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JOHN FUNCTIONS, QUADRATIC INTEGRAL FORMS AND O-MINIMAL STRUCTURES

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Abstract. Let Ω be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $\partial\Omega$ and $\delta_\Omega(x)$ denote, respectively, the boundary of Ω and the Euclidean distance of the point $x \in \Omega$ to $\mathbb{R}^n \setminus \Omega$. Denote by $K(\Omega)$ the John space of all C^1 functions $f : \Omega \to \mathbb{R}$ with $\sup_{x\in\Omega} \delta_\Omega(x)|\nabla f(x)| < +\infty$. We study $K(\Omega)$-functions via quadratic integral forms and o-minimal structures.

Introduction

Let Ω be a proper subdomain of the Euclidean space $\mathbb{R}^n$ ($n \geq 2$). In [Jo], John introduced the class $K(\Omega)$ of all C^1 functions $f : \Omega \to \mathbb{R}$ which have bounded expansion

$$\|f\|_{K(\Omega)} = \sup_{x\in\Omega} \delta_\Omega(x)|\nabla f(x)| < +\infty,$$

where, here and afterwards, ∇ and $\delta_\Omega(x)$ denote the gradient operator and the Euclidean distance of the point x to the boundary $\partial\Omega$ of Ω, respectively. This class is suggested by the well-known fact that uniformly bounded solutions to many elliptic differential equations belong to $K(\Omega)$, regardless of boundary conditions. Note, however, that not every $f \in K(\Omega)$ is bounded uniformly in Ω. An example is the function $\log|x|$ in the punctured unit disc. This example shows actually a general property of John's class: A function $f \in K(\Omega)$ can become unbounded at most like $\|f\|_{K(\Omega)} \log \delta_\Omega(x)$ as x tends to $\partial\Omega$.

On the other hand, while studying the structure of positive solutions to a Schrödinger equation $(-\Delta + V(x))u(x) = 0$, Murata [Mu] considered the quadratic integral form

$$\int_\Omega |\nabla\psi(x)|^2 G(x)dx,$$

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where $\sqrt{G}$ is a positive solution of the equation. This relates nicely to the Green's potential characterization of BMO functions on the hyperbolic domains in $\mathbb{R}^2$ (see [ALXZ] or [Go], for example). Thus we introduce the following definition.

Suppose that Ω is a proper subdomain of $\mathbb{R}^n$ with the Green function $g_\Omega(\cdot,\cdot)$ for the Laplacian Δ. For a C^1 function $f : \Omega \to \mathbb{R}$, we say that $f \in K_G(\Omega)$ provided

$$\|f\|_{K_G(\Omega)} = \sup_{y\in\Omega} \left(\int_\Omega |\nabla f(x)|^2 g_\Omega(x,y)dx \right)^{1/2} < +\infty.$$

In this paper we focus on the problem under what conditions on Ω John functions have the quadratic form defined above. We give an answer in Theorem 3.1 for a large class of proper subdomains of $\mathbb{R}^n$. The domains in question are definable in an o-minimal structure; in particular, the theorem applies to the semi-algebraic domains in $\mathbb{R}^n$. The most important feature of those domains (see Lemma 2.2) is that their boundary is "piece-wise Lipschitz", with the Lipschitz constant being arbitrary small. This enables us to construct finitely many John functions with a positive lower bound on the expansion sum (see Theorem 2.3). The construction, given in Section 2, suggests that one cannot expect that the type of elementary reasoning used in [Jo] can be extended to these special John functions. In Section 3 we apply the results of Section 2 to the $K_G(\Omega)$-characteristic of $K(\Omega)$ and relate them to certain geometrical properties involving either the Green potentials or the Carleson-like measures over the upper half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,+\infty)$ (as a typical o-minimal set). In the last section we consider the Harnack and Poincaré metric versions of the John functions, but also use a quadratic integral form (determined by the Green function and the Poincaré metric) to give a geometric condition which characterizes the uniformly perfect domains in the sense of Beardon and Pommerenke [BePo]. In the first section we gather some basic (old and new) properties of the John functions.

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1. Essential properties of John functions

1.1. Bounded mean oscillation. The first way to recognize the John functions is via John-Nirenberg's BMO-characterization (cf. [JoNi]), as shown in [Jo].

PROPOSITION 1.1. *Let Ω be a proper subdomain of $\mathbb{R}^n$. Then $f \in K(\Omega)$ has bounded mean oscillation in the sense that*

$$\|f\|_{BMO(\Omega)} = \sup_{B\subset\Omega} \frac{1}{|B|}\int_B \left| f(x) - \frac{1}{|B|}\int_B f(y)dy \right| dx < +\infty, \tag{1.1}$$

where the supremum is taken over all Euclidean balls $B \subset \Omega$ with volume $|B|$.

From now on, $H(\Omega)$ refers to the class of all real-valued harmonic functions on Ω. With this notation, we may remark that any function $f \in H(\Omega)$ satisfying (1.1) must lie in $K(\Omega)$; see [La] and [Os]. The concept of BMO occurs naturally in connection with PDE's and in many other areas; see, for example, [Car], [CDS], and [Ste].

1.2. Global Lipschitz continuity. The second property is that all John functions are Lipschitz continuous in a sufficiently small neighborhood of any point in Ω (cf. [Jo], [La]). This can also be understood via the global Lipschitz continuity with respect to the quasi-hyperbolic distance. Following [GeOs], we denote by $k_\Omega(x, y)$ the quasi-hyperbolic distance between two points x, y in Ω,

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma (\delta_\Omega(z))^{-1} ds(z),$$

where ds denotes the length element and the infimum ranges over all rectifiable curves $\gamma \subset \Omega$ joining x and y.

PROPOSITION 1.2. *Let Ω be a proper subdomain of $\mathbb{R}^n$. Then $f \in K(\Omega)$ if and only if there exists a constant $C > 0$ independent of $x, y \in \Omega$ such that*

$$|f(x) - f(y)| \le Ck_\Omega(x, y). \tag{1.2}$$

Proof. If $f \in K(\Omega)$, then, by [Mar], for any points $x, y \in \Omega$ there is a quasi-hyperbolic geodesic γ_{xy}, which may be supposed to be smooth in the arclength parameter. Let s denote arclength measured along γ_{xy} from x, and let $\zeta = \zeta(s)$ denote the corresponding representation for γ_{xy}. If l denotes the length of γ_{xy}, then

$$|f(x) - f(y)| \le \int_0^l \left| \nabla f(\zeta(s)) \frac{d\zeta(s)}{ds} \right| ds \le \|f\|_{K(\Omega)} \int_{\gamma_{xy}} (\delta_\Omega(z))^{-1} ds(z),$$

which implies (1.2).

Conversely, if (1.2) holds, then

$$c(f, k_\Omega) = \sup_{x,y\in\Omega, x\ne y} \frac{|f(x) - f(y)|}{k_\Omega(x, y)} < +\infty$$

is true. Since the quasi-hyperbolic distance $k_B(\cdot, \cdot)$ of the ball $B = B(x, r) \subset \Omega$ with the center x and radius r satisfies

$$k_B(x, y) = \log \frac{r}{r - |x - y|}, \quad y \in B,$$

one has $k_B(x,y) \ge k_\Omega(x,y)$ when $y \in B$, as well as

$$|f(x) - f(y)| \le c(f, k_\Omega) \log \frac{r}{r - |x-y|}, \quad y \in B.$$

This leads to $r|\nabla f(x)| \le nc(f, k_\Omega)$ which implies $f \in K(\Omega)$ by letting $r \to \delta_\Omega(x)$. □

1.3. Asymptotic behavior. The third property is the asymptotic behavior which shows that the John functions cannot asymptotically contain the image of a nonconstant affine function on $\mathbb{R}^n$ (see [Min] for the holomorphic $\mathbb{R}^2$-case). More precisely, we have:

Proposition 1.3. *Let Ω be a proper subdomain of $\mathbb{R}^n$. If $f \in K(\Omega)$, then there are no sequences $\{x_k\} \subset \Omega$ and $\{t_k\} \subset (0, +\infty)$ such that*

$$t_k/\delta_\Omega(x_k) \to 0; \quad f(x_k + t_k x) - f(x_k) \to \langle a, x\rangle, \tag{1.3}$$

where $\langle a, x\rangle$ is the standard scalar product of two points a, x in $\mathbb{R}^n$, and $|a| = 1$.

