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COARSE COHOMOLOGY FOR FOLIATIONS: THE GENERAL CASE

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Abstract. We show how to extend coarse cohomology for foliations to non-Hausdorff foliations and compute several examples of Reeb type.

1. Introduction

Coarse cohomology for metric spaces was introduced by Roe [R1], [R2], [R3]. It evolved from the study of the index theory of geometric (Dirac-type) operators on complete open manifolds, and it has yielded new invariants beyond the usual index. In [HH], we extended this theory to parametrized families of metric spaces, the most important examples being foliations of compact manifolds. Unfortunately, in order to apply our general theory to foliations, we had to assume that the graph of the foliation was Hausdorff, a very strong assumption. In this paper we show how to remove this assumption and so extend the notion of coarse cohomology to all foliated manifolds. We also compute several important examples. In a subsequent paper, we will show how this cohomology theory naturally pairs with leafwise elliptic differential operators to yield new invariants for foliated manifolds. This gives a new and potentially important tool for understanding leafwise elliptic operators on foliated manifolds in general, not just those with Hausdorff graph.

Coarse cohomology for foliations combines the usual cohomology of the ambient manifold with the coarse cohomology of the holonomy covers of the leaves of the foliation. In particular, it is the cohomology of the differential forms on M with coefficients in the coarse de Rham cochains of the holonomy coverings of the leaves of F. When interpreted as a sheaf cohomology it yields a theory which is eminently computable. The results presented here essentially reduce the computation of this cohomology for any foliation to a good understanding of the *local* structure of the graph. Our main technical tool, the Controlled Poincaré Lemma, should have numerous and wide applications in the computation of particular examples. As one such application,

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we compute the coarse cohomology for several foliations (all of Reeb type) which have non-Hausdorff graphs.

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2. Coarse de Rham cohomology for foliations

Let F be a codimension q foliation of a compact n dimensional manifold M without boundary. The *holonomy groupoid* $\mathcal{G}_F$ of F consists of equivalence classes $y = [\gamma]$ of leafwise paths $\gamma : [0,1] \to M$. Two such leafwise paths γ_1 and γ_2 are equivalent provided $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the holonomy germ along the two paths is the same at $\gamma_1(0)$. There are natural maps $s, r : \mathcal{G}_F \to M$ defined by $s(y) = \gamma(0)$, $r(y) = \gamma(1)$. $\mathcal{G}_F$ is a (generally non-Hausdorff) $2n - q$ dimensional manifold with the local charts given as follows. Let U and V be foliation charts of $s(y)$ and $r(y)$, respectively, and choose $\gamma \in y$. Then the local chart (U, γ, V) consists of all equivalence classes of leafwise paths which start in U, end in V, and which are homotopic to γ through a homotopy of leafwise paths whose end points remain in U and V, respectively. It is easy to see that if $U, V \simeq \boldsymbol{R}^{n-q} \times \boldsymbol{R}^q$, then $(U, \gamma, V) \simeq \boldsymbol{R}^{n-q} \times \boldsymbol{R}^{n-q} \times \boldsymbol{R}^q$.

When we defined coarse cohomology for metric families in [HH], we assumed that the family of metrics satisfied a semi-continuity property, which holds for a foliation only if its graph is Hausdorff. We now give an alternate definition of the coarse de Rham cohomology for F, which for a foliation with Hausdorff graph agrees with that given in [HH], and which is natural for all foliations, whether or not their graphs are Hausdorff.

For each $x \in M$, $s^{-1}(x) \simeq \widetilde{L}_x$, the holonomy cover of the leaf L_x containing x. Denote by $\mathcal{G}_\ell$ the submanifold of $\times_\ell \mathcal{G}_F$ consisting of those points $(y_1, \ldots, y_\ell)$ with $s(y_1) = s(y_j)$ for $j = 2, \ldots, \ell$. We also denote by s the map $s : \mathcal{G}_\ell \to M$ given by $s(y_1, \ldots, y_\ell) = s(y_1)$. Note that here $s^{-1}(x) \simeq \times_\ell \widetilde{L}_x$. Choose a metric on M. This induces a metric on each leaf L and so also on $\widetilde{L}$ and $\times_\ell \widetilde{L}$, which makes them complete Riemannian manifolds. Their quasi-isometry types are independent of the choice of metric on M since M is compact. Denote the metric on $s^{-1}(x)$ by d_x, and note that for foliations with non-Hausdorff graph, this metric is not continuous in x. Given $A \subseteq \mathcal{G}_\ell$ and $r > 0$, define

$$\operatorname{Pen}(A, r) = \Big\{ (\hat{y}_1, \ldots, \hat{y}_\ell) \in \mathcal{G}_\ell \,|\, \exists\ (y_1, \ldots, y_\ell) \in A \text{ with } s(y_1) = s(\hat{y}_1) \text{ and } d_{s(y_1)}(y_i, \hat{y}_i) < r \text{ for } i = 1, \ldots, \ell \Big\}.$$

For each $x \in M$ let $*x$ be the equivalence class of the constant path at x, and $*M_\ell$ the subset of $\mathcal{G}_\ell$ given by

$$*M_\ell = \bigcup_{x \in M} (*x, \ldots, *x).$$

Denote by $A_c^{k,\ell}(F)$ the space of smooth (that is, smooth in each coordinate chart) k forms ω on $\mathcal{G}_{\ell+1}$ such that for all $r > 0$ there exists $R > 0$ so that

$$\sup(\omega) \cap \mathrm{Pen}(\Delta_{\ell+1}, r) \subseteq \mathrm{Pen}(*M_{\ell+1}, R),$$

where $\Delta_{\ell+1} \simeq \mathcal{G}_F$ is the thin diagonal of $\mathcal{G}_{\ell+1}$.

When we defined the de Rham coarse cohomology of a foliation in [HH], we defined $A_c^{k,\ell}(F)$ by requiring that $\sup(\omega) \cap \mathrm{Pen}(\Delta_{\ell+1}, r)$ be relatively compact. Note that when $\mathcal{G}_F$ is Hausdorff, these two notions agree. For Hausdorff spaces, relative compactness is reasonably easy to check, but it is not so easy for non-Hausdorff spaces. For the application we have in mind (namely pairing coarse cohomology with leafwise elliptic operators), the crucial property is not the relative compactness of $\sup(\omega) \cap \mathrm{Pen}(\Delta_{\ell+1}, r)$, but rather the fact that it has finite volume so that integration over this intersection of a bounded differential form produces a finite quantity.

