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AMENABILITY AND THE BICROSSED PRODUCT CONSTRUCTION

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ABSTRACT. We study stability properties of amenable locally compact quantum groups under the bicrossed product construction. We obtain as our main result an equivalence between amenability of the bicrossed product and amenability of the matched quantum groups used as building ingredients of the bicrossed product. Finally, we give examples of non-amenable locally compact quantum groups obtained by a bicrossed product construction.

1. Introduction

The theory of locally compact quantum groups has been introduced by J. Kustermans and the third author [9][10], unifying compact quantum groups and Kac algebras. As the example of the quantum $SU_q(2)$-group, developed by Woronowicz, shows, the antipode of a compact quantum group need not be bounded and it need not respect the *-operation. For this reason, compact quantum groups are not always Kac algebras. The crucial difference between Kac algebras and locally compact quantum groups is the possible unboundedness of the antipode.

Taking into account the importance of amenable locally compact groups within the category of all locally compact groups, it is natural to consider amenability of locally compact quantum groups. In fact, the main results on amenability of Kac algebras have been developed by Enock and Schwartz [6], and their proofs can be repeated in the more general framework of locally compact quantum groups. However, there is still one open problem. Recall that there are many different characterizations of amenability of locally compact groups. A first characterization deals with the existence of an invariant mean on a suitable algebra of functions on the group G (bounded continuous

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functions or $L^\infty(G)$). Another characterization says that the trivial representation of G is weakly contained in the left regular representation. Other characterizations are most of the time closely related to one of these two definitions. These two properties can be formulated for locally compact quantum groups; in this way, one defines *weakly amenable* and *strongly amenable* locally compact quantum groups. It is known that all strongly amenable quantum groups are weakly amenable. The converse has been claimed by Enock and Schwartz [6] on the level of Kac algebras. However, their proof contains a gap as pointed out by Ruan [14]. In the latter paper, Ruan closed the gap for discrete Kac algebras. Recently, E. Blanchard and the third author extended the result of Ruan to arbitrary discrete quantum groups. Even more recently, Tomatsu [15] found an alternative proof of the same result. Also, for locally compact groups the two notions of amenability coincide; see, e.g., [7]. However, in the general case this remains an open problem.

Having defined both notions of amenability for locally compact quantum groups, one asks for examples. A systematic way of constructing examples of locally compact quantum groups has been developed by Majid [11], Baaj and Skandalis [2] and Vainerman and the third author [19]. In this paper, we precisely characterize when these locally compact quantum groups are amenable. We also give two examples of non-amenable locally compact quantum groups obtained by this so-called bicrossed product construction.

In [19], one also defines bicrossed products of quantum groups, and one makes the link with short exact sequences of locally compact quantum groups, called extensions. In this paper, we will characterize in this full generality when the bicrossed product is amenable; in fact, our result is a quantum version of the well known result that a locally compact group G with normal closed subgroup H is amenable if and only if H and G/H are amenable.

2. Preliminaries

We refer to [9] and [10] for the theory of locally compact quantum groups in the C^*-algebra, as well as in the von Neumann algebra language. For the non-specialists, [18] is a good starting point. We recall from [10] the definition of a von Neumann algebraic quantum group: (M, Δ) is called a *(von Neumann algebraic) locally compact quantum group* when

- M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- there exist normal, semi-finite, faithful (n.s.f.) weights φ and ψ on M such that
  - φ is left invariant, i.e., $\varphi\big((\omega\otimes\iota)\Delta(x)\big) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$,

– ψ is right invariant, i.e., $\psi\big((\iota\otimes\omega)\Delta(x)\big)=\psi(x)\omega(1)$ for all $x\in\mathcal{M}_\psi^+$ and $\omega\in M_*^+$.

Here we use the notation $\mathcal{M}_\varphi^+=\{x\in M^+ \mid \varphi(x)<+\infty\}$, and define analogously $\mathcal{M}_\psi^+$.

Fix a left invariant n.s.f. weight φ on (M,Δ) and represent M on the GNS-space of φ such that (H,ι,Λ) is a GNS-construction for φ. Then we can define a unitary W on $H\otimes H$ by

$$W^*(\Lambda(a)\otimes\Lambda(b))=(\Lambda\otimes\Lambda)(\Delta(b)(a\otimes 1))\quad\text{for all } a,b\in\mathcal{N}_\varphi\ .$$

Here $\Lambda\otimes\Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi\otimes\varphi$. One proves that W satisfies the pentagonal equation: $W_{12}W_{13}W_{23}=W_{23}W_{12}$. We say that W is a *multiplicative unitary.* The comultiplication can be given in terms of W by the formula $\Delta(x)=W^*(1\otimes x)W$ for all $x\in M$. Also the von Neumann algebra M can be written in terms of W as

$$M=\{(\iota\otimes\omega)(W)\mid\omega\in \mathrm{B}(H)_*\}^{-\ \sigma\text{-strong}^*}\quad .$$

Next, the locally compact quantum group (M,Δ) has an *antipode* S, which is the unique σ-strong* closed linear map from M to M satisfying $(\iota\otimes\omega)(W)\in D(S)$ for all $\omega\in\mathrm{B}(H)_*$, $S(\iota\otimes\omega)(W)=(\iota\otimes\omega)(W^*)$, and such that the elements $(\iota\otimes\omega)(W)$ form a σ-strong* core for S. The antipode S has a polar decomposition $S=R\tau_{-i/2}$, where R is an anti-automorphism of M and (τ_t) is a strongly continuous one-parameter group of automorphisms of M. We call R the *unitary antipode* and (τ_t) the *scaling group* of (M,Δ). It is known that $\sigma(R\otimes R)\Delta=\Delta R$, where σ denotes the flip map on $M\otimes M$.

We turn the predual M_* into a Banach algebra with product $\omega*\mu=(\omega\otimes\mu)\Delta$, for all $\omega,\mu\in M_*$.

We use the notation Δ^{op} to denote the opposite comultiplication defined by $\Delta^{\mathrm{op}}:=\sigma\Delta$.

The dual locally compact quantum group $(\hat{M},\hat{\Delta})$ is defined as follows. Its von Neumann algebra $\hat{M}$ is

$$\hat{M}=\{(\omega\otimes\iota)(W)\mid\omega\in\mathrm{B}(H)_*\}^{-\ \sigma\text{-strong}^*}$$

and the comultiplication is given by $\hat{\Delta}(x)=\Sigma W(x\otimes 1)W^*\Sigma$ for all $x\in\hat{M}$, where Σ denotes the flip map on the tensorproduct of Hilbert spaces. Set $\hat{W}=\Sigma W^*\Sigma$.

Since $(\hat{M},\hat{\Delta})$ is again a locally compact quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$ and the scaling group $(\hat{\tau}_t)$ exactly as we did it for (M,Δ). Also, we can again construct the dual of $(\hat{M},\hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$ with GNS-construction $(H,\iota,\hat{\Lambda})$. From the biduality theorem, we get that the bidual locally compact quantum group $(\hat{\hat{M}},\hat{\hat{\Delta}})$ is isomorphic to (M,Δ).