Proof. Assuming that such sequences exist, and putting $g_k(x) = f(x_k + t_k x) - f(x_k)$, $g(x) = a \cdot x$, $|a| = 1$, we have

$$1 = |\nabla g(0)| = \lim_{k\to+\infty} |\nabla g_k(0)| = \lim_{k\to+\infty} t_k(\delta_\Omega(x_k))^{-1}\delta_\Omega(x_k)|\nabla f(x_k)|.$$

This implies that $\delta_\Omega(x_k)|\nabla f(x_k)| \to +\infty$, and hence $f \notin K(\Omega)$, a contradiction. □

Moreover, it is worth mentioning that if the above non-existence result holds for $f \in H(\Omega)$, then f must lie in $K(\Omega)$. Indeed, if $\|f\|_{K(\Omega)} = +\infty$, then we will get a contradiction again. To see this, let $\{\Omega_m\}$ be a regular exhaustion of Ω, i.e., $\bigcup_{m=1}^{+\infty} \Omega_m = \Omega$, $\overline{\Omega}_m \subset \Omega_{m+1}$, and each $\overline{\Omega}_m$ is a compact subset of Ω. Set $C_m = \max_{x\in\Omega_m} \delta_{\Omega_m}(x)|\nabla f(x)|$. Because f is harmonic on $\overline{\Omega}_m$ and $\delta_{\Omega_m}(x) = 0$ for $x \in \partial\Omega_m$, there is a point $x_m \in \Omega_m$ such that $C_m = \delta_{\Omega_m}(x_m)|\nabla f(x_m)|$. It is easy to see that $C_m \le C_{m+1}$ and $C_m \to +\infty$. Set now

$$r_m = \delta_{\Omega_m}(x_m)/C_m, \quad g_m(x) = f(x_m + r_m x) - f(x_m).$$

Then $r_m/\delta_{\Omega_m}(x_m) \to 0$ and g_m is defined for $|x| < C_m$ with $g_m(0) = 0$, $|\nabla g_m(0)| = r_m|\nabla f(x_m)| = 1$. We will verify that $\{|\nabla g_m|\}$ is locally uniformly bounded. Fix any compact subset E of $\mathbb{R}^n$. Because $C_m \to +\infty$, there exists a constant $M = M(E)$ (depending only on E) such that $E \subset \{x \in \mathbb{R}^n : |x| < C_m\}$ for all $m \ge M$. For $x \in E$ and $m \ge M$ we have

$$|\nabla g_m(x)| = r_m|\nabla f(x_m + r_m x)| \le \frac{r_m C_m}{\delta_{\Omega_m}(x_m + r_m x)} \le \left(1 - \frac{|x|}{C_m}\right)^{-1},$$

due to the following Lipschitz continuity of $\delta_{\Omega_m}(\cdot)$:

$$|\delta_{\Omega_m}(x_m + r_m x) - \delta_{\Omega_m}(x_m)| \le r_m|x|, \quad \text{for} \quad x_m + r_m x,\ x_m \in \Omega_m.$$

The above estimates on $|\nabla g_m(x)|$ tell us that $\{|\nabla g_m|\}$ is uniformly bounded on E. Since $g_m(0) = 0$, it follows that $\{g_m\}$ is also locally uniformly bounded. Consequently, $\{g_m\}$ is a normal family. Thus, there is a subsequence $\{g_{m_k}\}$ which converges locally uniformly on $\mathbb{R}^n$ to a harmonic function g on $\mathbb{R}^n$. Clearly, $g(0) = 0$ and $|\nabla g(0)| = 1$ and $|\nabla g(x)| \leq 1$ for all $x \in \mathbb{R}^n$. The Liouville Theorem implies that $|\nabla g(x)|$ is a constant. Hence $|\nabla g(x)| \equiv 1$. It follows that $g(x) = \langle a, x\rangle = \sum_{j=1}^n a_j x_j$, where $|a|^2 = \sum_{j=1}^n |a_j|^2 = 1$. This obviously violates the non-existence assumption.

2. John functions on o-minimal domains

2.1. O-minimal structures. Recall first that a *semi-algebraic* set of $\mathbb{R}^n$ is a finite Boolean combination of the sets $\{f \geq 0\}$, where f is a polynomial on $\mathbb{R}^n$. The family of all semi-algebraic sets is stable under projections (Tarski's Theorem) and has nice finiteness properties (see, e.g., [BCR]). These properties are also shared by global subanalytic sets (i.e., projections of sets defined by analytic inequalities). Clearly, semi-algebraic domains are natural objects and one can deal with them practically. Many results in semi-algebraic (or subanalytic) geometry of $\mathbb{R}^n$ hold true in a more general setting, namely the theory of o-minimal structures on the real field. This has been of great interest since Wilkie [Wi] proved that a natural extension of the family of semi-algebraic sets which contains the exponential function is an o-minimal structure. For more information on this theory, see [Dr], [DMM], [Ku2], [LiRo], [Mil] and [Sh].

We say that the collection $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$ is an *o-minimal structure* on $(\mathbb{R}, +, \cdot)$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$, provided that:

(1) Each $\mathcal{M}_n$ is closed under finite set-theoretical operations.
(2) If $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$.
(3) If $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the projection on the first n coordinates of $\mathbb{R}^{n+m}$, then $\pi(A) \in \mathcal{M}_n$.
(4) If $f, g_1, \dots, g_k \in \mathbb{Q}[X_1, \dots, X_n]$, then
$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\} \in \mathcal{M}_n.$$
(5) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

For a fixed o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$ we say that A is an $\mathcal{M}$ definable set (or definable in $\mathcal{M}$) if $A \in \mathcal{M}_n$ for some $n \in \mathbb{N}$. We also say that a map $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is an $\mathcal{M}$ definable function if its graph is $\mathcal{M}$ definable.

Example 2.1 (cf. [DrMi]). The following are useful examples of the o-minimal structures:

(i) Semi-algebraic sets (by Tarski-Seidenberg): $\{x^4 + y^4 < 1\}$.
(ii) Global subanalytic sets (by Gabrielov): $\{0 < y < 1/\sin x,\ x \in (0, \pi)\}$.
(iii) $(\mathbb{R}, \exp)$ definable sets (by Wilkie): $\{0 < y < \exp(-1/x^2),\ x \in (0, 1)\}$.

(iv) ($\mathbb{R}_{\text{an}}, \exp$) definable sets (by van den Dries, Macintyre, Marker): $\{x^{\sqrt{2}} \ln(\sin y) < 1, x > 0, y \in (0, \pi)\}$.

(v) ($\mathbb{R}_{\text{an}}^{\mathbb{R}}$) definable sets (by Miller): $\{x^{\sqrt{2}} \exp(x/y) < 1,\, 0 < x\}$.

