beta)}_{\widetilde{Q_N},\,R} C_x \\ &\leq c K^{(\beta)}_{Q,\,\widetilde{Q_N}} C_x + K^{(\beta)}_{\widetilde{Q_N},\,R} C_x \leq c K^{(\beta)}_{Q,\,R} C_x. \end{aligned}$$

□

Proof of Theorem 1. For all $p \in (1,\, n/\alpha)$ we will prove the following sharp maximal function estimate:

$$M^{\#,\,(\alpha)}([b,\, I_\alpha]f)(x) \leq c_p \|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)} f(x) + M_{p,\,(3/2)}(I_\alpha f)(x) + I_\alpha(|f|)(x) \right).$$

Then, if we take r such that $1 < r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, we get

$$\begin{aligned} \|[b,\, I_\alpha]f\|_q &\leq \|N([b,\, I_\alpha]f)\|_q \leq c\|M^{\#,\,(\alpha)}([b,\, I_\alpha]f)\|_q \\ &\leq c\|b\|_* \left(\|M^{(\alpha)}_{r,\,(9/8)} f\|_q + \|M_{r,\,(3/2)}(I_\alpha f)\|_q + \|I_\alpha(|f|)\|_q \right) \\ &\leq c\|b\|_* \|f\|_p. \end{aligned}$$

Thus it remains to prove the above sharp maximal function estimate.

Let $\{b_Q\}_Q$ be a family of numbers satisfying

$$\int_Q |b - b_Q| d\mu \leq 2\mu(2Q)\|b\|_{**}$$

for any cube Q, and

$$|b_Q - b_R| \leq 2K_{Q,\,R}\|b\|_{**}$$

for all cubes $Q \subset R$. For any cube Q, we set

$$h_Q := m_Q \left(I_\alpha((b - b_Q) f \chi_{\mathbb{R}^d \setminus (4/3)Q}) \right).$$

We will prove that

$$\frac{1}{\mu((3/2)Q)}\int_Q |[b,\, I_\alpha]f - h_Q|\,d\mu \le c\|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)}f(x) + M_{p,\,(3/2)}(I_\alpha f)(x)\right) \tag{7}$$

for all x and Q with $x \in Q$, and

$$|h_Q - h_R| \le c\|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)}f(x) + I_\alpha(|f|)(x)\right) K_{Q,\,R}K^{(\alpha)}_{Q,\,R} \tag{8}$$

for all cubes $Q \subset R$ with $x \in Q$.

To get (7) for some fixed cube Q and x with $x \in Q$, we write $[b,\, I_\alpha]f$ in the form

$$[b,\, I_\alpha]f = (b - b_Q)I_\alpha f - I_\alpha((b - b_Q)f_1) - I_\alpha((b - b_Q)f_2), \tag{9}$$

where $f_1 = f\chi_{(4/3)Q}$ and $f_2 = f - f_1$.

Let us first estimate the term $(b - b_Q)I_\alpha f$:

$$\begin{aligned}\frac{1}{\mu((3/2)Q)}\int_Q |(b - b_Q)I_\alpha f|\,d\mu &\le \left(\frac{1}{\mu((3/2)Q)}\int_Q |b - b_Q|^{p'}d\mu\right)^{1/p'} \\ &\quad\times \left(\frac{1}{\mu((3/2)Q)}\int_Q |I_\alpha f|^p d\mu\right)^{1/p} \\ &\le c\|b\|_* M_{p,\,(3/2)}(I_\alpha f)(x).\end{aligned} \tag{10}$$

Next we are going to estimate the second term on the right hand side of (9). We take $s = \sqrt{p}$. Then we have