Define M_c to be the norm closure of the space

$$\{(\iota \otimes \omega)(W) \mid \omega \in \mathrm{B}(H)_*\}$$

and Δ_c to be the restriction of Δ to M_c. It was proven in [10] that the pair (M_c, Δ_c) is a reduced C^*-algebraic locally compact quantum group. We know that there is a bijective correspondence between reduced C^*-algebraic quantum groups and von Neumann algebraic quantum groups. So, the choice for the von Neumann algebra language is not a restriction.

A $*$-homomorphism $\varepsilon : M_c \to \mathbb{C}$ is called a *co-unit* of (M_c, Δ_c), if

$$(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \iota \ .$$

Classical locally compact groups appear as $M = L^\infty(G)$ with $\Delta(f)(p,q) = f(pq)$. The invariant weights are defined by integrating with respect to the left or the right Haar measure. The dual $\hat{M}$ can be identified with the group von Neumann algebra $\mathcal{L}(G)$.

Working with tensor products with more than two factors, we will sometimes use the leg-numbering notation. For example, if H, K and L are Hilbert spaces and $X \in \mathrm{B}(H \otimes L)$, we denote by X_{13} (respectively, $X_{12},\ X_{23}$) the operator $(1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)$ (respectively, $X \otimes 1,\ 1 \otimes X$) defined on $H \otimes K \otimes L$. If now $H = H_1 \otimes H_2$ is itself a tensor product of two Hilbert spaces, then we sometimes switch from the leg-numbering notation with respect to $H \otimes K \otimes L$ to the one with respect to the finer tensor product $H_1 \otimes H_2 \otimes K \otimes L$, for example, from X_{13} to X_{124}. There is no confusion here, because the number of legs changes. Weak and σ-weak convergence are denoted by $\xrightarrow{w}$, respectively $\xrightarrow{\sigma w}$.

3. Amenability

Let (M, Δ) be a von Neumann algebraic locally compact quantum group. A state $m \in M^*$ is said to be a *left invariant mean (LIM)* on (M, Δ) if

$$m((\omega \otimes \iota)\Delta(x)) = m(x)\omega(1),$$

for all $\omega \in M_*$ and $x \in M$. It is said to be a *right invariant mean (RIM)* if

$$m((\iota \otimes \omega)\Delta(x)) = m(x)\omega(1),$$

for all $\omega \in M_*$ and $x \in M$. Finally, if m is both a LIM and a RIM, we call m an *invariant mean (IM)*.

Definition 1. We call (M, Δ) *weakly amenable* if there exists a left invariant mean (LIM) on (M, Δ). We say that (M, Δ) is *weakly coamenable* if $(\hat{M}, \hat{\Delta})$ is weakly amenable.

Definition 2. We call (M, Δ) *strongly amenable* if there exists a bounded co-unit on $(\hat{M}_c, \hat{\Delta}_c)$. We say that (M, Δ) is *strongly coamenable* if $(\hat{M}, \hat{\Delta})$ is strongly amenable.

Classical locally compact groups appear as $(L^\infty(G), \Delta_G)$ in the theory of locally compact quantum groups. For these groups, weak and strong amenability coincide and agree with the usual notion of amenability of the group G. Other authors sometimes use a "dual" terminology. This difference originates from the choice of the quantum group associated with a locally compact group, $L^\infty(G)$ or $\mathcal{L}(G)$. Here we adopt the point of view of Enock and Schwartz [6] and Ruan [14] (i.e., we take $L^\infty(G)$ as the associated quantum group). The "dual" convention is used, amongst others, by Banica [3], Baaj and Skandalis [2] and Ng [12]. They use *coamenable* where we use *strongly amenable*. Whenever we cite a result of one of the papers mentioned with different terminology, it will be already translated to our setting.

M. Enock and J.M. Schwartz [6] proved that, for Kac algebras, the following statements are equivalent to the fact that a Kac algebra is strongly amenable:

(i) There exists a net $(\xi_j)_j$ of normalized vectors in H such that

$$(\iota \otimes \omega_{\xi_j})(W) \xrightarrow{w} 1.$$

(ii) There exists a bounded left (resp., right) approximate unit in $\hat{M}_*$.

It was proven in [6] that strong amenability implies weak amenability. The authors also claim that the opposite implication is true, but, as mentioned by Ruan [14], there is a gap in their proof. It is an important open question whether or not weak amenability implies strong amenability. Until now, this is only known to be true for locally compact groups (see, for example, [7]), for discrete Kac algebras [14], and recently, also for discrete quantum groups [15].

Enock and Schwartz further showed that the following statements are equivalent:

(i) There exists a LIM on (M, Δ) (resp., RIM).
(ii) There exists a net $(\omega_i)_i$ of states in M_* such that $\omega * \omega_i - \omega_i$ converges weakly to 0 (resp., $\omega_i * \omega - \omega_i$), for all $\omega \in M_*$ with $\omega(1) = 1$.
(iii) There exists a net $(\omega_i)_i$ of states in M_* such that $\|\omega * \omega_i - \omega_i\|$ converges to 0 (resp., $\|\omega_i * \omega - \omega_i\|$), for all $\omega \in M_*$ with $\omega(1) = 1$.

All of these results are also true for locally compact quantum groups. Not surprisingly, we can prove the following proposition.

PROPOSITION 3. *Let (M, Δ) be a locally compact quantum group. There exists a LIM on (M, Δ) if and only if there exists an invariant mean on (M, Δ).*

Proof. One implication is immediate.

Conversely, suppose there exists a LIM on (M, Δ). From the result mentioned above, we know that there exists a net of states $(\omega_i)_i$ in M_* such that $\|\omega * \omega_i - \omega_i\|$ converges to 0 for all $\omega \in M_*$ with $\omega(1) = 1$. It is obvious that this is equivalent to the existence of a net of states $(\omega_i^\circ)_i$ in M_* such that

$\|\omega_i^\circ * \omega - \omega_i^\circ\|$ converges to 0 for all $\omega \in M_*$ with $\omega(1) = 1$; take $\omega_i^\circ = \omega_i \circ R$. It is easy to prove that $\mu_k = (\omega_i * \omega_j^\circ)_{(i,j)}$ is a net of states such that

$$\|\mu_k * \omega - \mu_k\| \to 0 \ \text{ and } \ \|\omega * \mu_k - \mu_k\| \to 0$$

for all $\omega \in M_*$ with $\omega(1) = 1$. Let m be a weak-$*$ limit point of $(\mu_k)_k$ in the unit ball of M^*. It is obvious that m will be an invariant mean. □

4. Bicrossed products

In this section, we collect some results and definitions from [19].