2.2. Construction of John functions. For $A \subset \mathbb{R}^n$, define d_A as the Euclidean distance to the set A, and write $\partial A = \overline{A} \setminus A$. Let Ω be a proper subdomain of $\mathbb{R}^n$ definable in some o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$. Then the boundary $\partial\Omega$ is also definable in $\mathcal{M}$ (cf. [Dr], [Ku2] or [BCR] for the semi-algebraic case). Thus we can decompose $\partial\Omega$ into a disjoint finite union $\bigcup_{i=1}^{N} \Gamma_i$ of connected C^1 submanifolds Γ_i of $\mathbb{R}^n$ in such a way that each $\overline{\Gamma}_i \setminus \Gamma_i$ is union of some of Γ_j, $\dim \Gamma_j < \dim \Gamma_i$. Moreover the manifolds Γ_i, $i = 1, \dots, N$, are definable in $\mathcal{M}$ and verify some Lipschitz type conditions.

LEMMA 2.2. *There exist a decomposition $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ and functions $f_i : \mathbb{R}^n \setminus \overline{\Gamma}_i \to \mathbb{R}$ corresponding to Γ_i, $i = 1, \dots, N$, such that*

$$f_i \in K(\mathbb{R}^n \setminus \overline{\Gamma}_i) \subset K(\Omega) \tag{2.1}$$

and

$$\inf \left\{ d_{\Gamma_i}(x) |\nabla f_i(x)| : \, 0 < d_{\Gamma_i}(x) < c_i d_{\partial\Gamma_i}(x) \right\} \geq 1, \tag{2.2}$$

where $c_i > 0$ is a constant, and if $\partial\Gamma_i = \emptyset$, then $d_{\partial\Gamma_i} = +\infty$ by convention.

Proof. By a rather standard construction in semi-algebraic or subanalytic geometry (which is also valid in any o-minimal category; see [Dr], [DrMi]), we can partition the set $\partial\Omega$ into a finite, disjoint union of the sets Γ_i. Each Γ_i, after a suitable orthogonal change of variables in $\mathbb{R}^n$, is of the form

$$\Gamma_i = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \, : \, x'' = \gamma_i(x'),\, x' \in L_i\},$$

where L_i is an open, $\mathcal{M}$-definable subset of $\mathbb{R}^k$, and $\gamma_i : L_i \to \mathbb{R}^{n-k}$ is a C^1 map. Moreover, $\|d_{x'}\gamma_i\| \leq \eta$, $x' \in L_i$, where the constant $\eta > 0$ can be chosen (in advance) arbitrarily small. Here $d_{x'}\gamma_i$ denotes the differential of γ_i at the point $x' \in L_i$.

Now the crucial point is that by a result of Kurdyka [Ku1] we may assume that all L_i have the following property (Whitney's property): Any two points $x'_1, x'_2 \in L_i$ can be joined, in L_i, by a smooth arc of length $\leq M|x'_1 - x'_2|$, where $M = M(k) > 1$ is a constant depending only on the dimension k. Actually, by [KuOr] the constant M can be taken arbitrarily close to 1, but we do not need this. So, by the Mean Value Theorem, we conclude that γ_i is Lipschitz with a constant $\varepsilon = M\eta > 0$, which will be chosen sufficiently small.

Let us fix one set $\Gamma_i = \Gamma$ and set $L_i = L$ and $\gamma_i = \gamma$ in order to simplify the notations. We will denote the corresponding function by f_Γ (instead of f_i). In fact, the set L, which is an L-regular cell in the sense of Parusiński [Pa] (i.e., after a suitable orthogonal change of variables in $\mathbb{R}^k$), is of the form

$$L = \{(\tilde{x}, x_k) \in \mathbb{R}^{k-1} \times \mathbb{R} \, : \, \varphi_k(\tilde{x}) < x_k < \psi_k(\tilde{x}),\, \tilde{x} \in L^{k-1}\},$$

where L^{k-1} is an L-regular cell in $\mathbb{R}^{k-1}$, and $\varphi_k, \psi_k : L^{k-1} \to \mathbb{R}$ are Lipschitz functions such that $\varphi_k < \psi_k$ in L^{k-1}. One of the functions φ_k and ψ_k may be equal to $-\infty$ or to $+\infty$. It is not difficult to prove the following result:

FACT 1. *If Γ is unbounded, then $\mathbb{R}^n \setminus \overline{\Gamma}$ is simply connected.*

Whenever $\dim \Gamma = 0$, $\Gamma = \{a\}$ is a point we set $f_\Gamma(x) = \log|x-a|$. Suppose now that $\dim \Gamma = k > 0$. We first define a C^∞ vector field $\vartheta_\Gamma : \mathbb{R}^n \setminus \overline{\Gamma} \to \mathbb{R}^n$ which will be the gradient of our function f_Γ. To the end, we put

$$\vartheta_\Gamma(x) = \int_L \frac{x - (\xi, \gamma(\xi))}{|x - (\xi, \gamma(\xi))|^{k+2}}\, d\xi.$$

Note that the above integral converges absolutely (see (2.3) and (2.4) below). Observe also that the integrated vector field is a gradient (with respect to x) of $-k^{-1}|x-(\xi, \gamma(\xi))|^{-k}$. Hence it is easily seen that the 1-form corresponding to ϑ_Γ is closed.

Suppose that Γ is unbounded. Then, by Fact 1, it follows that $\mathbb{R}^n \setminus \overline{\Gamma}$ is simply connected. So, by the classical Poincaré Lemma, there exists a C^∞ function $f_\Gamma : \mathbb{R}^n \setminus \overline{\Gamma} \to \mathbb{R}$ such that $\vartheta_\Gamma = \nabla f_\Gamma$.

If Γ is bounded, then L is bounded too, and hence we can write f_Γ explicitly as follows:

$$f_\Gamma(x) = -\frac{1}{k} \int_L |x - (\xi, \gamma(\xi))|^{-k}\, d\xi.$$

Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, and let $r = d_\Gamma(x)$. Clearly, $|x-(\xi, \gamma(\xi))| \geq r$ for any $\xi \in B(x', r)$, where $B(x', r)$ is the Euclidean ball of radius r about center x'. Thus

$$\int_{L \cap B(x',r)} \frac{d\xi}{|x - (\xi, \gamma(\xi))|^{k+1}} \;\leq\; \chi_k r^{-1}, \tag{2.3}$$

where χ_k denotes the volume of the unit ball in $\mathbb{R}^k$. On the other hand, we have

$$|x - (\xi, \gamma(\xi))| \geq |x' - \xi|,\ \xi \in \mathbb{R}^k.$$

Consequently,

$$\int_{L \setminus B(x',r)} \frac{d\xi}{|x - (\xi, \gamma(\xi))|^{k+1}} \;\leq\; \sigma_k r^{-1}, \tag{2.4}$$

where σ_k denotes the volume of the unit sphere in $\mathbb{R}^k$. Obviously (2.3) and (2.4) imply that $f_\Gamma \in K(\mathbb{R}^n \setminus \overline{\Gamma})$, and hence we have (2.1).

Next, we prove (2.2). For this, we introduce the notation $D_\Gamma(y) = |y'' - \gamma(y')|$ for any $y = (y', y'') \in L \times \mathbb{R}^{n-k}$. Notice that D_Γ is (bi-Lipschitz) equivalent to the distance function d_Γ in $L \times \mathbb{R}^{n-k}$, since the function γ is Lipschitz.

Fact 2. *If $\varepsilon > 0$, the Lipschitz constant of γ, is small enough, then*

$$\inf_{x \in T(\Gamma)} D_\Gamma(x) |\nabla f_\Gamma(x)| = c > 0,$$

where $T(\Gamma) = \{y = (y', y'') \in L \times \mathbb{R}^{n-k} \,:\, 0 < D_\Gamma(y) < d_{\partial L}(y')\}$.

Before proving Fact 2 let us show that it implies (2.2). Indeed, since D_Γ is (bi-Lipschitz) equivalent to the distance function d_Γ, there exists a constant $c_\Gamma > 0$ such that

$$\{y \in \mathbb{R}^n : 0 < d_\Gamma(y) < c_\Gamma d_{\partial \Gamma}(y)\} \subset T(\Gamma).$$

Thus multiplying, if necessary, f_Γ by a large constant and using the fact that D_Γ is equivalent to d_Γ, we obtain (2.2).