We define two differentials on $A_c^{k,\ell}(F)$ as follows. The usual exterior derivative d does not increase supports, so it is clear that it maps $A_c^{k,\ell}(F)$ to $A_c^{k+1,\ell}(F)$. Define $\delta : A_c^{k,\ell}(F) \to A_c^{k,\ell+1}(F)$ by

$$\delta\omega = \sum_{j=1}^{\ell+2} (-1)^{j+1} \pi_j^* \omega,$$

where $\pi_j : \mathcal{G}_{\ell+2} \to \mathcal{G}_{\ell+1}$ deletes the jth entry. To see that δ actually does map $A_c^{k,\ell}(F)$ to $A_c^{k,\ell+1}(F)$ we have the following proposition whose proof is immediate.

Proposition 2.1. *If* $\sup(\omega) \cap \mathrm{Pen}(\Delta_{\ell+1}, r) \subseteq \mathrm{Pen}(*M_{\ell+1}, R)$, *then* $\sup(\pi_j^*\omega) \cap \mathrm{Pen}(\Delta_{\ell+2}, r) \subseteq \mathrm{Pen}(*M_{\ell+2}, 2R + 2r)$.

Definition 2.2. The coarse de Rham cohomology $HX^*(F)$ of F is the cohomology of the bicomplex $\{A_c^{*,*}(F), \delta, d\}$.

As in [H], this cohomology theory is a sheaf cohomology theory given as follows. The *coarse presheaf* L^* of F is the differential presheaf which associates to each open set $U \subset M$ and each non-negative integer q the space

$$L^q(U) = \sum_{k+\ell=q} A_c^{k,\ell}(U), \quad \text{where} \quad A_c^{k,\ell}(U) = \{\omega\,|_{s^{-1}(U)} \ \mid\ \omega \in A_c^{k,\ell}(F)\}.$$

The differential $D : L^q(U) \to L^{q+1}(U)$ is given by $D \,|\, A_c^{k,\ell}(U) = d + (-1)^k \delta$.

The *coarse sheaf* $\mathcal{L}^*$ of F is the differential sheaf associated to the differential presheaf L^*.

For each q, $\mathcal{L}^q$ is a fine sheaf, so the Čech bicomplex associated to $\mathcal{L}^*$ computes the coarse de Rham cohomology of F. Theorem 2.1 on p. 132 of [B] gives a spectral sequence which converges to $HX^*(F)$. Its E_2 term is

$$E_2^{p,q} = H^p(M; \mathcal{H}^q(\mathcal{L}^*)),$$

where $\mathcal{H}^q(\mathcal{L}^*)$ is the homology sheaf of the differential sheaf $\mathcal{L}^*$. In particular,

$$\mathcal{H}^q(\mathcal{L}^*) = \mathrm{Ker}(D : \mathcal{L}^q \to \mathcal{L}^{q+1}) / \mathrm{Im}(D : \mathcal{L}^{q-1} \to \mathcal{L}^q).$$

This reduces the computation of the coarse cohomology of a foliation to understanding the local structure of the map $s : \mathcal{G} \to M$ and the analysis of the spectral sequence given above.

3. Some Reeb type examples

In this section, we compute the coarse cohomology of several foliations of S^3 of Reeb type.

3.1. The Reeb foliation. The Reeb foliation is the foliation F of $S^3 = S^1 \times D^2 \cup S^1 \times D^2$, where each copy of $S^1 \times D^2$ is a Reeb component; see [MS, p. 41]. This foliation has a single compact leaf which is diffeomorphic to T^2, and all the other leaves are diffeomorphic to $\boldsymbol{R}^2$. For each $x \in T^2$, the holonomy cover $\widetilde{L}_x$ is quasi isometric to $\boldsymbol{R}^2$ with the usual metric. For $x \in M - T^2$, $\widetilde{L}_x$ is isometric to $\boldsymbol{R}^2$ with a metric making it coarsely equivalent to $[0, \infty)$ with the usual metric.

Theorem 3.1. *The coarse cohomology of the Reeb foliation is trivial, that is, $HX^*(F) = 0$.*

Here and in the succeeding examples, we will be computing the stalks of certain sheaves. In all the cases we consider the stalks will be either 0 or $\boldsymbol{R}$, and it will be easy to see that there is no twisting in any of the sheaves. Thus all of the sheaves we consider will be trivial $\boldsymbol{R}$ sheaves over subsets of S^3 extended by zero to all of S^3.

Recall (see [HH]) that a metric family $\mathcal{F} = \{G, d, \pi, U\}$ consists of a paracompact Hausdorff space G, a metric space U, a continuous map $\pi \colon G \to U$, and a family of fiberwise metrics $\{d_x \mid x \in U\}$ satisfying certain continuity and properness conditions. In [HH], we also defined the notion of coarsely equivalent metric families. Roughly speaking, two such families $\{G_1, d_1, \pi_1, U\}$ and $\{G_2, d_2, \pi_2, U\}$ with the same base space U are coarsely equivalent if there is a map $\phi : G_1 \to G_2$ covering the identity which induces a coarse equivalence on each of the fibers of the maps $\pi_j \colon G_j \to U$. Coarse cohomology is defined for metric families and coarsely equivalent families have the same coarse cohomology. These notions and results extend (using our new definition of coarse cohomology) to the metric families defined by taking U to be an open subset in any foliated manifold M (whether or not $\mathcal{G}_F$ is Hausdorff), G to be $s^{-1}(U) \subset \mathcal{G}_F$, $\pi = s$, and d_x the metric on $\widetilde{L}_x$ induced from a metric on M.

To prove Theorem 3.1 we need only show that for all x and q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$. There are two separate cases to consider, namely $x \in S^3 - T^2$, and $x \in T^2$.