Definition 4. We call a pair $(\alpha,\mathcal{U})$ a *cocycle action* of a locally compact quantum group (M,Δ) on a von Neumann algebra N if

$$\alpha : N \to M \otimes N$$

is a normal, injective and unital $*$-homomorphism,

$$\mathcal{U} \in M \otimes M \otimes N$$

is a unitary, and if α and $\mathcal{U}$ satisfy

$$(\iota \otimes \alpha)\alpha(x) = \mathcal{U}\,(\Delta \otimes \iota)\alpha(x)\mathcal{U}^* \ \text{ for all } \ x \in N,$$
$$(\iota \otimes \iota \otimes \alpha)(\mathcal{U})(\Delta \otimes \iota \otimes \iota)(\mathcal{U}) = (1 \otimes \mathcal{U})(\iota \otimes \Delta \otimes \iota)(\mathcal{U}).$$

If $\mathcal{U}$ is trivial, i.e., $\mathcal{U} = 1$, we call α an *action*.

Notation 5. If $(\alpha,\mathcal{U})$ is a cocycle action of (M,Δ) on a von Neumann algebra N, we introduce the notation

$$\tilde{W} = (W \otimes 1)\,\mathcal{U}^*;$$

then, $\tilde{W}$ is a unitary in $M \otimes \mathrm{B}(H) \otimes N$.

Given a cocycle action $(\alpha,\mathcal{U})$ of a locally compact quantum group (M,Δ) on a von Neumann algebra N, we construct the crossed product $M\,{}_{\alpha,\mathcal{U}}\!\ltimes N$. This is the von Neumann subalgebra of $\mathrm{B}(H) \otimes N$ generated by

$$\alpha(N) \text{ and } \{(\omega \otimes \iota \otimes \iota)((W \otimes 1)\,\mathcal{U}^*) \mid \omega \in M_*\}.$$

When $\mathcal{U}$ is trivial, the crossed product is denoted by $M\,{}_\alpha\!\ltimes N$. There is a unique action $\hat{\alpha}$ of $(\hat{M},\hat{\Delta}^{\mathrm{op}})$ on $M\,{}_{\alpha,\mathcal{U}}\!\ltimes N$ such that, for all $x \in N$,

$$\hat{\alpha}\big(\alpha(x)\big) = 1 \otimes \alpha(x) \text{ and } (\iota \otimes \hat{\alpha})\big((W \otimes 1)\,\mathcal{U}^*\big) = W_{12}W_{13}\mathcal{U}^*_{134}. \tag{1}$$

We call this action $\hat{\alpha}$ the dual action. It was proven in [19] that the fixed point algebra is

$$(M\,{}_{\alpha,\mathcal{U}}\!\ltimes N)^{\hat{\alpha}} = \{x \in M\,{}_{\alpha,\mathcal{U}}\!\ltimes N \mid \hat{\alpha}(x) = 1 \otimes x\} = \alpha(N).$$

Definition 6. A pair $(M_1,\Delta_1),(M_2,\Delta_2)$ is said to be a *matched pair* of locally compact quantum groups if there exists a triple $(\tau,\mathcal{U},\mathcal{V})$ (called a *cocycle matching*) satisfying the following conditions:

- $\mathcal{U} \in M_1 \otimes M_1 \otimes M_2$ and $\mathcal{V} \in M_1 \otimes M_2 \otimes M_2$ are both unitaries;
- $\tau : M_1 \otimes M_2 \to M_1 \otimes M_2$ is a faithful $*$-homomorphism;
- defining $\alpha(y) = \tau(1 \otimes y)$ and $\beta(x) = \tau(x \otimes 1)$ we have:
  - $(\alpha, \mathcal{U})$ is a cocycle action of (M_1, Δ_1) on M_2,
  - $(\sigma\beta, \mathcal{V}_{321})$ is a cocycle action of (M_2, Δ_2) on M_1,
  - $(\alpha, \mathcal{U})$ and $(\beta, \mathcal{V})$ are matched in the following sense:

$$\tau_{13}(\alpha \otimes \iota)\Delta_2(y) = \mathcal{V}_{132}(\iota \otimes \Delta_2)\alpha(y)\mathcal{V}^*_{132},$$
$$\tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(x) = \mathcal{U}(\Delta_1 \otimes \iota)\beta(x)\,\mathcal{U}^*,$$
$$(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{U}^*) =$$
$$(\mathcal{U}^* \otimes 1)(\iota \otimes \tau\sigma \otimes \iota)\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)(\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)(1 \otimes \mathcal{V}).$$

Given a cocycle matching $(\tau, \mathcal{U}, \mathcal{V})$ of (M_1, Δ_1) and (M_2, Δ_2), one is able to construct the cocycle bicrossed product (M, Δ). By definition $M = M_1 \, {}_{\alpha,\mathcal{U}}\!\ltimes M_2$ and $\Delta(x) = W^*(1 \otimes x)W$ with $W = \Sigma \hat{W}^* \Sigma$ and

$$\hat{W} = (\beta \otimes \iota \otimes \iota)\big((W_1 \otimes 1)\,\mathcal{U}^*\big)(\iota \otimes \iota \otimes \alpha)\big(\mathcal{V}(1 \otimes \hat{W}_2)\big)$$
$$\in M_1 \otimes \mathrm{B}(H_2) \otimes \mathrm{B}(H_1) \otimes M_2,$$

with W_1 and W_2 the multiplicative unitaries of (M_1, Δ_1) and (M_2, Δ_2), respectively. It was proven in [19] that (M, Δ) is a locally compact quantum group and that W is its multiplicative unitary.

In Section 5, (M_1, Δ_1) and (M_2, Δ_2) will always be two locally compact quantum groups matched by $(\tau, \mathcal{U}, \mathcal{V})$ and their cocycle bicrossed product locally compact quantum group will be denoted by (M, Δ). All the objects associated with a quantum group (e.g., W, Δ,…) will be denoted with an index, when they refer to (M_1, Δ_1) and (M_2, Δ_2), respectively, and without an index when they refer to (M, Δ). So we have that W_1 (resp., W_2) is the multiplicative unitary of (M_1, Δ_1) (resp., (M_2, Δ_2)) and

$$\tilde{W}_1 = (W_1 \otimes 1)\,\mathcal{U}^*.$$

From Propositions 2.4 and 2.5 of [19] we know how the comultiplication Δ works on the generators $\alpha(x)$ and $(\omega \otimes \iota \otimes \iota)(\tilde{W}_1)$ of (M, Δ):

$$\Delta(\alpha(x)) = (\alpha \otimes \alpha)\Delta_2(x)$$
$$(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}_1) = (\tilde{W}_1 \otimes 1 \otimes 1)((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W}_1)(\iota \otimes \alpha \otimes \alpha)(\mathcal{V}) \tag{2}$$

Define $\hat{M}$ as the von Neumann subalgebra of $M_1 \otimes B(H_2)$ generated by $\beta(M_1)$ and $\{(\iota \otimes \iota \otimes \omega)(\mathcal{V}(1 \otimes \hat{W}_2)) \mid \omega \in M_{2*}\}$. We define $\hat{\Delta}(z) = \hat{W}^*(1 \otimes z)\hat{W}$, for all $z \in \hat{M}$. In [19, Sec. 3.3.2] it was shown that $(\hat{M}, \hat{\Delta})$ is the dual locally compact quantum group of (M, Δ). So, if we interchange the roles of α and β, M_1 and M_2, respectively, then we find, as the cocycle bicrossed product, the dual of the original cocycle bicrossed product.