Hence it remains to find an $\varepsilon > 0$ such that Fact 2 holds. Recall that the mapping $\gamma : L \to \mathbb{R}^{n-k}$ is ε-Lipschitz. Thus for $x_0 = (x', \gamma(x'))$, Γ (the graph of γ) is contained in the affine cone $C^\varepsilon_{x_0}$, i.e.,

$$\Gamma \subset C^\varepsilon_{x_0} = \{(y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \,: |y'' - \gamma(x')| \leq \varepsilon |y' - x'|\}.$$

We will often use this fact without referring to it explicitly. Let $x = (x', x'') \in L \times \mathbb{R}^{n-k}$, $x'' \neq \gamma(x')$. Put $v = x - (x', \gamma(x'))$ and $r = |v|$. We can take $\varepsilon > 0$ so small that

$$\langle x - (\xi, \gamma(\xi)), |v|^{-1} v\rangle \geq 2^{-1} r, \quad |x - (\xi, \gamma(\xi))| \leq 2r, \tag{2.5}$$

for any $\xi \in B(x', r)$, where $B(x', r)$ is any ball contained in L. Once again, $\langle \cdot\, , \cdot \rangle$ stands for the standard scalar product in $\mathbb{R}^n$.

For brevity, let

$$\psi(\xi) = \frac{x - (\xi, \gamma(\xi))}{|x - (\xi, \gamma(\xi))|^{k+2}}.$$

We can write $\vartheta_\Gamma(x) = \nabla f_\Gamma(x)$ as a sum of the following three vectors:

$$v_1 = \int_{B(x', r)} \psi(\xi)\, d\xi,$$
$$v_2 = \int_{L \cap (B(x', pr) \setminus B(x', r))} \psi(\xi)\, d\xi,$$
$$v_3 = \int_{L \setminus B(x', pr)} \psi(\xi)\, d\xi,$$

where $p > 1$ will be determined below.

It is obvious, by (2.5), that

$$\langle v_1, |v|^{-1} v\rangle \geq 2^{-(k+3)} \chi_k r^{-1}.$$

Note that

$$|v_3| \leq M(\varepsilon)(pr)^{-1},$$

where $M(\varepsilon)$ is an increasing function of ε. Clearly, we may assume that $\varepsilon < 1$. Now we take $p > 1$ large enough so that $M(1) \leq p\chi_k 2^{-(k+4)}$. Finally, we take $\varepsilon > 0$ small enough so that $\langle x - (\xi, \gamma(\xi)), v\rangle > 0$ for any $\xi \in L \cap (B(x', pr) \setminus B(x', r))$. Put $c = \chi_k 2^{-(k+4)}$. Thus we obtain

$$|\nabla f_\Gamma(x)| \geq \langle v_1 + v_2 + v_3, |v|^{-1}v\rangle \geq cr^{-1} \geq c(D_\Gamma(x))^{-1},$$

for any $x = (x', x'') \in L \times \mathbb{R}^{n-k}$ such that $d_{\partial L}(x) < D_\Gamma(x')$. This proves Fact 2 and hence Lemma 2.2. □

2.3. Exhaustive John functions. We state now the main result of this section.

THEOREM 2.3. *Let Ω be a proper subdomain of $\mathbb{R}^n$. If Ω is definable in some o-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$, then there exist finitely many C^∞ functions $f_i \in K(\Omega)$, $i = 1, \ldots, N$, such that*

$$\inf_{x\in\Omega} \delta_\Omega(x) \sum_{i=1}^{N} |\nabla f_i(x)| > 0. \tag{2.6}$$

Proof. We apply Lemma 2.2 to get that each Γ_i, $i = 1, \ldots, N$, is associated with a function $f_i : \mathbb{R}^n \setminus \overline{\Gamma}_i \to \mathbb{R}$ such that $f_i \in K(\Omega)$ and $d_{\Gamma_i}(x)|\nabla f_i(x)| \geq 1$ holds in $\{x \in \mathbb{R}^n : 0 < d_{\Gamma_i}(x) < c_i d_{\partial\Gamma_i}(x)\}$, where $c_i > 0$ is a constant. If $\partial\Gamma_i = \emptyset$, then, by convention, we put $d_{\partial\Gamma_i} = +\infty$.

We now prove (2.6) by induction on N. From the partition constructed in Lemma 2.2 it follows that at least one of the sets Γ_i is a point. Hence the case $N = 1$ is trivial, since $\partial\Omega = \{a\}$ for some $a \in \mathbb{R}^n$ and (2.6) holds for $\Omega = \mathbb{R}^n \setminus \{a\}$.

Suppose that $N > 1$ and that Γ_N has a maximal dimension. Then Γ_N must be open in $\partial\Omega$ and consequently $\partial\Omega' = \bigcup_{i=1}^{N-1} \Gamma_i$ is closed in $\partial\Omega$. Hence $\Omega' = \mathbb{R}^n \setminus \partial\Omega'$ is open in $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$, $\delta_\Omega(x) > 0$. To prove (2.6) we will consider three cases.

Case 1. If $\delta_\Omega(x) = d_{\Gamma_N}(x)$ and $d_{\Gamma_N}(x) < c_N d_{\partial\Gamma_N}(x)$, then, by (2.2), we have

$$d_{\Gamma_N}(x)|\nabla f_N(x)| \geq 1.$$

Case 2. If $\delta_\Omega(x) = d_{\Gamma_N}(x)$ and $d_{\Gamma_N}(x) \geq c_N d_{\partial\Gamma_N}(x)$, then $\delta_\Omega(x) \geq c_N \delta_{\Omega'}(x)$ since $d_{\partial\Gamma_N}(x) \geq d_{\partial\Omega'}(x) = \delta_{\Omega'}(x)$. So, by induction, we have

$$\delta_\Omega(x) \sum_{i=0}^{N} |\nabla f_i(x)| \geq c_N \delta_{\Omega'}(x) \sum_{i=0}^{N-1} |\nabla f_i(x)| \geq c_N \lambda_{N-1},$$

where $\lambda_{N-1} > 0$ is the infimum of (2.6) corresponding to Ω'.

Case 3. If $\delta_\Omega(x) = d_{\Gamma_i}(x)$ for some $i \leq N-1$, then $\delta_\Omega(x) = \delta_{\Omega'}(x)$, and hence by induction

$$\delta_\Omega(x) \sum_{i=0}^{N} |\nabla f_i(x)| \geq \lambda_{N-1}.$$

This completes the proof of Theorem 2.3. □

Remark 2.4. Theorem 2.3 is an o-minimal generalization of the key result, Proposition 5.4, in [RU] (which is established by using complex analysis on the unit disc of $\mathbb{R}^2$). It is clear that there exists an essential difference between two situations (differentiable versus holomorphic).

3. Quadratic forms via Green potentials

3.1. The general cases. In general, the Green function of a proper subdomain Ω of $\mathbb{R}^n$ for the Laplacian Δ is defined by $g_\Omega(x,y) = -p(x-y) - q_y(x)$, where

$$p(x) = \begin{cases} \log|x|, & n = 2, \\ -|x|^{2-n}, & n \geq 3, \end{cases} \tag{3.1}$$

and $q_y(x)$ is a function (differentiable on Ω and continuous on $\overline{\Omega}$) solving the boundary value problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u(x) = -p(x-y), & x \in \partial\Omega. \end{cases}$$

It is very difficult to give an explicit formula for $g_\Omega(\cdot,\cdot)$. However, if Ω is either a ball or the upper half space of $\mathbb{R}^n$, then $g_\Omega(\cdot,\cdot)$ can be computed explicitly; see [AiEs, p. 65], for example. Clearly, not all domains definable in an o-minimal structure have the Green functions, but there are still many o-minimal domains (explained below) which have their Green functions.