Proposition 3.2. *For all $x \in S^3 - T^2$, and for all q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.*

Proof. Each $x \in S^3 - T^2$ has a neighborhood $U \subset S^3 - T^2$, $U \simeq D^3$, so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the product metric family $\{U \times [0,\infty), d^U, \pi, U\}$, where for each $u \in U$, d^U_u is the usual metric on $\{u\} \times [0,\infty)$. The coarse cohomology of $[0,\infty)$ is trivial, so the spectral sequence of [HH] gives that the coarse cohomology of $\{U \times [0,\infty), d^U, \pi, U\}$ is also trivial. Thus the coarse cohomology of $\{s^{-1}(U), d, s, U\}$ is trivial, so for all $x \in S^3 - T^2$ and all q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$. □

We now compute $\mathcal{H}^q(\mathcal{L}^*)_x$ for $x \in T^2$. Each neighborhood of $x \in T^2$ contains a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the metric family $\mathcal{F}_x = \{G, d^U, \pi, U\}$, where $G = \widehat{G}/\boldsymbol{Z}$, and $\widehat{G} \subset D^3 \times \boldsymbol{R}^2$ is the set

$$\widehat{G} = \{(u_1,u_2,u_3,t_1,t_2) \,|\, \text{if } u_3 \geq 0,\ t_1 > -u_3^{-2},\ \text{and if } u_3 \leq 0,\ t_2 > -u_3^{-2}\}.$$

We use the convention that if $u_3 = 0$, then $-u_3^{-2} = -\infty$. The action of $\boldsymbol{Z}$ is generated by:

$$\begin{aligned}&\text{if } u_3 > 0,\ \text{then } (u_1,u_2,u_3,t_1,t_2) \to (u_1,u_2,u_3,t_1,t_2+1);\\ &\text{if } u_3 < 0,\ \text{then } (u_1,u_2,u_3,t_1,t_2) \to (u_1,u_2,u_3,t_1+1,t_2).\end{aligned}$$

The map $\pi : G \to U$ is the natural projection and for each $u \in U$ the metric d^U_u on $\pi^{-1}(u)$ is induced from the restriction of the natural metric on $\{u\}\times\boldsymbol{R}^2$. If u has $u_3 \neq 0$, then $\pi^{-1}(u) \simeq S^1 \times (-u_3^{-2},\infty)$ with the usual metric, while if $u_3 = 0$, $\pi^{-1}(u) \simeq \boldsymbol{R}^2$ with the usual metric. Note that the set $U \cap T^2$ equals $\{u \,|\, u_3 = 0\}$.

Denote by $\widehat{G}_\ell \subset U \times \boldsymbol{R}^{2\ell}$ the set

$$\widehat{G}_\ell = \{(u, t_1, \dots, t_{2\ell}) \,|\, \text{if } u_3 \geq 0,\ t_{2j+1} > -u_3^{-2};\ \text{if } u_3 \leq 0,\ t_{2j} > -u_3^{-2}\},$$

and set

$$G_\ell = \widehat{G}_\ell/\boldsymbol{Z}^\ell,$$

where the action of $\boldsymbol{Z}^\ell$ is the natural extension of the action of $\boldsymbol{Z}$ above. As above, we have the natural projection $\pi : G_\ell \to U$. For each $u \in U$ with $u_3 \neq 0$, $\pi^{-1}(u) \simeq \times_\ell(S^1 \times (u_3^{-2},\infty))$ with the usual metric, and for each $u \in U$ with $u_3 = 0$, $\pi^{-1}(u) \simeq (\boldsymbol{R}^2)^\ell$ with the usual metric.

Denote by $\Delta_\ell \simeq G$ the thin diagonal of G_ℓ, and by $*U_\ell \subset G_\ell$ the subset given by the image in G_ℓ of the set $U \times \{(0.\dots,0)\} \subset U \times \boldsymbol{R}^{2\ell}$. Then the coarse cohomology of the metric family $\mathcal{F}_x$ may be computed just as the de Rham coarse cohomology for a foliation is computed, with G_ℓ substituted for $\mathcal{G}_\ell$. In particular, we have the spaces $A_c^{k,\ell}(\mathcal{F}_x)$ of k forms ω on the space $G_{\ell+1}$ such that for all $r > 0$, there exists $R > 0$ so that

$$\sup(\omega) \cap \operatorname{Pen}(\Delta_{\ell+1}, r) \subseteq \operatorname{Pen}(*U_{\ell+1}, R).$$

The differentials d and δ are as above, and the coarse cohomology of $\mathcal{F}_x$ is the cohomology of the bicomplex $\{A_c^{*,*}(\mathcal{F}_x), \delta, d\}$.

PROPOSITION 3.3. *For all $x \in T^2$, and for all q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.*

Proof. $HX^0(\mathcal{F}_x)$ consists of constant functions on G_1 with compact support. As G_1 is not compact, $HX^0(\mathcal{F}_x) = 0$, which implies $\mathcal{H}^0(\mathcal{L}^*)_x = 0$.

To compute the higher coarse cohomology of the metric family $\mathcal{F}_x$, we need the following lemmas.

LEMMA 3.4. *For all $\omega \in A_c^{k,\ell}(\mathcal{F}_x)$, $\sup(\omega) \cap \pi^{-1}(U \cap T^2) = \emptyset$.*

Proof. The $\boldsymbol{Z}^{\ell+1}$ action on $\widehat{G}_{\ell+1}$ induces two different actions of $\boldsymbol{Z}^{\ell+1}$ on $\pi^{-1}(T^2) \simeq T^2 \times \boldsymbol{R}^{2\ell+2}$, one being the natural action on the odd coordinates of $\boldsymbol{R}^{2\ell+2}$, called the odd action, and the other being the natural action on the even coordinates. Suppose there is $z \in \sup(\omega) \cap \pi^{-1}(T^2)$. Then there is a sequence, $\{z_n\}$, in $G_{\ell+1}$, which converges to z with $\omega(z_n) \neq 0$, and we may assume without loss of generality that $u_3(z_n) < 0$ for all n. Then the sequence $\{z_n\}$ also converges to all the points in the orbit of z under the odd action of $\boldsymbol{Z}^{\ell+1}$. Let the coordinates of z be $z = (y, (t_1, t_2), \ldots, (t_{2\ell+1}, t_{2\ell+2}))$. Now z is in some bounded neighborhood of the diagonal in $y \times (\boldsymbol{R}^2)^{\ell+1}$, so there is $r > 0$ so that for all $a, b \in \{1, 2, \ldots, \ell+1\}$, $d_y^U((t_{2a-1}, t_{2a}), (t_{2b-1}, t_{2b})) < r$. But then the sequence of points $(y, (t_1 + n, t_2), \ldots, (t_{2\ell+1} + n, t_{2\ell+2}))$ also satisfies $d_y^U((t_{2a-1} + n, t_{2a}), (t_{2b-1} + n, t_{2b})) < r$, so it is also in the same bounded neighborhood of the diagonal. This sequence is in the orbit of z under the odd action, so it is also in $\sup(\omega) \cap \pi^{-1}(T^2)$. As this sequence increases without bound as $n \to \infty$, the support condition is violated. □