$$\begin{aligned}\frac{1}{\mu((3/2)Q)}\int_Q |I_\alpha((b - b_Q)f_1)|\,d\mu &\le \frac{\mu(Q)^{1-1/r}}{\mu((3/2)Q)}\|I_\alpha((b - b_Q)f_1)\|_{L^r(\mu)} \\ &\le c\frac{\mu(Q)^{1-1/r}}{\mu((3/2)Q)}\|(b - b_Q)f_1\|_{L^s(\mu)} \quad (1/r = 1/s - \alpha/n) \\ &\le c\frac{\mu(Q)^{1-1/r}}{\mu((3/2)Q)}\left(\int_{(4/3)Q} |(b - b_Q)f_1|^s d\mu\right)^{1/s} \\ &\le c\frac{1}{\mu((3/2)Q)^{1/r}}\left(\int_{(4/3)Q} |b - b_Q|^{ss'}d\mu\right)^{1/ss'}\left(\int_{(4/3)Q} |f|^p d\mu\right)^{1/p} \\ &\le c\left(\frac{1}{\mu((3/2)Q)}\int_{(4/3)Q} |b - b_Q|^{ss'}d\mu\right)^{1/ss'} \\ &\quad\times\left(\frac{1}{\mu((3/2)Q)^{1-\alpha p/n}}\int_{(4/3)Q} |f|^p d\mu\right)^{1/p} \\ &\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).\end{aligned} \tag{11}$$

By (9), (10) and (11), to get (7) it remains to estimate the difference $|I_\alpha((b-b_Q)f_2)-h_Q|$. For $y_1,\, y_2\in Q$ we have

(12)

$$\begin{aligned}
&|I_\alpha((b-b_Q)f_2)(y_1)-I_\alpha((b-b_Q)f_2)(y_2)|\\
&\quad\le c\int_{\mathbb{R}^d\setminus(4/3)Q}\frac{|y_2-y_1|}{|z-y_1|^{n+1-\alpha}}|b(z)-b_Q||f(z)|d\mu(z)\\
&\quad\le c\sum_{k=1}^{\infty}\int_{2^k(4/3)Q\setminus 2^{k-1}(4/3)Q}\frac{l(Q)}{|z-y_1|^{n+1-\alpha}}\Bigg(|b(z)-b_{2^k(4/3)Q}|\\
&\qquad\qquad+|b_Q-b_{2^k(4/3)Q}|\Bigg)|f(z)|d\mu(z)\\
&\quad\le c\sum_{k=1}^{\infty}2^{-k}\frac{1}{l(2^kQ)^{n-\alpha}}\int_{2^k(4/3)Q}|b(z)-b_{2^k(4/3)Q}||f(z)|d\mu(z)\\
&\qquad\qquad+c\sum_{k=1}^{\infty}k2^{-k}\|b\|_*\frac{1}{l(2^kQ)^{n-\alpha}}\int_{2^k(4/3)Q}|f(z)|d\mu(z)\\
&\quad\le c\sum_{k=1}^{\infty}2^{-k}\left(\frac{1}{\mu(2^k(3/2)Q)}\int_{2^k(4/3)Q}|b-b_{2^k(4/3)Q}|^{p'}d\mu\right)^{1/p'}\\
&\qquad\qquad\quad\times\left(\frac{1}{\mu(2^k(3/2)Q)^{1-\alpha p/n}}\int_{2^k(4/3)Q}|f|^pd\mu\right)^{1/p}\\
&\qquad\qquad+c\sum_{k=1}^{\infty}k2^{-k}\|b\|_*\left(\frac{1}{\mu(2^k(3/2)Q)^{1-\alpha p/n}}\int_{2^k(4/3)Q}|f|^pd\mu\right)^{1/p}\\
&\quad\le c\sum_{k=1}^{\infty}2^{-k}\|b\|_*M^{(\alpha)}_{p,\,(9/8)}f(x)+c\sum_{k=1}^{\infty}k2^{-k}\|b\|_*M^{(\alpha)}_{p,\,(9/8)}f(x)\\
&\quad\le c\|b\|_*M^{(\alpha)}_{p,\,(9/8)}f(x),
\end{aligned}$$

where we used the fact that

$$|b_Q-b_{2^k(4/3)Q}|\le 2K_{Q,\,2^k(4/3)Q}\|b\|_{**}\le ck\|b\|_*.$$

Taking the mean over $y_2\in Q$, we get

$$\begin{aligned}
|I_\alpha((b-b_Q)f_2)(y_1)-h_Q| &= |I_\alpha((b-b_Q)f_2)(y_1)-m_Q(I_\alpha((b-b_Q)f_2))|\\
&\le c\|b\|_*M^{(\alpha)}_{p,\,(9/8)}f(x).
\end{aligned}$$

Thus

$$\frac{1}{\mu((3/2)Q)}\int_Q |I_\alpha((b-b_Q)f_2)(y_1)-h_Q|d\mu(y_1)\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x), \tag{13}$$

and so (7) holds.