Recall that L is an *open cell* in $\mathbb{R}^n$ if L is of the form

$$L = \{(\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \varphi_n(\tilde{x}) < x_n < \psi_n(\tilde{x}),\ \tilde{x} \in L^{n-1}\},$$

where L^{n-1} is an L-regular cell in $\mathbb{R}^{n-1}$, $\varphi_n, \psi_n : L^{n-1} \to \mathbb{R}$ are C^1 functions such that $\varphi_n < \psi_n$ in L^{n-1}. One or both functions φ_n and ψ_n may be equal to $-\infty$ or to $+\infty$. In $\mathbb{R}$ the open cells are open intervals. If $\mathcal{M}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$ we say that L is *definable in* $\mathcal{M}$ if $L \in \mathcal{M}$. This is equivalent to the condition that L^{n-1} and φ_n, ψ_n are definable in $\mathcal{M}$. Furthermore, it is known that every open subset of $\mathbb{R}^n$ definable in $\mathcal{M}$ is a finite union of open cells which are definable in $\mathcal{M}$. For these facts see [Dr].

Let Ω be an open set in $\mathbb{R}^n$. The classical sufficient condition for Ω to be regular (with respect to the Dirichlet Problem) is the following: For any

$x \in \partial\Omega$ there exists an open cone Λ with the vertex at x and a neighborhood U of x such that

$$(*) \qquad \overline{\Lambda} \cap U \cap \overline{\Omega} = \{x\}.$$

Every open cell $L \subset \mathbb{R}^n$ is regular, so, in particular, it has the Green function. This can be verified by showing by induction that ∂L has the above cone property $(*)$. In fact, if $n = 1$, then $(*)$ is trivial since L is an open interval. Let $x \in \partial L$. Then $x \in \partial L^{n-1} \times \mathbb{R}$ or x belongs to the graph of φ_n or ψ_n. In the first case, $x = (\tilde{x}, x_n)$, where $\tilde{x} \in \partial L^{n-1}$, so we find (by induction) a cone $\Lambda' \subset \mathbb{R}^{n-1}$ which satisfies $(*)$. Take $B = \{(\tilde{y}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y_n - x_n| \le |\tilde{y} - \tilde{x}|\}$. Then $\Lambda = (\Lambda' \times \mathbb{R}) \cap B$ is a cone with vertex at x, which implies condition $(*)$. The second case is obvious since φ_n and ψ_n are C^1 functions.

To see when $K(\Omega)$ is of the quadratic integral form mentioned in the introduction, we recall the definition of $K_G(\Omega)$. Given a proper subdomain Ω of $\mathbb{R}^n$ with the Green function $g_\Omega(\cdot,\cdot)$, a C^1 function $f : \Omega \to \mathbb{R}$ belongs to $K_G(\Omega)$ if and only if

$$\|f\|_{K_G(\Omega)} = \sup_{y\in\Omega} \left(\int_\Omega |\nabla f(x)|^2 g_\Omega(x,y)dx \right)^{1/2} < +\infty.$$

THEOREM 3.1. *Let Ω be a proper subdomain of $\mathbb{R}^n$ with the Green function $g_\Omega(\cdot,\cdot)$. Suppose that Ω is definable in some o-minimal structure $\mathcal{M}$ on $(\mathbb{R},+,\cdot)$. Then there exists a constant $C > 0$ depending only on Ω such that $\|f\|_{K_G(\Omega)} \le C\|f\|_{K(\Omega)}$ for all $f \in K(\Omega)$ if and only if*

$$(3.2) \qquad C_g(\Omega) = \sup_{y\in\Omega} \int_\Omega (\delta_\Omega(x))^{-2} g_\Omega(x,y)dx < +\infty.$$

Proof. The sufficiency is simple. To prove the necessity, we apply Theorem 2.3 to obtain N functions $f_1, f_2, \dots, f_N \in K(\Omega)$ such that

$$(3.3) \qquad m = \inf_{x\in\Omega} \delta_\Omega(x)\big(|\nabla f_1(x)| + |\nabla f_2(x)| + \cdots + |\nabla f_N(x)|\big) > 0.$$

If $\|f\|_{K_G(\Omega)} \le C\|f\|_{K(\Omega)}$ holds for all $f \in K(\Omega)$ and some constant $C > 0$ depending only on Ω, then $\|f_j\|_{K_G(\Omega)} \le C\|f_j\|_{K(\Omega)}$, $j = 1, 2, \dots, N$. From (3.3) it follows that

$$\begin{aligned} m^2 \int_\Omega (\delta_\Omega(x))^{-2} g_\Omega(x,y)dx &\le 2^N \int_\Omega \sum_{j=1}^N |\nabla f_j(x)|^2 g_\Omega(x,y)dx \\ &\le C^2 2^N \sum_{j=1}^N \|f_j\|^2_{K(\Omega)}, \end{aligned}$$

which implies (3.2). The proof is complete. □

REMARK 3.2. It is clear that the sufficiency part of Theorem 3.1 holds for general domains with the Green functions. More importantly, the condition $C_g(\Omega) < +\infty$ defines a class of proper subdomains of $\mathbb{R}^n$. Those domains with the property (3.2) are called *GP-domains (Green potential domains).*

COROLLARY 3.3. *Let Ω be a proper subdomain of $\mathbb{R}^n$ with the Green function $g_\Omega(\cdot,\cdot)$. Then $K(\Omega)\cap H(\Omega) \supset K_G(\Omega)\cap H(\Omega)$. Moreover, $K(\Omega)\cap H(\Omega) = K_G(\Omega)\cap H(\Omega)$ whenever Ω is also a GP-domain.*

Proof. Fix a point $y \in \Omega$ and its Euclidean ball $B = B(y,r)$ with radius $r = \delta_\Omega(y)$. The Green function of B obeys

$$g_B(x,y) = \begin{cases} \log \dfrac{r}{|x-y|}, & n = 2, \\ \dfrac{1}{|x-y|^{n-2}} - \dfrac{1}{r^{n-2}}, & n \geq 3. \end{cases} \tag{3.4}$$

Let $f \in K_G(\Omega)\cap H(\Omega)$. Then an elementary estimation and the subharmonicity of $|\nabla f|^2$ show that there exists a constant $C_1 > 0$ depending only on n such that

$$\|f\|^2_{K_G(\Omega)} \geq \int_{B(y,r/2)} |\nabla f(x)|^2 g_B(x,y)dx \geq C_1\big(r|\nabla f(y)|\big)^2.$$

This means $f \in K(\Omega)\cap H(\Omega)$ and thus $K(\Omega)\cap H(\Omega) \supset K_G(\Omega)\cap H(\Omega)$. Further, if Ω is a GP-domain, with the help of Theorem 3.1, we then have $K(\Omega)\cap H(\Omega) \subset K_G(\Omega)\cap H(\Omega)$, and hence $K(\Omega)\cap H(\Omega) = K_G(\Omega)\cap H(\Omega)$. This completes the proof. □

3.2. The upper half space. In the sequel, we consider $\mathbb{R}^n_+$, a typical $\mathcal{M}$-definable domain. For this purpose, we introduce a generalized Carleson measure on $\mathbb{R}^n_+$. For $p \in (0,+\infty)$, a positive Borel measure $d\mu$ on $\mathbb{R}^n_+$ is said to be a p-Carleson measure provided

$$\sup \frac{\mu(S(I))}{\big(\ell(I)\big)^{p(n-1)}} < +\infty, \tag{3.5}$$

where the supremum is taken over all Carleson boxes $S(I) = I \times (0, \ell(I)] \subset \mathbb{R}^n_+$ based on cubes $I \subset \mathbb{R}^{n-1}$ with edges parallel to the coordinate axes of $\mathbb{R}^{n-1}$, where $\ell(I)$ stands for the edge length of I. The case $p = 1$ is the so-called Carleson measure. Moreover, if the supremum in (3.5) is taken over all $a(\geq 0)$-Carleson boxes $S_a(I) = I \times (a, a+\ell(I)]$, then $d\mu$ is called a strong p-Carleson measure. Obviously, a strong p-Carleson measure must be a p-Carleson measure, but not conversely. For the case of the unit disc, see [Zhu, Ex.6, p. 188].