Denote by $\widehat{d} : A_c^{k,\ell}(\mathcal{F}_x) \to A_c^{k+1,\ell}(\mathcal{F}_x)$ the exterior derivative with respect to the u coordinates only. Strictly speaking this does not make sense. However, we may pull back any form ω on $G_{\ell+1}$ to $\widehat{G}_{\ell+1}$ by the natural projection $\rho : \widehat{G}_{\ell+1} \to G_{\ell+1}$, and $\widehat{d}(\rho^*(\omega))$ does make sense. As $\widehat{d}(\rho^*(\omega))$ is invariant under the action of $\boldsymbol{Z}^{\ell+1}$, it induces a well defined form on $G_{\ell+1}$. Define this to be $\widehat{d}(\omega)$.

LEMMA 3.5 (Controlled Poincaré Lemma). *Suppose $x \in T^2$, $k > 0$, and $\omega_{k,\ell} \in A_c^{k,\ell}(\mathcal{F}_x)$. Assume that each term of $\rho^*(\omega_{k,\ell})$ contains at least one du_i and that $\widehat{d}(\omega_{k,\ell}) = 0$. Then there is $\omega_{k-1,\ell} \in A_c^{k-1,\ell}(\mathcal{F}_x)$ such that $\widehat{d}(\omega_{k-1,\ell}) = \omega_{k,\ell}$.*

Proof. We will work on the space $\widehat{G}_{\ell+1}$ using objects which are invariant under the action of $\boldsymbol{Z}^{\ell+1}$. Let $\omega \in A_c^{k-1,\ell}(\mathcal{F}_x)$ and suppose that $\rho^*(\omega)$ on $\widehat{G}_{\ell+1}$ is a monomial

$$\rho^*(\omega) = f(u,t)du_{i_1} \wedge \ldots \wedge du_{i_r} \wedge dt_{j_1} \wedge \ldots \wedge dt_{j_{k-(r+1)}}.$$

Set

$$\phi(\rho^*(\omega))(u,t) = \left(\int_0^1 s^{r-1} f(su,t)ds\right) du_{i_1} \wedge \ldots \wedge du_{i_r} \wedge dt_{j_1} \wedge \ldots \wedge dt_{j_{k-(r+1)}}.$$

Let i_X be interior multiplication by the vector field $u_1\partial/\partial u_1 + u_2\partial/\partial u_2 + u_3\partial/\partial u_3$ on $\widehat{G}_{\ell+1}$. Then the proof of the usual Poincaré lemma (see [W, p. 155]) shows that the form $\widehat{\omega}_{k-1,\ell} = \phi(i_X\rho^*(\omega_{k,\ell}))$ satisfies $\widehat{d}(\widehat{\omega}_{k-1,\ell}) = \rho^*(\omega_{k,\ell})$. In addition, $\widehat{\omega}_{k-1,\ell}$ is invariant by the action of $\boldsymbol{Z}^{\ell+1}$, so it induces a well defined form $\omega_{k-1,\ell}$ on $G_{\ell+1}$ and $\widehat{d}(\omega_{k-1,\ell}) = \omega_{k,\ell}$.

It remains to check that $\omega_{k-1,\ell}$ satisfies the support condition required of an element of $A_c^{k-1,\ell}(\mathcal{F}_x)$. First, note that for all $0 < s \leq 1$,

$$(u,t) \in \rho^{-1}(\mathrm{Pen}(\Delta_{\ell+1}, r)) \Leftrightarrow (su,t) \in \rho^{-1}(\mathrm{Pen}(\Delta_{\ell+1}, r))$$

and

$$(u,t) \in \rho^{-1}(\mathrm{Pen}(*M_{\ell+1}, R)) \Leftrightarrow (su,t) \in \rho^{-1}(\mathrm{Pen}(*M_{\ell+1}, R)).$$

Now suppose that $\rho(u,t) \in \sup(\omega_{k-1,\ell}) \cap \mathrm{Pen}(\Delta_{\ell+1}, r)$. It follows immediately that $(u,t) \in \sup(\widehat{\omega}_{k-1,\ell}) \cap \rho^{-1}(\mathrm{Pen}(\Delta_{\ell+1}, r))$. This implies that there is $0 < s \leq 1$ so that $(su,t) \in \sup(\rho^*\omega_{k,\ell}) \cap \rho^{-1}(\mathrm{Pen}(\Delta_{\ell+1}, r))$, for if not $\widehat{\omega}_{k-1,\ell}$ would be zero on a neighborhood of (u,t). Because $\sup(\rho^*\omega_{k,\ell}) \cap \rho^{-1}(\mathrm{Pen}(\Delta_{\ell+1}, r)) = \rho^{-1}(\sup(\omega_{k,\ell}) \cap \mathrm{Pen}(\Delta_{\ell+1}, r))$ and $\omega_{k,\ell}$ satisfies the support condition, we may choose $R > 0$, so that $(su,t) \in \rho^{-1}(\mathrm{Pen}(*M_{\ell+1}, R))$. Thus we have that $(u,t) \in \rho^{-1}(\mathrm{Pen}(*M_{\ell+1}, R))$, so $\rho(u,t) \in \mathrm{Pen}(*M_{\ell+1}, R)$ and the support condition is satisfied.

Finally, extend this construction to general $\omega_{k,\ell}$ by linearity. □

Let $\alpha = [\sum_{i+j=k} \omega_{i,j}]$ be a $k > 0$ dimensional coarse class for $\mathcal{F}_x$, where $\omega_{i,j} \in A_c^{i,j}(\mathcal{F}_x)$. We first show that we may assume the $\omega_{i,j}$ are independent of u. Since $D(\sum_{i+j=k} \omega_{i,j}) = 0$, $d\omega_{k,0} = 0$. Write $\omega_{k,0} = \omega_u + \omega_t$, where ω_u is the part of $\omega_{k,0}$ of highest u form degree. (Strictly speaking, we should say this about their pullbacks by ρ. We will no longer make this distinction.) The fact that $d(\omega_{k,0}) = 0$ implies that $\widehat{d}(\omega_u) = 0$. The Controlled Poincaré Lemma gives a $k-1$ form $\omega_{k-1,0} \in A_c^{k-1,0}(\mathcal{F}_x)$ so that $\widehat{d}(\omega_{k-1,0}) = \omega_u$. As $\alpha = [\sum_{i+j=k} \omega_{i,j} - D(\omega_{k-1,0})]$, we may replace $\sum_{i+j=k} \omega_{i,j}$ by $\sum_{i+j=k} \omega_{i,j} - D(\omega_{k-1,0})$. This has the effect of reducing the highest u form degree of $\omega_{k,0}$ by 1. By induction, we may assume that the u form degree of $\omega_{k,0}$ is 0. The fact that $d\omega_{k,0} = 0$ combined with the fact that $\omega_{k,0}$ contains no du_i's means that $\omega_{k,0}$ is completely independent of u.