Now we have to check the regularity condition (8) for the numbers $\{h_Q\}_Q$. Consider two cubes $Q\subset R$ with $x\in Q$. We set $N=N_{Q,\,R}+1$. We write the difference $|h_Q-h_R|$ in the form

$$\begin{aligned}
&|m_Q(I_\alpha((b-b_Q)f\chi_{\mathbb{R}^d\setminus(4/3)Q}))-m_R(I_\alpha((b-b_R)f\chi_{\mathbb{R}^d\setminus(4/3)R}))|\\
&\quad\le |m_Q(I_\alpha((b-b_Q)f\chi_{2Q\setminus(4/3)Q}))|+|m_Q(I_\alpha((b_Q-b_R)f\chi_{\mathbb{R}^d\setminus 2Q}))|\\
&\qquad+|m_Q(I_\alpha((b-b_R)f\chi_{2^NQ\setminus 2Q}))|+|m_R(I_\alpha((b-b_R)f\chi_{2^NQ\setminus(4/3)R}))|\\
&\qquad+|m_Q(I_\alpha((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ}))-m_R(I_\alpha((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ}))|\\
&\quad:=U_1+U_2+U_3+U_4+U_5.
\end{aligned}$$

Let us estimate U_1. For $y\in Q$ we have

$$\begin{aligned}
|I_\alpha((b-b_Q)f\chi_{2Q\setminus(4/3)Q})(y)|&\le \frac{c}{l(Q)^{n-\alpha}}\int_{2Q}|b-b_Q||f|d\mu\\
&\le \frac{c}{l(Q)^{n-\alpha}}\left(\int_{2Q}|b-b_Q|^{p'}d\mu\right)^{1/p'}\left(\int_{2Q}|f|^p d\mu\right)^{1/p}\\
&\le c\left(\frac{1}{\mu(3Q)}\int_{2Q}|b-b_Q|^{p'}d\mu\right)^{1/p'}\left(\frac{1}{\mu((9/4)Q)^{1-\alpha p/n}}\int_{2Q}|f|^p d\mu\right)^{1/p}\\
&\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).
\end{aligned}$$

Hence we obtain

$$U_1\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).$$

Next, consider the term U_2. For $x,\,y\in Q$, it is easily seen that

$$|I_\alpha(f\chi_{\mathbb{R}^d\setminus 2Q})(y)|\le I_\alpha(|f|)(x)+cM^{(\alpha)}_{p,\,(9/8)}f(x).$$

Thus

$$\begin{aligned}
U_2&=\left|\frac{1}{\mu(Q)}\int_Q (b_Q-b_R)I_\alpha(f\chi_{\mathbb{R}^d\setminus 2Q})(y)d\mu\right|\\
&\le cK_{Q,\,R}\|b\|_*\left(I_\alpha(|f|)(x)+M^{(\alpha)}_{p,\,(9/8)}f(x)\right).
\end{aligned}$$

The term U_4 is easy to estimate. Calculations similar to those carried out for U_1 yield

$$U_4\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).$$

Let us now turn to the term U_5. Arguing as in (12), for any $y,\,z\in R$, we get

$$|I_\alpha((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})(y)-I_\alpha((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})(z)|\le c\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).$$

Taking the mean over Q for y and over R for z, we obtain

$$U_5 \leq c\|b\|_* M^{(\alpha)}_{p,\,(9/8)} f(x).$$

Finally, it remains to deal with U_3. For $y \in Q$ we have

$$\begin{aligned}|I_\alpha((b-b_R)f\chi_{2^NQ\setminus 2Q})(y)| &\leq c\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^{n-\alpha}}\int_{2^{k+1}Q\setminus 2^kQ}|b-b_R||f|d\mu \\ &\leq c\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^{n-\alpha}}\left(\int_{2^{k+1}Q}|b-b_R|^{p'}d\mu\right)^{1/p'}\left(\int_{2^{k+1}Q}|f|^p d\mu\right)^{1/p}.\end{aligned}$$