In the rest of this section, the notations $g(x,y)$ and $\delta(x)$ will stand for the Green function of $\mathbb{R}^n_+$ and the distance of the point $x \in \mathbb{R}^n_+$ to the boundary

$\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, respectively. Also, $\tilde{y}$ stands for the symmetric point of $y \in \mathbb{R}^n_+$ with respect to $\mathbb{R}^{n-1}$, that is to say, if $y = (y_1, \dots, y_n)$, then $\tilde{y} = (y_1, \dots, -y_n)$.

LEMMA 3.4. *Let $d\mu$ be a positive Borel measure on $\mathbb{R}^n_+$ and let $p \in (0, +\infty)$. Then $d\mu$ is a p-Carleson measure if and only if*

$$\|\mu\|_p = \sup_{y \in \mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \left(\frac{\delta(y)}{|x-\tilde{y}|^n} \right)^p d\mu(x) \right)^{1/2} < +\infty. \tag{3.6}$$

Proof. See Lemma 4.1 in [EJPX]. □

THEOREM 3.5. *Let f be a C^1 function on $\mathbb{R}^n_+$ and let $d\mu_f(x) = |\nabla f(x)|^2 \delta(x) dx$. If $f \in K_G(\mathbb{R}^n_+)$, then $d\mu_f$ is a 1-Carleson measure. Conversely, if $d\mu_f$ is a strong 1-Carleson measure, then $f \in K_G(\mathbb{R}^n_+)$.*

Proof. Note that

$$g(x,y) = \begin{cases} \log \dfrac{|x-\tilde{y}|}{|x-y|}, & n = 2, \\ \dfrac{1}{|x-y|^{n-2}} - \dfrac{1}{|x-\tilde{y}|^{n-2}}, & n \geq 3, \end{cases} \tag{3.7}$$

and that there is a constant $C_2 > 0$ depending only on n such that

$$\begin{cases} (1) \quad g(x,y) \geq \dfrac{2\delta(x)\delta(y)}{|x-\tilde{y}|^n}, & n \geq 2, \\ (2) \quad g(x,y) \leq \dfrac{C_2 \delta(x)\delta(y)}{|x-\tilde{y}|^2 |x-y|^{n-2}}, & n \geq 3, \\ (3) \quad g(x,y) \leq \left(\dfrac{-2\log c}{1-c^2} \right) \dfrac{\delta(x)\delta(y)}{|x-\tilde{y}|^2}, & 0 < c < 1,\ c^2 < \frac{|x-y|}{|x-\tilde{y}|},\ n = 2; \end{cases} \tag{3.8}$$

see [AiEs, p. 68] and [Ga, p. 289].

If $f \in K_G(\mathbb{R}^n_+)$, then (1) of (3.8), together with Lemma 3.4, gives immediately the desired assertion. Conversely, if $d\mu_f$ is a strong 1-Carleson measure, then

$$|||\mu_f|||_1 := \sup_{S_a(I) \subset \mathbb{R}^n_+} \left(\frac{\mu_f(S_a(I))}{\big(\ell(I)\big)^{n-1}} \right)^{1/2} < +\infty.$$

Since the case $n = 2$ is similar, it is enough to consider the cases $n \geq 3$. By (3.6) and (2) of (3.8), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} |\nabla f(x)|^2 g(x,y)dx \\
&\le C_2 \sum_{k=0}^{+\infty} \int_{2^{-(k+1)}\le|x-y|/|x-\tilde{y}|<2^{-k}} |\nabla f(x)|^2 \left(\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^2|x-y|^{n-2}}\right) dx \\
&\le C_2 \sum_{k=0}^{+\infty} 2^{(n-2)(k+1)} \int_{2^{-(k+1)}\le|x-y|/|x-\tilde{y}|<2^{-k}} |\nabla f(x)|^2 \left(\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^n}\right) dx \\
&\le C_2 \sum_{k=2}^{+\infty} 2^{(n-2)(k+1)} \int_{|x-y|\le 2^{-k+1}\delta(y)/(1-2^{-k})} |\nabla f(x)|^2 \left(\frac{\delta(x)\delta(y)}{(\delta(y))^n}\right) dx \\
&\qquad + C_2 2^{n-2} \int_{2^{-2}\le|x-y|/|x-\tilde{y}|<1} |\nabla f(x)|^2 \left(\frac{\delta(x)\delta(y)}{|x-\tilde{y}|^n}\right) dx \\
&\le C_3(\delta(y))^{1-n} \sum_{k=2}^{+\infty} 2^{(n-2)(k+1)} \int_{|x-y|\le 2^{-k+2}\delta(y)} |\nabla f(x)|^2 \delta(x)dx + C_3\|\mu_f\|_1^2 \\
&= C_3(\delta(y))^{1-n} \sum_{k=2}^{+\infty} 2^{(n-2)(k+1)} \mu_f(B(y, 2^{2-k}\delta(y))) + C_3\|\mu_f\|_1^2 \\
&\le C_4 \sum_{k=0}^{+\infty} 2^{-k} |||\mu_f|||_1^2 + C_3\|\mu_f\|_1^2.
\end{aligned}$$

Here C_3 and C_4 are positive constants independent of y. As a result, $f \in K_G(\mathbb{R}^n_+)$. □

Corollary 3.6. *Let $f \in H(\mathbb{R}^n_+)$ and $d\nu_f(x) = |\nabla f(x)|^2(\delta(x))^2 dx$. Then the following statements are equivalent:*

(i) $f \in K(\mathbb{R}^n_+)$.
(ii) *$d\nu_f$ is an $n/(n-1)$-Carleson measure.*
(iii) *f satisfies*

$$\sup_{y\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} |\nabla f(x)|^2 \big(g(x,y)\big)^{\frac{n}{n-1}} (\delta(x))^{\frac{n-2}{n-1}} dx < +\infty.$$

Proof. (i)⇔(ii). If $f \in K(\mathbb{R}^n_+)$, then for any Carleson box $S(I) \subset \mathbb{R}^n_+$,

$$\int_{S(I)} |\nabla f(x)|^2 (\delta(x))^2 dx \le \|f\|^2_{K(\mathbb{R}^n_+)} (\ell(I))^n,$$

which implies that $d\nu_f$ is an $n/(n-1)$-Carleson measure.

Conversely, if $d\nu_f$ is an $n/(n-1)$-Carleson measure, then

$$|||\nu_f|||_{n/(n-1)} := \sup_{S(I)\subset\mathbb{R}^n_+} \left(\frac{\mu_f(S(I))}{\big(\ell(I)\big)^n}\right)^{1/2} < +\infty.$$

The Submean Value Property of $|\nabla f|^2$ shows that there exists a constant $C_5 > 0$ depending on the dimension n so that for the Carleson box $S(I)$ with center $y \in \mathbb{R}^n_+$ and edge length $2\delta(y)$,

$$(\ell(I))^n \||\nu_f\||^2_{n/(n-1)} \geq \int_{S(I)} |\nabla f(x)|^2 (\delta(x))^2 dx \geq C_5 |\nabla f(y)|^2 \int_0^{\delta(y)} t^{n+1} dt.$$

Hence f lies in $K(\mathbb{R}^n_+)$.