By induction, we have a representative $\sum_{i+j=k}\omega_{i,j}$ of α which has $\omega_{i,j}$ completely independent of u for $i > r$. The fact that $D(\sum_{i+j=k}\omega_{i,j}) = 0$ implies that $d\omega_{r,s} = \pm\delta\omega_{r+1,s-1}$, which is entirely independent of u. This implies that $\widehat{d}\omega_{r,s} = 0$. Proceed exactly as above to replace $\sum_{i+j=k}\omega_{i,j}$ by a new representative of α with $\omega_{i,j}$ completely independent of u for $i \geq r$. Thus we may assume that $\sum_{i+j=k}\omega_{i,j}$ is completely independent of u. But on $\pi^{-1}(U \cap T^2)$, $\sum_{i+j=k}\omega_{i,j} = 0$, so $\sum_{i+j=k}\omega_{i,j} = 0$, and $\alpha = 0$.

This completes the proof of Theorem 3.1. □

We now compute the coarse cohomology for three other foliations derived from the Reeb foliation. We do this by replacing the compact leaf T^2 by $T^2 \times I$ foliated in three different ways: first by a foliation with all leaves diffeomorphic to T^2; second by foliations with all leaves diffeomorphic to $S^1 \times \boldsymbol{R}$; and third by foliations with all leaves diffeomorphic to $\boldsymbol{R}^2$. These will be called the T^2 Reeb foliation, the $S^1 \times \boldsymbol{R}$ Reeb foliations, and the $\boldsymbol{R}^2$ Reeb foliations, respectively.

3.2. The T^2 Reeb foliation. To obtain this foliation, replace the T^2 leaf of the Reeb foliation by a family of such leaves, i.e., by the manifold $T^2 \times [0,1]$ foliated by the fibers of the fibration $T^2 \times [0,1] \to [0,1]$. Any point in the interior of $T^2 \times [0,1]$ has a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the product metric family $\{U \times T^2, d^U, \pi, U\}$, where the metric on each $\{u\} \times T^2$ is the usual metric. As the fibers of the projection are compact, this family is coarsely equivalent to the trivial metric family $\{U, d, \pi, U\}$, with points as fibers. The coarse cohomology of a point is just its usual cohomology, so the spectral sequence of [HH] gives:

Proposition 3.6. *For all $x \in T^2 \times (0,1)$, $\mathcal{H}^0(\mathcal{L}^*)_x = \boldsymbol{R}$; otherwise $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.*

The computation of $\mathcal{H}^q(\mathcal{L}^*)_x$ for $x \in S^3 - (T^2 \times [0,1])$ is the same as in the Reeb case so we have:

Proposition 3.7. *For all $x \in S^3 - (T^2 \times [0,1])$, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$ for all q.*

For x in the boundary of $T^2 \times [0,1]$, we have:

Proposition 3.8. *For all $x \in \partial(T^2 \times [0,1])$, and for all q, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.*

Proof. Any such x has a neighborhood $U \simeq D^3$ with the metric family $\{s^{-1}(U), d, s, U\}$ coarsely equivalent to the metric family $\mathcal{F}_x = \{G, d^U, \pi, U\}$, where $G = \widehat{G}/\boldsymbol{Z}^2$, and $\widehat{G} \subset D^3 \times \boldsymbol{R}^2$ is the set

$$\widehat{G} = \{(u_1, u_2, u_3, t_1, t_2) \,|\, \text{if } u_3 \geq 0,\ t_1 > -u_3^{-2}\}.$$

We continue to use the convention that if $u_3 = 0$, then $-u_3^{-2} = -\infty$. The action of $\boldsymbol{Z}^2$ is given by:

$$\begin{aligned} &\text{if } u_3 \geq 0, \text{ then } (a,b)((u_1,u_2,u_3,t_1,t_2)) = (u_1,u_2,u_3,t_1,t_2+b);\\ &\text{if } u_3 < 0, \text{ then } (a,b)((u_1,u_2,u_3,t_1,t_2)) = (u_1,u_2,u_3,t_1+a,t_2+b). \end{aligned}$$

The map $\pi : G \to U$ is the natural projection. For each $u \in U$ with $u_3 \geq 0$, $\pi^{-1}(u) \simeq S^1 \times (-u_3^{-2}, \infty)$ with the usual metric, and for each $u \in U$ with $u_3 < 0$, $\pi^{-1}(u) \simeq T^2$ with the usual metric. Finally note that $U \cap \partial(T^2 \times [0,1]) = \{u \mid u_3 = 0\}$.

Since G_1 is not compact, we have as above that $\mathcal{H}^0(\mathcal{L}^*)_x = 0$.

Lemma 3.9. *Any $\omega \in A_c^{k,\ell}(\mathcal{F}_x)$ has C^∞ contact with 0 on the set $\pi^{-1}(U \cap \partial(T^2 \times [0,1]))$.*

Proof. If not, there is a sequence $\{z_n\}$ with $u_3(z_n) < 0$ and $\omega(z_n) \neq 0$. Proceeding as in the proof of Lemma 3.4, we have that the support condition is violated. □

Thus any coarse class α may be written as $\alpha = \alpha_1 + \alpha_2$, where α_1 has a representative which is zero for $u_3 \leq 0$ and α_2 has a representative which is zero for $u_3 \geq 0$.

Suppose $\sum_{i+j=k} \omega_{i,j}$ is a representative of α_1 which is zero for $u_3 \leq 0$. The argument for the Reeb foliation that $\sum_{i+j=k} \omega_{i,j}$ may be assumed to be entirely independent of u works equally well here, and so $\sum_{i+j=k} \omega_{i,j} = 0$ and $\alpha_1 = 0$.