Definition 7. A cocycle action $(\alpha,\mathcal{U})$ of (M,Δ) on a von Neumann algebra N is said to be *stabilizable* with a unitary $X \in M \otimes N$ if

$$(1 \otimes X)(\iota \otimes \alpha)(X) = (\Delta \otimes \iota)(X)\mathcal{U}^* \,.$$

Proposition 8. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N which is stabilizable with a unitary $X \in M \otimes N$. Then the formulas*

$$\beta : N \to M \otimes N : \beta(x) = X\alpha(x)X^* \quad \textit{and} \quad \Phi : z \mapsto X^* z X$$

define, respectively, an action of (M,Δ) on N and a *-isomorphism from $M \,{}_\beta\!\ltimes N$ onto $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ satisfying

$$\hat{\alpha} \circ \Phi = (\iota \otimes \Phi) \circ \hat{\beta} \,.$$

The next proposition shows that many cocycle actions are stabilizable.

Proposition 9. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N. Then $(\alpha \otimes \iota, \mathcal{U} \otimes 1)$ is a cocycle action of (M,Δ) on $N \otimes B(H)$ which is stabilizable.*

5. Amenability and the bicrossed product construction

In this section, we investigate the relation between weak, respectively strong, amenability of the cocycle bicrossed product quantum group (M,Δ), and of its building ingredients (M_1,Δ_1) and (M_2,Δ_2).

We start with a technical remark about slicing with non-normal functionals. Let N and L be von Neumann algebras, $n \in N^*$ and $X \in N \otimes L$.

If $n \in N_*$, then it is obvious that $(n \otimes \iota)(X) \in L$. This remains true for $n \in N^*$, even if n is not normal. Indeed, consider the map $L_* \to \mathbb{C} : \omega \mapsto n((\iota \otimes \omega)(X))$. It is obvious that this is a bounded linear functional and since $L = (L_*)^*$, we know that there exists a unique $Y \in L$ such that $\omega(Y) = n((\iota \otimes \omega)(X)$ for all $\omega \in L_*$. Set $Y = (n \otimes \iota)(X)$.

Suppose that $\Phi : L \to K$ is a normal $*$-homomorphism of von Neumann algebras. Since for all $\omega \in K_*$

$$\begin{aligned}\omega(\Phi(n \otimes \iota)(X)) &= n((\iota \otimes \omega \circ \Phi)(X)) \\ &= n((\iota \otimes \omega)(\iota \otimes \Phi)(X)) \\ &= \omega((n \otimes \iota)(\iota \otimes \Phi)(X)),\end{aligned}$$

we may conclude that for all Φ

$$\Phi((n \otimes \iota)(X)) = (n \otimes \iota)(\iota \otimes \Phi)(X).$$

This will be used several times in the sequel, where n will be an invariant mean and $\Phi = \alpha, \Delta, \ldots$.

Definition 10. If α is an action of (M,Δ) on a von Neumann algebra N, we define an *α-invariant mean* to be a state $m \in N^*$ such that

$$m((\omega \otimes \iota)\alpha(x)) = m(x)\omega(1)$$

for all $\omega \in M_*$ and $x \in N$.

PROPOSITION 11. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on a von Neumann algebra N, $M \,_{\alpha,\mathcal{U}}\!\ltimes N$ the cocycle crossed product and $\hat{\alpha}$ the dual action. Then $(\hat{M},\hat{\Delta})$ is weakly amenable if and only if there exists a $\hat{\alpha}$-invariant mean on $M \,_{\alpha,\mathcal{U}}\!\ltimes N$.*

Proof. Suppose that $\hat{m}$ is an invariant mean on $(\hat{M},\hat{\Delta})$. Then we argue that there exists an $\hat{\alpha}$-invariant mean on $M \,_{\alpha,\mathcal{U}}\!\ltimes N$. This can be done by generalizing a result in [6] from the Kac algebra level to the general setting. However, the proof in [6] is based on a non-constructive argument. We construct explicitly an $\hat{\alpha}$-invariant mean on $M \,_{\alpha,\mathcal{U}}\!\ltimes N$. The dual weight construction is the source of inspiration. Define $T : M \,_{\alpha,\mathcal{U}}\!\ltimes N \to M \,_{\alpha,\mathcal{U}}\!\ltimes N$ by $T(z) := (\hat{m} \otimes \iota)\hat{\alpha}(z)$. We prove that $T(z) \in \alpha(N)$ for all $z \in M \,_{\alpha,\mathcal{U}}\!\ltimes N$. Since $\alpha(N)$ is the fixed point algebra of $\hat{\alpha}$, it is sufficient to show that $\hat{\alpha}(T(z)) = 1 \otimes T(z)$.

Observe that, since $\hat{\alpha}$ is an action, $\hat{\alpha}(T(z)) = (\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)$. So we have to prove that, for all $\omega \in (\hat{M} \otimes M \,_{\alpha,\mathcal{U}}\!\ltimes N)_*$,

$$\omega((\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)) = \omega(1 \otimes (\hat{m} \otimes \iota)\hat{\alpha}(z)).$$

But it is sufficient to check this for normal functionals of the form $\mu \otimes \nu$ with $\mu \in \hat{M}_*$ and $\nu \in (M \,_{\alpha,\mathcal{U}}\!\ltimes N)_*$. Using the fact that $\hat{m}$ is an invariant mean on $(\hat{M},\hat{\Delta})$ and hence an invariant mean on $(\hat{M},\hat{\Delta}^{\mathrm{op}})$, we get

$$\begin{aligned}(\mu \otimes \nu)((\hat{m} \otimes \iota \otimes \iota)(\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)) &= \hat{m}((\iota \otimes \mu \otimes \nu)(\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)) \\ &= \mu(1)\hat{m}((\iota \otimes \nu)(\hat{\alpha}(z))) \\ &= (\mu \otimes \nu)(1 \otimes (\hat{m} \otimes \iota)\hat{\alpha}(z)).\end{aligned}$$

So we may conclude that $T(z) \in \alpha(N)$ for all $z \in M \,_{\alpha,\mathcal{U}}\!\ltimes N$.

Choose a state $\eta \in N^*$. Define $m(z) = \eta(\alpha^{-1}(T(z)))$. We will prove that m is $\hat{\alpha}$-invariant. For all $\omega \in \hat{M}_*$ and $z \in M \,_{\alpha,\mathcal{U}}\!\ltimes N$, we get that

$$\begin{aligned}m((\omega \otimes \iota)\hat{\alpha}(z)) &= \eta(\alpha^{-1}((\hat{m} \otimes \iota)\hat{\alpha}((\omega \otimes \iota)\hat{\alpha}(z)))) \\ &= \eta(\alpha^{-1}((\hat{m} \otimes \iota)(\omega \otimes \iota \otimes \iota)((\iota \otimes \hat{\alpha})\hat{\alpha}(z)))) \\ &= \eta(\alpha^{-1}((\hat{m} \otimes \iota)(\omega \otimes \iota \otimes \iota)((\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)))) \\ &= \eta(\alpha^{-1}((\hat{m} \otimes \iota)\hat{\alpha}(z)))\omega(1) = m(z)\omega(1).\end{aligned}$$

Conversely, suppose that m is a $\hat{\alpha}$-invariant mean on $M \,_{\alpha,\mathcal{U}}\!\ltimes N$. We have to prove that $(\hat{M},\hat{\Delta})$ is weakly amenable. We consider three cases:

Case 1: $\mathcal{U}$ is trivial.