(i)$\Leftrightarrow$(iii). If (iii) holds, then (3.8) and Lemma 3.4 imply that (ii) holds. Hence (i) follows. On the other hand, if $f \in K(\mathbb{R}^n_+)$, then an elementary calculation shows that $d\nu_f$ is a strong $n/(n-1)$-Carleson measure. Moreover, the argument of Theorem 3.5 yields (iii) at once. □

REMARK 3.7. We have actually proved that the results on $K(\mathbb{R}^n_+) \cap H(\mathbb{R}^n_+)$ in Corollary 3.6 correspond nicely to the analogous results on $BMOH(\mathbb{R}^n_+)$, the class of the Poisson harmonic extensions to $\mathbb{R}^n_+$ of functions in $BMO(\mathbb{R}^{n-1})$; see Carleson [Car] and Leutwiler [Le2].

4. Harnack metric and uniformly perfect domains

4.1. The Harnack metric. For a proper subdomain Ω of $\mathbb{R}^n$, write $H^+(\Omega)$ for the set of all positive harmonic functions on Ω. The Harnack density on Ω is given by

$$\eta_\Omega(x) = \sup_{f \in H^+(\Omega)} |\nabla \log f(x)|. \tag{4.1}$$

Since the Harnack and quasi-hyperbolic densities are comparable on balls of $\mathbb{R}^n$ (cf. [Ko] or [Le1]), it follows that

$$\eta_\Omega(x) \leq n(\delta_\Omega(x))^{-1}. \tag{4.2}$$

We now use the Harnack density to define $K_\eta(\Omega)$ as the space of all C^1 functions f on Ω satisfying

$$\|f\|_{K_\eta(\Omega)} = \sup_{x \in \Omega} (\eta_\Omega(x))^{-1} |\nabla f(x)| < +\infty.$$

From (4.2) it follows that if $f \in H^+(\Omega)$ then $\log f$ is in $K(\Omega)$, and that $K_\eta(\Omega) \subset K(\Omega)$ with $\|\cdot\|_{K(\Omega)} \leq n\|\cdot\|_{K_\eta(\Omega)}$. However, we will see that not all John functions have the $K_\eta(\Omega)$-property.

THEOREM 4.1. *Let Ω be a proper subdomain of $\mathbb{R}^n$. Then there exists a constant $C > 0$ depending only on Ω such that $\|f\|_{K_\eta(\Omega)} \leq C\|f\|_{K(\Omega)}$ for all $f \in K(\Omega)$ if and only if*

$$C_\eta(\Omega) = \inf_{x \in \Omega} \eta_\Omega(x)\delta_\Omega(x) > 0. \tag{4.3}$$

Proof. The sufficiency is obvious. As to the necessity, assume that there is a constant $C > 0$ depending only on Ω such that for all $f \in K(\Omega)$,

$$\|f\|_{K_\eta(\Omega)} \leq C\|f\|_{K(\Omega)}. \tag{4.4}$$

Now, fix $x_0 \in \Omega$ and pick a point $y_0 \in \partial\Omega$ such that $\delta_\Omega(x_0) = |x_0 - y_0|$. Then the function $f_0(x) = \log|x - y_0|$ belongs to $K(\Omega)$ and $\|f_0\|_{K(\Omega)} \leq 1$. By (4.4) we get $\|f_0\|_{K_\eta(\Omega)} \leq C$ and $\eta_\Omega(x)|x - y_0| \geq C^{-1}$, which, in particular, implies (4.3) (by choosing $x = x_0$). □

Remark 4.2. (4.3) actually defines a class of proper subdomains of $\mathbb{R}^n$. We call a domain satisfying (4.3) an *HM-domain (Harnack metric domain)*. Obviously, the unit ball and the upper half space of $\mathbb{R}^n$ are HM-domains. In the case $n \geq 3$, many Hölder domains are HM-domains (cf. [SmSt]). In fact, more is true: If $n \geq 3$, then every proper subdomain of $\mathbb{R}^n$ is an *HM*-domain. To see this, let $x_0 \in \Omega$, and choose $y_0 \in \partial\Omega$ with $\delta_\Omega(x_0) = |x_0 - y_0|$. If $f(x) = |x - y_0|^{2-n}$ for $x \in \Omega$, then $f \in H^+(\Omega)$ and hence $|\nabla \log f(x_0)| = (n-2)/|x_0 - y_0|$. As a consequence, $\eta_\Omega(x_0)\delta_\Omega(x_0) \geq n-2$ and so $C_\eta(\Omega) \geq n-2$. The authors thank the referee for pointing out this argument.

4.2. The Poincare metric. We next consider planar domains. In particular, we find that the quasi-hyperbolic metric and the Poincaré metric enable us to distinguish the John functions.

From now on, $\mathbb{R}^2$ is identified with the finite complex plane $\mathbb{C}$ and x and y are viewed as complex numbers. A proper subdomain Ω of $\mathbb{R}^2$ is called *hyperbolic* if its universal covering surface is the unit disk $\mathbb{D}$. Suppose that $\lambda_\Omega(x)$ is the Poincaré density on Ω, determined by

$$\lambda_\Omega(p(y))|p'(y)| = \lambda_{\mathbb{D}}(y) = (1 - |y|^2)^{-1}, \quad y \in \mathbb{D}. \tag{4.5}$$

Note that $x = p(y)$ is a universal covering map from $\mathbb{D}$ onto Ω and (4.5) is independent of the choice of y. The Schwarz Lemma easily yields that this density is decreasing; i.e., if two hyperbolic domains Ω_1 and Ω_2 satisfy $\Omega_1 \subset \Omega_2$, then $\lambda_{\Omega_2}(x) \leq \lambda_{\Omega_1}(x)$ for $x \in \Omega_1$ (cf. [BePo]). The following inequalities on $\delta_\Omega(x)$, $\eta_\Omega(x)$ and $\lambda_\Omega(x)$ are well known:

$$\delta_\Omega(x) \leq (\lambda_\Omega(x))^{-1} \leq 2(\eta_\Omega(x))^{-1}; \tag{4.6}$$

see, for instance, [GeOs] and [Ko]. Nevertheless, when Ω is simply connected, $\delta_\Omega(x)$, $(\lambda_\Omega(x))^{-1}$ and $(\eta_\Omega(x))^{-1}$ are comparable.

As before, we use the Poincaré density to define the space $K_\lambda(\Omega)$ consisting of all C^1 functions on the hyperbolic domain Ω in $\mathbb{R}^2$ with

$$\|f\|_{K_\lambda(\Omega)} = \sup_{x\in\Omega}(\lambda_\Omega(x))^{-1}|\nabla f(x)| < +\infty.$$

Theorem 4 of [Os] shows that $\log \lambda_\Omega \in K(\Omega)$. It is clear that $K_\lambda(\Omega) \subset K(\Omega)$ with $\|\cdot\|_{K(\Omega)} \leq \|\cdot\|_{K_\lambda(\Omega)}$. Therefore it is natural to compare $K(\Omega)$ with $K_\lambda(\Omega)$.