Now suppose $\sum_{i+j=k} \omega_{i,j}$ represents α_2 and is zero for $u_3 \geq 0$. If the support of $\sum_{i+j=k} \omega_{i,j}$ intersects $\pi^{-1}(U \cap \partial(T^2 \times [0,1]))$ non-trivially, then the support condition will be violated by the same argument as in the Reeb case. If the support of $\sum_{i+j=k} \omega_{i,j}$ does not intersect $\pi^{-1}(U \cap \partial(T^2 \times [0,1]))$ non-trivially, then because of the compactness of T^2 there is $\epsilon < 0$ so that for all z with $u_3 > \epsilon$, $\sum_{i+j=k} \omega_{i,j}(z) = 0$. As we are computing the sheaf cohomology at x, the coarse class determined by $\sum_{i+j=k} \omega_{i,j}$ is the same as that determined by its restriction to $u_3 > \epsilon$. Thus we may suppose that $\sum_{i+j=k} \omega_{i,j} = 0$ so $\alpha_2 = 0$ also and $\alpha = 0$. □

Given any subset $A \subset S^3$, denote by $\mathcal{R}_A$ the trivial $\boldsymbol{R}$ sheaf over A extended by 0 to all of S^3. Combining Propositions 3.6, 3.7, and 3.8 we have:

Proposition 3.10. *For the T^2 Reeb foliation, $\mathcal{H}^0(\mathcal{L}) = \mathcal{R}_{T^2 \times (0,1)}$; otherwise $\mathcal{H}^q(\mathcal{L}) = \mathbf{0}$.*

Now the spectral sequence of [B, Theorem 2.1, p. 132], applied to this foliation, has E_2 term

$$E_2^{p,q} = H^p(S^3; \mathcal{H}^q(\mathcal{L}^*)) = 0 \text{ for all } q > 0;$$
$$E_2^{p,0} = H^p(S^3; \mathcal{H}^0(\mathcal{L}^*)) = H^p(S^3; \mathcal{R}_{T^2\times(0,1)}).$$

Thus the coarse cohomology of the T^2 Reeb foliation is

$$HX^*(F) = H^*(S^3; \mathcal{R}_{T^2\times(0,1)}).$$

Now consider the short exact sequence $\mathcal{R}_{T^2\times(0,1)} \to \mathcal{R}_{S^3} \to \mathcal{R}_B$, where B is the complement of the open subset $T^2 \times (0,1)$. Since B is closed, the map $H^*(S^3; \mathcal{R}_{S^3}) \to H^*(S^3; \mathcal{R}_B)$ in the associated long exact sequence is the usual restriction homomorphism $H^*(S^3; \boldsymbol{R}) \to H^*(B; \boldsymbol{R})$. Thus $H^*(S^3; \mathcal{R}_{T^2\times(0,1)})$ is the same as the third term of the long exact sequence for the inclusion $B \to S^3$, namely the relative cohomology $H^*(S^3, B; \boldsymbol{R}) \simeq H^{*-1}(T^2; \boldsymbol{R})$, and we have:

THEOREM 3.11. *For the T^2 Reeb foliation of S^3, $HX^{*+1}(F) \simeq H^*(T^2; \boldsymbol{R})$.*

An isomorphism is given as follows. Choose a smooth non-negative function $\phi(x)$ on $(0,1)$ which is positive on some subset of the interior and zero on a neighborhood of the boundary. On $T^2 \times (0,1)$ we have coordinates (θ_1, θ_2, x). Over $T^2 \times (0,1)$, $\mathcal{G}_1 \simeq T^2 \times (0,1) \times T^2$, with coordinates $(\theta_1, \theta_2, x, t_1, t_2)$. Let ξ be a differential form on T^2 representing a class in $H^k(T^2; \boldsymbol{R})$ and consider the form $s^*(\phi\,\xi\,dx) \in A^{k+1,0}(F)$, which we extend to all of $\mathcal{G}_1$ by defining it to be zero off $s^{-1}(T^2 \times (0,1))$. Then $d(s^*(\phi\,\xi\,dx)) = 0$, and since $s^*(\phi\,\xi\,dx)$ is independent of (t_1, t_2), $\delta(s^*(\phi\,\xi\,dx)) = 0$ also, so $D(s^*(\phi\,\xi\,dx)) = 0$. The isomorphism we seek is $A([\xi]) = [s^*(\phi\,\xi\,dx)]$, where [ ] indicates taking cohomology classes.

Note that if $[\xi] \in H^k(T^2; \boldsymbol{R})$, where $k = 0, 1$, there is a k-form β on S^3 so that $d(\beta) = \phi\,\xi\,dx$. Consider the form $s^*(\beta)$ on $\mathcal{G}_1$. It satisfies $d(s^*(\beta)) = s^*(\phi\,\xi\,dx)$. In addition, as $s^*(\beta)$ is independent of (t_1, t_2), it follows that $\delta(s^*(\beta)) = 0$. Thus $D(s^*(\beta)) = s^*(\phi\,\xi\,dx)$. This does not contradict the fact that A is an isomorphism, since the form $s^*(\beta)$ in not in $A_c^{k,0}(F)$ as it will not satisfy the support condition off $T^2 \times (0,1)$, where the leaves of F are not compact.

3.3. The $S^1 \times \boldsymbol{R}$ Reeb foliations. To obtain these foliations, replace the T^2 leaf of the Reeb foliation by $T^2 \times [0,1]$ with a foliation which has the two boundary components as leaves and the interior foliated by leaves diffeomorphic to $S^1 \times \boldsymbol{R}$. There are two distinct ways to foliate $S^1 \times \boldsymbol{R}$ in this way. The first is obtained by foliating the strip $\{(x,y) \in \boldsymbol{R}^2 \mid 0 \le y \le 1\}$ by horizontal lines, taking the product of this with S^1, and then identifying (x, y, θ) with $(x+1, \varphi(y), \theta)$. The map $\varphi : [0,1] \to [0,1]$ is a diffeomorphism so that on $(0,1)$, $\varphi(y) > y$. The second way is to foliate $\{(x,y) \in \boldsymbol{R}^2 \mid 0 \le y \le 1\}$

by copies of $\boldsymbol{R}$ so that the boundary components are leaves, and the leaves in the interior have both their ends at $x = +\infty$; see the picture at the top of p. 41 of [MS]. We assume that the foliation is invariant under translations in the x variable. Take the product of this foliation with S^1 and identify (x, y, θ) with $(x+1, y, \theta)$ to obtain a foliation of $T^2 \times [0,1]$. The coarse cohomology of the foliations we construct below does not depend on which of these two foliations we use.