We know that $M \,_{\alpha}\!\ltimes N$ is generated by $\alpha(N)$ and $\hat{M} \otimes \mathbb{C}$. Define $\hat{m}(\hat{x}) := m(\hat{x} \otimes 1)$ for all $\hat{x} \in \hat{M}$. Using the formula $\hat{\alpha}(\hat{x} \otimes 1) = \hat{\Delta}^{\mathrm{op}}(\hat{x}) \otimes 1$, we get that for all $\omega \in \hat{M}_*$

$$\begin{aligned}\hat{m}((\omega \otimes \iota)\hat{\Delta}^{\mathrm{op}}(\hat{x})) &= m((\omega \otimes \iota \otimes \iota)(\hat{\Delta}^{\mathrm{op}}(\hat{x}) \otimes 1)) \\ &= m((\omega \otimes \iota \otimes \iota)\hat{\alpha}(\hat{x} \otimes 1)) \\ &= m(\hat{x} \otimes 1)\omega(1) = \hat{m}(\hat{x})\omega(1).\end{aligned}$$

So we may conclude that $\hat{m}$ is a left invariant mean on $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$.

Case 2: $(\alpha,\mathcal{U})$ *is stabilizable.*
We know from Proposition 8 that, in this case, there exists an action β of (M, Δ) on N and a $*$-isomorphism

$$\Phi : M \,{}_{\beta}\!\ltimes N \to M \,{}_{\alpha,\mathcal{U}}\!\ltimes N,$$

such that $\hat{\alpha} \circ \Phi = (\iota \otimes \Phi) \circ \hat{\beta}$.

Define $\tilde{m} := m \circ \Phi$. Then it is easy to prove that $\tilde{m}$ is $\hat{\beta}$-invariant. Using the first case, we may conclude that the restriction of $\tilde{m}$ to $\hat{M}$ will be a LIM on $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$.

General case: Arbitrary $(\alpha,\mathcal{U})$.
In general, $(\alpha \otimes \iota, \mathcal{U} \otimes 1)$ will be a cocycle action of (M, Δ) on $N \otimes \mathrm{B}(H)$, and we know from Proposition 9 that it will be stabilizable. It is not too difficult to show that its corresponding cocycle crossed product factorizes as $(M \,{}_{\alpha,\mathcal{U}}\!\ltimes N) \otimes \mathrm{B}(H)$, as well as the dual action, which is given by $\hat{\alpha} \otimes \iota$.

Choose a normalized vector $\xi \in H$. Then we have for all $z \in (M \,{}_{\alpha,\mathcal{U}}\!\ltimes N) \otimes \mathrm{B}(H)$ that

$$\begin{aligned}(\iota \otimes m \otimes \omega_\xi)((\hat{\alpha} \otimes \iota)(z)) &= (\iota \otimes m)\hat{\alpha}((\iota \otimes \omega_\xi)(z)) \\ &= m((\iota \otimes \omega_\xi)(z))1 \\ &= (m \otimes \omega_\xi)(z)1.\end{aligned}$$

We find that $m \otimes \omega_\xi$ is $(\hat{\alpha} \otimes \iota)$-invariant and from the second case, we may conclude that $(\hat{M}, \hat{\Delta})$ is weakly amenable. $\square$

With this theorem in mind, we are going to prove our main results, generalizing a result of Ng [13]. Ng showed in [13] that the bicrossed product with trivial cocycles of two locally compact groups G_1 and G_2 is strongly amenable if G_2 is amenable. Notice that for any group G_1, $(L^\infty(G_1), \Delta_1)$ is always strongly coamenable.

To prove our main theorem, we need a lemma. We can get this result from Propositions 3.1 and 3.4 of [19], but we have chosen to give a straightforward proof.

Lemma 12. *Let* $(\tau, \mathcal{U}, \mathcal{V})$ *be a cocycle matching of* (M_1, Δ_1) *and* (M_2, Δ_2), *and let* (M, Δ) *be the cocycle bicrossed product. Then*

$$(\iota \otimes \Delta^{\mathrm{op}})\hat{\alpha}(z) = (\hat{\alpha} \otimes \iota)\Delta^{\mathrm{op}}(z) \text{ for all } z \in M. \tag{3}$$

Proof. It suffices to check (3) on the generators. Choose $x \in M_2$. Observe that

$$\begin{aligned}(\iota \otimes \Delta^{\mathrm{op}})\hat{\alpha}\big(\alpha(x)\big) &= (\iota \otimes \Delta^{\mathrm{op}})\big(1 \otimes \alpha(x)\big) \\ &= 1 \otimes \Delta^{\mathrm{op}}\big(\alpha(x)\big) \\ &= 1 \otimes (\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(x) \\ &= (\hat{\alpha} \otimes \iota)\big((\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(x)\big) \\ &= (\hat{\alpha} \otimes \iota)\Delta^{\mathrm{op}}\big(\alpha(x)\big).\end{aligned}$$

Next, we will prove that $(\iota \otimes \iota \otimes \Delta^{\mathrm{op}})(\iota \otimes \hat{\alpha})(\tilde{W}_1) = (\iota \otimes \hat{\alpha} \otimes \iota)\big((\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}_1)\big)$. Using equation (1) we get

$$(\iota \otimes \iota \otimes \Delta^{\mathrm{op}})(\iota \otimes \hat{\alpha})(\tilde{W}_1) = (\iota \otimes \iota \otimes \Delta^{\mathrm{op}})\big((W_1)_{12}(\tilde{W}_1)_{134}\big).$$

Finally, observe that, as operators on $H_1 \otimes H_1 \otimes H_1 \otimes H_2 \otimes H_1 \otimes H_2$,

$$\begin{aligned}&(\iota \otimes \hat{\alpha} \otimes \iota)\big((\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}_1)\big) \\ &\quad = (\iota \otimes \hat{\alpha} \otimes \iota)\big((\tilde{W}_1 \otimes 1 \otimes 1)\big((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W}_1)(\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\big) \\ &\quad = (W_1)_{12}(\tilde{W}_1)_{134}((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W}_1)_{13456}(\iota \otimes \alpha \otimes \alpha)(\mathcal{V})_{13456} \\ &\quad = (W_1)_{12}(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}_1)_{13456} \\ &\quad = (W_1)_{12}(\iota \otimes \iota \otimes \Delta^{\mathrm{op}})\big((\tilde{W}_1)_{134}\big) \\ &\quad = (\iota \otimes \iota \otimes \Delta^{\mathrm{op}})\big((W_1)_{12}(\tilde{W}_1)_{134}\big),\end{aligned}$$

where we used equation (2) in the first line. □

THEOREM 13. *Let $(\tau, \mathcal{U}, \mathcal{V})$ be a cocycle matching of (M_1, Δ_1) and (M_2, Δ_2) and let (M, Δ) be the cocycle bicrossed product. Then (M, Δ) is weakly amenable if and only if $(\hat{M}_1, \hat{\Delta}_1)$ and (M_2, Δ_2) are weakly amenable.*

Proof. The proof is divided into three parts.