THEOREM 4.3. *Let Ω be a hyperbolic domain in $\mathbb{R}^2$. Then there is a constant $C > 0$ depending only on Ω such that $\|f\|_{K_\lambda(\Omega)} \le C\|f\|_{K(\Omega)}$ for all $f \in K(\Omega)$ if and only if*

$$C_\lambda(\Omega) = \inf_{x\in\Omega} \lambda_\Omega(x)\delta_\Omega(x) > 0. \tag{4.7}$$

Proof. The sufficiency is trivial, so it remains to show the necessity. Assume that there exists a constant $C > 0$ depending only on Ω such that $\|f\|_{K_\lambda(\Omega)} \le C\|f\|_{K(\Omega)}$ for all $f \in K(\Omega)$. Fix a point $x_0 \in \Omega$ and pick a point $y_0 \in \partial\Omega$ such that $\delta_\Omega(x_0) = |x_0 - y_0|$. It is known that the function $f_0(x) = \log|x - y_0|$ is a member of $K(\Omega)$. Thus, the above hypothesis implies that $f_0 \in K_\lambda(\Omega)$ with $\|f_0\|_{K_\lambda(\Omega)} \le C\|f_0\|_{K(\Omega)}$, and (4.7) follows. □

REMARK 4.4. A hyperbolic domain in $\mathbb{R}^2$ is called *uniformly perfect (UP)* if (4.7) holds. Obviously, an HM-domain is a UP-domain. Also, the proof of Theorem 4.3 actually reveals that $K_\lambda(\Omega) \cap H(\Omega) = K(\Omega) \cap H(\Omega)$ if and only if Ω is a UP-domain. In other words, even harmonic functions can distinguish between the quasi-hyperbolic metric and the Poincaré metric.

The concept of UP-domains comes originally from [BePo]. Several characterizations of such domains can be found in [Po]. However, the following result gives a special geometric description of these domains.

THEOREM 4.5. *Let Ω be a finitely connected hyperbolic domain in $\mathbb{R}^2$. Then Ω is a UP-domain if and only if every component of $\partial\Omega$ contains at least two points.*

Proof. The sufficiency is essentially known (cf. [Mas1] and [Mas2]), so it remains to prove the necessity, which is quite complicated. To this end, suppose that $C_\lambda(\Omega)$ is positive. Thus, λ_Ω and δ_Ω^{-1} are comparable. Further, such a domain cannot have any isolated boundary point; otherwise, if y were an isolated boundary point, by taking a punctured Euclidean ball $B(y, r) \setminus \{y\} \subset \Omega$ we would get that

$$\lambda_\Omega(x)\delta_\Omega(x) \le r(\log r - \log|x - y|)^{-1} \to 0$$

as $x \to y$, and hence $C_\lambda(\Omega) = 0$, a contradiction.

In what follows, we will prove

$$C_{\lambda,g}(\Omega) = \sup_{y\in\Omega} \int_\Omega \big(\lambda_\Omega(x)g_\Omega(x, y)\big)^2 dx < +\infty, \tag{4.8}$$

where, here and afterwards, $g_\Omega(x, y)$ denotes still the Green function of Ω for the Laplacian Δ.

Observe that $C_\lambda(\Omega)$ and $C_{\lambda,g}(\Omega)$ are conformally equivalent. Thus, without loss of generality, we may suppose that Ω is a regular domain, i.e., a hyperbolic domain bounded by finitely many simple closed analytic curves. Furthermore, we put $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ and $\overline{\mathbb{R}^2} \setminus \Omega = \bigcup_{j=1}^m G_j$, where $\{G_j\}_{j=1}^m$ are all the components of $\overline{\mathbb{R}^2} \setminus \Omega$. Hence each G_j is simply connected, but also has at least three boundary points.

Now, choose compact subsets $\{E_j\}_{j=1}^m$ and $\{F_j\}_{j=1}^m$ of $\overline{\Omega}$ such that

$$\begin{cases} E_j \cap \Omega \subset \mathrm{Int} F_j \quad (\text{interior of } F_j), \quad j = 1, 2, \dots, m, \\ F_j \cap G_k = \emptyset, \quad j \neq k, \\ \bigcup_{j=1}^m E_j = \overline{\Omega}. \end{cases}$$

We will show that for each $j = 1, 2, \dots, m$,

$$M_j = \sup\{\lambda_\Omega(x) g_\Omega(x, y) : (x, y) \in (\Omega \setminus F_j) \times (E_j \cap \Omega)\} < +\infty. \tag{4.9}$$

Assume that ϕ_k is a conformal map from $\mathbb{D}$ onto $\Omega_k = \overline{\mathbb{R}^2} \setminus G_k$, $k = 1, 2, \dots, m$. For $y \in \partial E_j \cap \Omega$ and $x \in \Omega$ sufficiently close to G_k, we have $\delta_\Omega(x) = \delta_{\Omega_k}(x)$ and

$$\lambda_\Omega(x) g_\Omega(x, y) \leq 4\lambda_{\mathbb{D}}(u) g_{\mathbb{D}}(u, v) |\phi_k'(u)|^{-1},$$

where $x = \phi_k(u)$ and $y = \phi_k(v)$. Since ϕ_k can be extended continuously and conformally beyond $\partial\mathbb{D}$ (the unit circle) and

$$\lim_{u \to e^{i\theta}} \lambda_{\mathbb{D}}(u) g_{\mathbb{D}}(u, v) = (1 - |v|^2)/(2|e^{i\theta} - v|^2),$$

(4.9) follows. Here we have used the fact that $g_\Omega(x, y) = 0$ whenever $y \in \partial G_j$ (owing to the regularity of Ω).

Since $y \in \Omega$, we can pick j so that $y \in E_j$ and

$$\int_{\Omega \setminus F_j} \big(\lambda_\Omega(x) g_\Omega(x, y)\big)^2 dx \leq |\Omega| \sup_{1 \leq k \leq m} M_k^2. \tag{4.10}$$

We also have

$$\int_{F_j} \big(\lambda_\Omega(x) g_\Omega(x, y)\big)^2 dx \leq \left(\sup_{x \in F_j} \frac{\lambda_\Omega(x)}{\lambda_{\Omega_j}(x)}\right)^2 \int_{F_j} \big(\lambda_{\Omega_j}(x) g_{\Omega_j}(x, y)\big)^2 dx. \tag{4.11}$$

Noting that

$$\frac{\lambda_\Omega(x)}{\lambda_{\Omega_j}(x)} \leq \frac{\delta_{\Omega_j}(x)}{\delta_\Omega(x)}, \quad x \in \Omega,$$

and $\delta_\Omega(x) = \delta_{\Omega_j}(x)$ as $x \to \partial\Omega \cap F_j$, we get

$$N_j = \sup_{x \in F_j} \frac{\lambda_\Omega(x)}{\lambda_{\Omega_j}(x)} < +\infty. \tag{4.12}$$

However, it follows from the conformal invariance of λ_{Ω_j} and g_{Ω_j} that

$$\int_{F_j} \big(\lambda_{\Omega_j}(x) g_{\Omega_j}(x,y)\big)^2 dx \leq \int_{\mathbb{D}\setminus\{v\}} \big(\lambda_{\mathbb{D}}(u) g_{\mathbb{D}}(u,v)\big)^2 du \leq 2\pi \int_0^1 \left(\frac{\log t}{1-t^2}\right)^2 dt. \tag{4.13}$$

Using (4.10)–(4.13), we get (4.8).

Let us return to the proof that every component of $\partial\Omega$ contains at least two points. If not, then there would be a component of $\partial\Omega$ consisting of a single point, say $\{y_0\}$. By taking a small Euclidean ball $B(y_0, r)$ for which $B(y_0, r) \setminus \{y_0\} \subset \Omega$, and using a remark in [Mas1], we obtain

$$C(\Omega) = \inf_{x \in B(y_0,r)\setminus\{y_0\}} \lambda_\Omega(x)\delta_\Omega(x) \log \delta_\Omega(x) > 0.$$

For $y \in B(y_0, r) \setminus \{y_0\}$ and a suitable small r, we get

$$\int_\Omega \big(\lambda_\Omega(x) g_\Omega(x,y)\big)^2 dx \geq (C(\Omega))^2 \int_{B(y_0,r)\setminus\{y_0\}} \left(\frac{g_{B(y_0,r)\setminus\{y_0\}}(x,y)}{|x-y_0| \log |x-y_0|}\right)^2 dx.$$

The last integral tends to $+\infty$ as y approaches y_0. This contradicts (4.8). Therefore the proof is complete. □

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