For $x \notin \partial(T^2 \times [0,1])$, we have:

Proposition 3.12. *If $x \in S^3 - (T^2 \times [0,1])$, then for all q, $\mathcal{H}^q(\mathcal{L})_x = 0$. If $x \in T^2 \times (0,1)$, then $\mathcal{H}^1(\mathcal{L})_x = \boldsymbol{R}$; otherwise $\mathcal{H}^q(\mathcal{L})_x = 0$.*

Proof. The first statement follows from the proof in the Reeb case. For the second, note that any point $x \in T^2 \times (0,1)$ has a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the product metric family $\mathcal{F}_x = \{U \times \boldsymbol{R}, d^U, \pi, U\}$, where the metric on each $\{u\} \times \boldsymbol{R}$ is the usual metric. This is because $s^{-1}(U) \simeq U \times (\boldsymbol{R} \times S^1)$ and each $\{u\} \times \boldsymbol{R} \times S^1$ has a metric coarsely equivalent to the usual metric, which makes it coarsely equivalent to $\boldsymbol{R}$ with the usual metric. $HX^1(\boldsymbol{R}) = \boldsymbol{R}$ and is 0 otherwise, so the spectral sequence of [HH] gives that $HX^1(\mathcal{F}_x) = \boldsymbol{R}$ and is 0 otherwise. Thus for $x \in T^2 \times (0,1)$, $\mathcal{H}^1(\mathcal{L})_x = \boldsymbol{R}$ and is 0 otherwise. □

We now compute $\mathcal{H}^q(\mathcal{L})_x$ for $x \in \partial(T^2 \times [0,1])$. There are infinitely many ways we can glue $T^2 \times [0,1]$ to the two copies of $S^1 \times D^2$ to obtain S^3. In fact, by using all possible gluings, we obtain foliations of all three manifolds which can be obtained from S^3 by Dehn surgery on the unknot. The computations here can be adapted to give the coarse cohomologies for all these foliations. We leave the details to the reader.

Let ϕ be the generator of $\pi_1(T^2)$ which has trivial holonomy for the $S^1 \times \boldsymbol{R}$ foliation of $T^2 \times [0,1]$. Let α and β be generators of $\pi_1(T^2)$ so that β has trivial holonomy for the Reeb foliation of $S^1 \times D^2$. Suppose the gluing map takes ϕ to $a\alpha + b\beta$. If $a = 0$, there is no one-sided holonomy, so the graph of the foliation is Hausdorff at all points in this component of $\partial(T^2 \times [0,1])$. On the other hand, if $a \neq 0$, the graph of the foliation is non-Hausdorff at all points in this component of $\partial(T^2 \times [0,1])$.

3.3.1. *The Hausdorff case:* $a = 0$. Any such x has a neighborhood $U \simeq D^3$, with the associated metric family $\{s^{-1}(U), d, s, U\}$ coarsely equivalent to the metric family $\{G, d^U, \pi, U\}$, $G = \widehat{G}/\boldsymbol{Z}$, and $\widehat{G} \subset D^3 \times \boldsymbol{R}^2$ is the set

$$\widehat{G} = \{(u_1, u_2, u_3, t_1, t_2) \mid \text{if } u_3 > 0,\ t_1 > -u_3^{-2}\}.$$

The action of $\boldsymbol{Z}$ is generated by:

$$(u_1, u_2, u_3, t_1, t_2) \to (u_1, u_2, u_3, t_1, t_2 + 1).$$

The map $\pi : G \to U$ is the natural projection. For each $u \in U$ with $u_3 > 0$, $\pi^{-1}(u) \simeq S^1 \times (-u_3^{-2}, \infty)$ with the usual metric, and for each $u \in U$ with $u_3 \le 0$, $\pi^{-1}(u) \simeq S^1 \times \boldsymbol{R}$ with the usual metric. This in turn is coarsely equivalent to the metric family $\{G_1, d^U, \pi, U\}$, where $G_1 \subset U \times \boldsymbol{R}$,

$$G_1 = \{(u_1, u_2, u_3, t) \,|\, \text{if } u_3 > 0,\ t > -u_3^{-2}\},$$

the projection is the natural one, and the fibers have the metric induced from $U \times \boldsymbol{R}$. In [H], we computed the coarse cohomology of the Double Reeb foliation of $S^1 \times S^2$ and in so doing showed that for all x in the single compact leaf ($= T^2$), $\mathcal{H}^1(\mathcal{L})_x = \boldsymbol{R}$, and for all other q, $\mathcal{H}^q(\mathcal{L})_x = 0$. The proof there works here also, so we have:

Proposition 3.13. *If x is in a component of $\partial(T^2 \times [0,1])$ where the graph is Hausdorff, $\mathcal{H}^1(\mathcal{L})_x = \boldsymbol{R}$; otherwise $\mathcal{H}^q(\mathcal{L})_x = 0$.*

Note that if one of the gluing maps satisfies $a = 0$, the other cannot satisfy this condition and still have the resulting manifold be S^3. This is because the composition of the two gluing maps must send β in the first $\partial(S^1 \times \boldsymbol{D}^2)$ to $\pm(\alpha + n\beta)$ in the second $\partial(S^1 \times \boldsymbol{D}^2)$, where $\alpha + n\beta$ is contractible in the second $S^1 \times \boldsymbol{D}^2$ if the resulting manifold is S^3. Thus the second map must sent ϕ to $\pm(\alpha + n\beta)$. So for $S^1 \times \boldsymbol{R}$ foliations of S^3 it is only possible to have one component of $\partial(T^2 \times [0,1])$ over which the graph is Hausdorff. We call such foliations half-Hausdorff $S^1 \times \boldsymbol{R}$ foliations of S^3. Foliations where the graph is non-Hausdorff on both components of $\partial(T^2 \times [0,1])$ will be called non-Hausdorff $S^1 \times \boldsymbol{R}$ foliations of S^3.