(1) *If (M, Δ) is weakly amenable, then $(\hat{M}_1, \hat{\Delta}_1)$ is weakly amenable.* Let m be an invariant mean on (M, Δ). From Proposition 11, we know that it is sufficient to show that m is $\hat{\alpha}$-invariant. If we apply $\iota \otimes \iota \otimes m$ on the result in Lemma 12, we get that, for all $z \in M$,

$$(\iota \otimes m)\hat{\alpha}(z) \otimes 1 = m(z)1 \otimes 1.$$

From this we conclude that $(\iota \otimes m)\hat{\alpha}(z) = m(z)1$ and therefore that m is $\hat{\alpha}$-invariant.

(2) *If (M, Δ) is weakly amenable, then (M_2, Δ_2) is weakly amenable.* Suppose that m is a LIM on (M, Δ). Define $m_2 \in M_2^*$ by $m_2(x) = m(\alpha(x))$. Since $(\alpha \otimes \alpha)\Delta_2 = \Delta \circ \alpha$ and $M_{2*} = \{\omega \circ \alpha \mid \omega \in M_*\}$, it is obvious that m_2 will be a LIM on (M_2, Δ_2).

(3) *If* $(\hat{M}_1, \hat{\Delta}_1)$ *and* (M_2, Δ_2) *are weakly amenable, then* (M, Δ) *is weakly amenable.*

Suppose that $\hat{m}_1$ and m_2 are left invariant means on $(\hat{M}_1, \hat{\Delta}_1)$ and (M_2, Δ_2), respectively. Consider the dual action $\hat{\alpha} : M \to \hat{M}_1 \otimes M$. Define $T(z) := (\hat{m}_1 \otimes \iota)\hat{\alpha}(z)$ for all $z \in M$. From the proof of Proposition 11, we know that $T(z) \in \alpha(M_2)$.

Define $m := m_2 \circ \alpha^{-1} \circ T$. We prove that m is a left invariant mean on M. Choose any $z \in M$. Applying $\hat{m}_1 \otimes \iota \otimes \iota$ on both sides of the result of Lemma 12 we get

$$\Delta^{\mathrm{op}}(T(z)) = (T \otimes \iota)\Delta^{\mathrm{op}}(z).$$

So, we have for all $\mu \in M_*$

$$(\iota \otimes \mu)\Delta^{\mathrm{op}}(T(z)) = T((\iota \otimes \mu)\Delta^{\mathrm{op}}(z)).$$

Take $y \in M_2$ such that $T(z) = \alpha(y)$. Since

$$(\iota \otimes \mu)\Delta^{\mathrm{op}}(\alpha(y)) = (\iota \otimes \mu)(\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(y) = \alpha((\iota \otimes \mu \circ \alpha)\Delta_2^{\mathrm{op}}(y)),$$

we get

$$T((\iota \otimes \mu)\Delta^{\mathrm{op}}(z)) = \alpha((\iota \otimes \mu \circ \alpha)\Delta_2^{\mathrm{op}}(y)). \tag{4}$$

Applying $m_2 \circ \alpha^{-1}$ on both sides of equation (4), we get

$$m((\iota \otimes \mu)\Delta^{\mathrm{op}}(z)) = m_2((\iota \otimes \mu \circ \alpha)\Delta_2^{\mathrm{op}}(y)).$$

Now we use the left invariance of m_2 and obtain for all $\mu \in M_*$

$$\begin{aligned} m((\iota \otimes \mu)\Delta^{\mathrm{op}}(z)) &= m_2((\iota \otimes \mu \circ \alpha)\Delta_2^{\mathrm{op}}(y)) \\ &= \mu(\alpha(1))m_2(y) \\ &= \mu(1)m_2(\alpha^{-1}(T(z))) \\ &= \mu(1)m(z). \end{aligned}$$

Therefore, m is a left invariant mean on (M, Δ). □

A natural question is whether or not the strong version of Theorem 13, i.e., Theorem 13 with weak amenability replaced by strong amenability, is true. We can only give a partial answer. First of all, it is not too difficult to see that $(\hat{M}_1, \hat{\Delta}_1)$ is strongly amenable if (M, Δ) is. Just suppose that the net $(\hat{\mu}_k)_k$ is an approximate unit of $\hat{M}_*$. Define $\mu_{1k} := \hat{\mu}_k \circ \beta$. Then $(\mu_{1k})_k$ is an approximate unit of M_{1*}. Hence we arrive at the following proposition.

Proposition 14. *Let* $(\tau, \mathcal{U}, \mathcal{V})$ *be a cocycle matching of* (M_1, Δ_1) *and* (M_2, Δ_2) *and let* (M, Δ) *be the cocycle bicrossed product. If* (M, Δ) *is strongly amenable, then* $(\hat{M}_1, \hat{\Delta}_1)$ *is strongly amenable.*

Next, we prove the strong version of Theorem 13 in the case where the cocycles are trivial: $\mathcal{U} = \mathcal{V} = 1$. We do not know whether or not the same result holds with non-trivial cocycles.

THEOREM 15. *Let (M, Δ) be the bicrossed product of (M_1, Δ_1) and (M_2, Δ_2) with trivial cocycles. Then (M, Δ) is strongly amenable if and only if $(\hat{M}_1, \hat{\Delta}_1)$ and (M_2, Δ_2) are strongly amenable.*

Proof. We will first prove that if $(\hat{M}_1, \hat{\Delta}_1)$ and (M_2, Δ_2) are strongly amenable, then (M, Δ) is strongly amenable. Suppose that $(\omega_i)_i$ is a bounded two-sided approximate unit for M_{1*}. It is sufficient to show that

$$((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) \xrightarrow{\sigma w} 1. \tag{5}$$

Indeed, by definition, $\hat{W} = ((\beta \otimes \iota)(W_1) \otimes 1)(1 \otimes (\iota \otimes \alpha)(\hat{W}_2)) \in M_1 \otimes \mathrm{B}(H_2) \otimes \mathrm{B}(H_1) \otimes M_2$ and so

$$(\omega_i \otimes \iota \otimes \iota \otimes \iota)(\hat{W}) = (((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) \otimes 1)((\iota \otimes \alpha)(\hat{W}_2)).$$

Using equation (5) we get that

$$(\omega_i \otimes \iota \otimes \iota \otimes \iota)(\hat{W}) \xrightarrow{\sigma w} (\iota \otimes \alpha)(\hat{W}_2).$$

Because (M_2, Δ_2) is strongly amenable, we can take a net $(\xi_j)_j$ of normalized vectors in H_2 such that $(\mu_{\xi_j} \otimes \iota)(\hat{W}_2) \xrightarrow{\sigma w} 1$.