Note that

$$\begin{aligned}&\left(\int_{2^{k+1}Q}|b-b_R|^{p'}d\mu\right)^{1/p'} \\ &\qquad\leq \left(\int_{2^{k+1}Q}|b-b_{2^{k+1}Q}|^{p'}d\mu\right)^{1/p'} + \mu(2^{k+1}Q)^{1/p'}|b_{2^{k+1}Q}-b_R| \\ &\qquad\leq cK_{Q,\,R}\|b\|_*\mu(2^{k+2}Q)^{1/p'}.\end{aligned}$$

Thus

$$\begin{aligned}&|I_\alpha((b-b_R)f\chi_{2^NQ\setminus 2Q})(y)| \\ &\quad\leq cK_{Q,\,R}\|b\|_*\sum_{k=1}^{N-1}\frac{\mu(2^{k+2}Q)^{1/p'}}{l(2^kQ)^{n-\alpha}}\left(\int_{2^{k+1}Q}|f|^p d\mu\right)^{1/p} \\ &\quad\leq cK_{Q,\,R}\|b\|_*\sum_{k=1}^{N_{Q,\,R}}\frac{\mu(2^{k+2}Q)^{1-\alpha/n}}{l(2^kQ)^{n-\alpha}}\left(\frac{1}{\mu(2^{k+2}Q)^{1-\alpha p/n}}\int_{2^{k+1}Q}|f|^p d\mu\right)^{1/p} \\ &\quad\leq cK_{Q,\,R}K^{(\alpha)}_{Q,\,R}\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).\end{aligned}$$

Taking the mean over Q, we get

$$U_3 \leq cK_{Q,\,R}K^{(\alpha)}_{Q,\,R}\|b\|_* M^{(\alpha)}_{p,\,(9/8)}f(x).$$

From the estimates on U_1, U_2, U_3, U_4 and U_5, the regularity condition (8) follows.

Let us see how from (7) and (8) one obtains the sharp maximal function estimate. By (7), if Q is a doubling cube and $x \in Q$, we have

$$\begin{aligned}|m_Q([b,\,I_\alpha]f) - h_Q| &\leq \frac{1}{\mu(Q)}\int_Q |[b,\,I_\alpha]f - h_Q|d\mu \\ &\leq c\|b\|_*\left(M^{(\alpha)}_{p,\,(9/8)}f(x) + M_{p,\,(3/2)}(I_\alpha f)(x)\right).\end{aligned} \tag{14}$$

Also, for any cube Q with $x \in Q$, $K_{Q,\widetilde{Q}} \leq c$ and $K^{(\alpha)}_{Q,\widetilde{Q}} \leq c$, we have, by (7) and (8),

$$
\begin{aligned}
(15)\quad & \frac{1}{\mu((3/2)Q)} \int_Q \Big| [b,\, I_\alpha] f - m_{\widetilde{Q}}([b,\, I_\alpha] f) \Big| \, d\mu \\
& \leq \frac{1}{\mu((3/2)Q)} \int_Q |[b,\, I_\alpha] f - h_Q| \, d\mu + |h_Q - h_{\widetilde{Q}}| + |h_{\widetilde{Q}} - m_{\widetilde{Q}}([b,\, I_\alpha] f)| \\
& \leq c \|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)} f(x) + M_{p,\,(3/2)}(I_\alpha f)(x) + I_\alpha(|f|)(x) \right).
\end{aligned}
$$

On the other hand, for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K^{(\alpha)}_{Q,\,R} \leq P'_\alpha$, where P'_α is the constant in Lemma 6, we have by (8)

$$|h_Q - h_R| \leq c K_{Q,\,R} \|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)} f(x) + I_\alpha(|f|)(x) \right) P'_\alpha.$$

Hence by Lemma 6 we get

$$|h_Q - h_R| \leq c K^{(\alpha)}_{Q,\,R} \|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)} f(x) + I_\alpha(|f|)(x) \right)$$

for all doubling cubes $Q \subset R$ with $x \in Q$, and using (14) again, we obtain

$$
\begin{aligned}
& |m_Q([b,\, I_\alpha] f) - m_R([b,\, I_\alpha] f)| \\
& \quad \leq c K^{(\alpha)}_{Q,\,R} \|b\|_* \left(M^{(\alpha)}_{p,\,(9/8)} f(x) + M_{p,\,(3/2)}(I_\alpha f)(x) + I_\alpha(|f|)(x) \right).
\end{aligned}
$$

From this estimate and (15) we get the sharp maximal function estimate. □

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