Choose $\mu \in M_*$. Observe that for all i, j

$$|\mu((\mu_{\xi_j} \otimes \iota \otimes \iota)(\omega_i \otimes \iota \otimes \iota \otimes \iota)(\hat{W}))| \leq \|(\iota \otimes \iota \otimes \mu)(\hat{W})\|.$$

Taking first the limit over i and then the limit over j, we get

$$|\mu(1)| \leq \|(\iota \otimes \iota \otimes \mu)(\hat{W})\|.$$

Define $\hat{\varepsilon}((\iota \otimes \mu)(\hat{W})) = \mu(1)$. Thus, $\hat{\varepsilon}$ is a bounded co-unit for $(\hat{M}_c, \hat{\Delta}_c)$.

It remains to prove (5). The definition of matched pairs implies that

$$\tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(x) = (\Delta_1 \otimes \iota)\beta(x).$$

Applying $\omega_i \otimes \iota \otimes \iota$ on both sides we get

$$\tau\sigma(((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x)) = (\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x)). \tag{6}$$

For all $\omega \in M_{1*}$ and $\nu \in M_{2*}$ we have that

$$(\omega \otimes \nu)(\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x)) = (\omega_i * \omega \otimes \nu)\beta(x) \to (\omega \otimes \nu)(\beta(x)).$$

By linearity and the fact that $(\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x))$ is uniformly bounded in i, we get that

$$(\omega_i \otimes \iota \otimes \iota)((\Delta_1 \otimes \iota)\beta(x)) \xrightarrow{\sigma w} \beta(x).$$

Using equation (6) and the normality of $\tau\sigma$ we find that

$$\tau\sigma(((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x)) \xrightarrow{\sigma w} \beta(x) = \tau\sigma(1 \otimes x).$$

Now, $\tau\sigma$ is an injective and normal $*$-homomorphism, and therefore it will be homeomorphic onto its image for the σ-weak topology (see [5, p. 60]). From this, we get

$$((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1(x) \xrightarrow{\sigma w} 1 \otimes x. \tag{7}$$

Applying $\beta \otimes \iota \otimes \iota$ on $(\Delta_1 \otimes \iota)(W_1) = W_{1,13} W_{1,23}$ we get

$$((\beta \otimes \iota)\Delta_1 \otimes \iota)(W_1) = ((\beta \otimes \iota)(W_1))_{124} W_{1,34}$$

and

$$(((\omega_i \otimes \iota)\beta \otimes \iota)\Delta_1 \otimes \iota)(W_1) = (((\omega_i \otimes \iota)\beta \otimes \iota)(W_1))_{13} W_{1,23}.$$

Using equation (7), we may conclude that

$$(((\omega_i \otimes \iota)\beta \otimes \iota)(W_1))_{13} W_{1,23} \xrightarrow{\sigma w} 1 \otimes W_1 = W_{1,23}.$$

As W_1 is invertible, this implies that

$$((\omega_i \otimes \iota)\beta \otimes \iota)(W_1) \xrightarrow{\sigma w} 1. \tag{8}$$

This concludes the first part of the proof.

By taking trivial cocycles in Proposition 14, it is immediately clear that $(\hat{M}_1, \hat{\Delta}_1)$ is strongly amenable if (M, Δ) is strongly amenable.

It remains to show that if (M, Δ) is strongly amenable, then (M_2, Δ_2) is strongly amenable. Using the biduality theorem, it is sufficient to prove that if $(\hat{M}, \hat{\Delta})$ is strongly amenable, then (M_1, Δ_1) is strongly amenable. Suppose that $(\omega_i)_i$ is a bounded two-sided approximate unit for M_*. We know that now

$$M = (\alpha(M_2) \cup \{(\omega \otimes \iota)(W_1) \otimes 1 \mid \omega \in M_{1*}\})'' .$$

Using equation (2), we get

$$(\iota \otimes \Delta^{\mathrm{op}})(W_1 \otimes 1) = (W_1 \otimes 1 \otimes 1 \otimes 1)((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(W_1 \otimes 1),$$

so

$$(\iota \otimes \iota \otimes \omega_i)(\iota \otimes \Delta^{\mathrm{op}})(W_1 \otimes 1) = (W_1 \otimes 1)(\iota \otimes \alpha)\beta((\iota \otimes \omega_i)(W_1 \otimes 1)). \tag{9}$$

Using the fact that $(\omega_i)_i$ is an approximate unit of M_*, we have

$$(\iota \otimes \iota \otimes \omega_i)(\iota \otimes \Delta^{\mathrm{op}})(W_1 \otimes 1) \xrightarrow{\sigma w} W_1 \otimes 1$$

and thus, by equation (9),

$$(\iota \otimes \alpha)\beta((\iota \otimes \omega_i)(W_1 \otimes 1)) \xrightarrow{\sigma w} 1.$$

But $(\iota \otimes \alpha)\beta$ is a normal and injective $*$-homomorphism and therefore

$$(\iota \otimes \omega_i)(W_1 \otimes 1) \xrightarrow{\sigma w} 1. \tag{10}$$

Define $\mu_i \in \hat{M}_{1*}$ such that $\mu_i(z) = \omega_i(z \otimes 1)$ for all $z \in \hat{M}_1$, so $(\iota \otimes \mu_i)(W_1) = (\iota \otimes \omega_i)(W_1 \otimes 1)$. Using equation (10) we get that

$$(\iota \otimes \mu_i)(W_1) \xrightarrow{\sigma w} 1.$$

This concludes the proof. $\square$

6. Examples

In order to construct these examples, we rely on the extension procedure of locally compact quantum groups as developed in [2], [11], [19]. All the bicrossed product locally compact quantum groups in [19] are weakly amenable. This is easily seen, since the groups from which one starts in the examples are both amenable. We give two examples of non-amenable locally compact quantum groups, obtained by a bicrossed product construction. From Theorem 13 we know that if we take, as one of the ingredients, a non-amenable group, the bicrossed product locally compact quantum group will either be non-amenable or non-coamenable. In the first case, we take $SL_2(\mathbb{R})$ as the non-amenable group, and in the second case (a double cover of) $SU(1,1)$. It is known that these groups are not amenable, since they are non-compact, almost connected, semi-simple Lie groups; see [7].

We briefly review what is needed from the extension procedure.

Let G, G_1 and G_2 be locally compact groups with fixed left invariant Haar measures. Let $i : G_1 \to G$ be a homomorphism and $j : G_2 \to G$ an antihomomorphism such that both have a closed image and are homeomorphisms onto these images. Suppose moreover that the mapping

$$\theta : G_1 \times G_2 \to \Omega \subset G : (g,s) \mapsto i(g)j(s)$$

is a homeomorphism of $G_1 \times G_2$ onto an open subset Ω of G having a complement of measure zero. Then we call G_1 and G_2 a *matched pair of locally compact groups*. From this data, one constructs a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ with trivial cocycles as follows. Let $\rho : G_1 \times G_2 \to \Omega^{-1}$ be the homeomorphism given by $\rho(g,s) = j(s)i(g)$. Let $\mathcal{O} = \theta^{-1}(\Omega \cap \Omega^{-1})$ and for $(g,s) \in \mathcal{O}$ define $\beta_s(g) \in G_1$ and $\alpha_g(s) \in G_2$ by

$$\rho^{-1}(\theta(g,s)) = (\beta_s(g), \alpha_g(s)).$$

Finally, one can define a $*$-isomorphism

$$\tau : L^\infty(G_1) \otimes L^\infty(G_2) \to L^\infty(G_1) \otimes L^\infty(G_2)$$

by $\tau(f)(g,s) = f(\beta_s(g), \alpha_g(s)$. Then $(\tau, 1, 1)$ gives a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ with trivial cocycles.

EXAMPLE 1.

$$G = \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R},\ x, y \in \mathbb{R} \right\}$$

So, G is a Lie-subgroup of $SL_3(\mathbb{R})$.

$$G_1 = \mathbb{R}^2, + \,,$$
$$G_2 = SL_2(\mathbb{R}).$$

Further, take embeddings i and j defined by

$$i((x,y)) = \begin{pmatrix} 1 & 0 & -x \\ -x & 1 & -y+\frac{1}{2}x^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) .$$

Then the mutual actions are given by

$$\alpha_{(x,y)}(A) = \begin{pmatrix} a+bx & b \\ c+dx-(a+bx)(ax+b(y+\frac{1}{2}x^2)) & d-b(ax+b(y+\frac{1}{2}x^2)) \end{pmatrix}$$

and

$$\beta_A((x,y)) = \left(ax+by+\frac{b}{2}x^2, cx+d\left(y+\frac{1}{2}x^2\right)-\frac{1}{2}\left(ax+b\left(y+\frac{1}{2}x^2\right)\right)^2\right).$$

We take trivial cocycles and construct the bicrossed locally compact quantum group (M,Δ). Denote by δ, δ_1 and δ_2 the modular functions of the groups G, G_1 and G_2, respectively. It is not difficult to show that δ_1 and δ_2 are trivial and $\delta(A,(x,y)) = \det A = 1$. Therefore, the bicrossed product is a Kac algebra. One might think that there is a hope to leave the Kac algebra "world" by working with the general linear groups (GL) instead of the special linear groups (SL). Unfortunately, the determinant will be α-invariant. So, we will also find that the bicrossed product is a Kac algebra.

Now, one can construct the infinitesimal Hopf algebra of the bicrossed product quantum group in the sense of [19]. It is an algebraic version of the same quantum group.

In this example the infinitesimal Hopf algebra has generators X, Y, A, B, C and D satisfying $AD - BC = 1$ and the following relations:

$$\begin{array}{lll} [A,B]=0, & [A,C]=0, & [A,D]=0, \\ [B,C]=0, & [B,D]=0, & [C,D]=0, \\ [A,X]=B, & [A,Y]=0, & \\ [B,X]=0, & [B,Y]=0, & \\ [C,X]=D-A^2, & [C,Y]=-AB, & \\ [D,X]=-AB, & [D,Y]=-B^2, & \end{array}$$

$$\begin{aligned}
\Delta(A) &= A \otimes A + B \otimes C, \\
\Delta(B) &= B \otimes D + A \otimes B, \\
\Delta(C) &= C \otimes A + D \otimes C, \\
\Delta(D) &= D \otimes D + C \otimes B, \\
\Delta(X) &= 1 \otimes X + X \otimes A + Y \otimes C, \\
\Delta(Y) &= 1 \otimes Y + X \otimes B + Y \otimes D.
\end{aligned}$$

EXAMPLE 2. Now, we will construct a non-amenable locally compact quantum group that is not a Kac algebra.

$$\begin{aligned}
G_1 &= \{(x,z) \mid x \in \mathbb{R}, x \neq 0, z \in \mathbb{C}\} \text{ with } (x,z)(y,u) = (xy, z + xu), \\
G_2 &= \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \;\middle|\; a, c \in \mathbb{C}, |a|^2 - |c|^2 = \pm 1 \right\}, \\
G &= \{\ (2 \times 2)\text{-matrices over } \mathbb{C} \text{ with determinant } \pm 1\}\ .
\end{aligned}$$

Define $\mathrm{Sq}(x) = \mathrm{Sgn}(x)\sqrt{|x|}$ for all $x \in \mathbb{R}$. Take embeddings i and j defined by

$$i : (x,z) \mapsto \frac{1}{\mathrm{Sq}(x)} \begin{pmatrix} x & -z \\ 0 & 1 \end{pmatrix}, \quad j : \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}^{-1}.$$

The mutual actions are given by

$$\begin{aligned}
\alpha_{(x,z)}(a,c) &= \frac{D}{\mathrm{Sq}(|cz + \bar{a}x|^2 - |c|^2)} (\bar{c}\bar{z} + ax, c), \\
\beta_{(a,c)}(x,z) &= \frac{D}{|x|}(|cz + \bar{a}x|^2 - |c|^2, (az + \bar{c}x)(\bar{c}\bar{z} + ax) - a\bar{c})
\end{aligned}$$

with

$$D = D(x,z,a,c) = \frac{x}{|x|}(|a|^2 - |c|^2).$$

Taking $\mathcal{U} = \mathcal{V} = 1$, we can construct the bicrossed product locally compact quantum group (M, Δ). Since δ and δ_2 are trivial and $\delta_1(x,z) = 1/x^2$, we conclude, using Propositions 2.17 and 4.16 of [19], that (M, Δ) is not a Kac algebra, is non-compact and non-discrete. As far as we know, there was, until now, no example of a non-discrete non-amenable quantum group that is not a group.

Now, the infinitesimal Hopf $*$-algebra is generated as a $*$-algebra by normal elements A, C and Y, an anti-selfadjoint element X and a selfadjoint element U satisfying the following commutation relations:

$$[A,C] = [A,C^*] = 0, \qquad A^*A - C^*C = U, \qquad U^2 = 1,$$
$$[X,Y] = Y,$$
$$[A,X] = -UACC^*, \qquad [C,X] = -UAA^*C,$$
$$[A,Y] = 2C^* - UAA^*C^*, \qquad [C,Y] = -UA^*C^*C,$$
$$[A,Y^*] = UA^2C, \qquad [C,Y^*] = UAC^2.$$

Furthermore, the comultiplication is given by

$$\Delta(A) = A \otimes A + C^* \otimes C,$$
$$\Delta(C) = C \otimes A + A^* \otimes C,$$
$$\Delta(X) = X \otimes U(A^*A + C^*C) + Y \otimes UA^*C - Y^* \otimes UAC^* + 1 \otimes X,$$
$$\Delta(Y) = 1 \otimes Y + X \otimes 2UA^*C^* + Y \otimes U(A^*)^2 - Y^* \otimes U(C^*)^2.$$

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