3.3.2. *The non-Hausdorff case: $a \neq 0$.* Here x has a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the metric family $\mathcal{F}_x = \{G, d^U, \pi, U\}$, where $G = \widehat{G}/\boldsymbol{Z}$, and $\widehat{G} \subset D^3 \times \boldsymbol{R}^2$ is the set

$$\widehat{G} = \{(u_1, u_2, u_3, t_1, t_2) \,|\, \text{if } u_3 > 0,\ t_1 > -u_3^{-2}\}.$$

The action of $\boldsymbol{Z}$ is generated by:

$$\text{if } u_3 > 0, \text{ then } (u_1, u_2, u_3, t_1, t_2) \to (u_1, u_2, u_3, t_1, t_2 + 1);$$
$$\text{if } u_3 < 0, \text{ then } (u_1, u_2, u_3, t_1, t_2) \to (u_1, u_2, u_3, t_1 + a, t_2 + b).$$

The map $\pi : G \to U$ is the natural projection. For each $u \in U$ with $u_3 > 0$, $\pi^{-1}(u) \simeq S^1 \times (-u_3^{-2}, \infty)$ with the usual metric; for each u with $u_3 < 0$, $\pi^{-1}(u) \simeq S^1 \times \boldsymbol{R}$ with the usual metric; and for each $u \in U$ with $u_3 = 0$, $\pi^{-1}(u) \simeq \boldsymbol{R}^2$ with the usual metric. This is essentially the same situation we encountered in computing $\mathcal{H}^q(\mathcal{L})_x$ for $x \in T^2$ for the Reeb foliation, and the result is the same.

Proposition 3.14. *If x is in a component of $\partial(T^2 \times [0,1])$, where the graph is non-Hausdorff, $\mathcal{H}^q(\mathcal{L})_x = 0$ for all q.*

Combining Propositions 3.12, 3.13, and 3.14 we have:

PROPOSITION 3.15. *For any half-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3, $\mathcal{H}^1(\mathcal{L}) = \mathcal{R}_{T^2\times[0,1)}$; otherwise $\mathcal{H}^q(\mathcal{L}) = \mathbf{0}$.*

For any non-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3, $\mathcal{H}^1(\mathcal{L}) = \mathcal{R}_{T^2\times(0,1)}$; otherwise $\mathcal{H}^q(\mathcal{L}) = \mathbf{0}$.

Applying the spectral sequence as above, we get that the coarse cohomology of any half-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3 is

$$HX^*(F) = H^{*-1}(S^3; \mathcal{R}_{T^2\times[0,1)}),$$

and for any non-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3 it is

$$HX^*(F) = H^{*-1}(S^3; \mathcal{R}_{T^2\times(0,1)}).$$

To compute $H^*(S^3; \mathcal{R}_{T^2\times[0,1)})$, write S^3 as the disjoint union $S^3 = B_0 \cup T^2 \times [0,1) \cup B_1$, where $B_1 \simeq D^2 \times S^1$, $B_0 \simeq D_0^2 \times S^1$, and D_0^2 is the interior of the closed two disc D^2. Denote the complement of B_0 by CB_0 and consider the short exact sequence $\mathcal{R}_{T^2\times[0,1)} \to \mathcal{R}_{CB_0} \to \mathcal{R}_{B_1}$. As both B_1 and CB_0 are closed, $H^*(S^3; \mathcal{R}_{CB_0}) \to H^*(S^3; \mathcal{R}_{B_1})$ in the associated long exact sequence is just the restriction homomorphism $H^*(CB_0; \boldsymbol{R}) \to H^*(B_1; \boldsymbol{R})$, which is an isomorphism. Thus, $H^*(S^3; \mathcal{R}_{T^2\times[0,1)}) = 0$. Combining this with the computation for $H^*(S^3; \mathcal{R}_{T^2\times(0,1)})$ above we have:

THEOREM 3.16. *If F is a half-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3, then $HX^*(F) = 0$.*

If F is a non-Hausdorff $S^1 \times \boldsymbol{R}$ Reeb foliation of S^3, then $HX^{*+2}(F) \simeq H^*(T^2; \boldsymbol{R})$.

Representatives of the classes determined by the elements of $H^*(T^2; \boldsymbol{R})$ are given as follows. The coarse cohomology of $\boldsymbol{R}$ is non-zero only in dimension 1, where it is isomorphic to $\boldsymbol{R}$. A generator (in the extended coarse cohomology of $\boldsymbol{R}$) is given by $[\omega_{1,0} + \omega_{0,1}]$, where $\omega_{i,j} \in A_c^{i,j}(\boldsymbol{R})$, and

$$\omega_{1,0}(t) = \psi(t)dt, \quad \omega_{0,1}(t_1, t_2) = \int_{t_1}^{t_2} \psi(z)dz,$$

where $\psi(t)$ is any non-negative smooth function on $\boldsymbol{R}$ with compact support and positive total integral. Let ξ represent a class in $H^k(T^2; \boldsymbol{R})$, and let ϕ and dx be as in the T^2 Reeb foliation. Then $s^*(\phi\xi dx)\omega_{1,0}$ defines an element of $A_c^{k+2,0}$ over $T^2 \times (0,1)$ which is zero in a neighborhood of the boundary, so extending it by zero defines a class in $A_c^{k+2,0}$ over all of S^3. Similarly, $s^*(\phi\xi dx)\omega_{0,1}$ defines an element of $A_c^{k+1,1}$ over $T^2 \times (0,1)$ which may be extended by zero to all of S^3. The coarse class corresponding to the class of ξ is then the class of $s^*(\phi\xi dx)\omega_{1,0} + s^*(\phi\xi dx)\omega_{0,1}$.

3.4. The $\boldsymbol{R}^2$ Reeb foliations. To obtain these foliations, replace the T^2 leaf of the Reeb foliation by $T^2 \times [0,1]$ with a foliation which has the two boundary components as leaves and the interior foliated by leaves all diffeomorphic to $\boldsymbol{R}^2$. There are essentially two distinct ways to do this also. The first is obtained by foliating the set $\{(x,y,z) \in \boldsymbol{R}^3 \,|\, 0 \le y \le 1\}$ by planes parallel to the $y=0$ plane and then identifying (x,y,z) with $(x+1, \varphi(y), z)$ and with $(x, \psi(y), z+1)$. The maps $\varphi, \psi : [0,1] \to [0,1]$ are commuting diffeomorphisms so that on $(0,1)$, $\varphi(y) > y$, $\psi(y) > y$, and if $0 < y < 1$, $\varphi^k(y) = \psi^\ell(y) \iff k = \ell = 0$. The second way is to foliate $\{(x,y) \in \boldsymbol{R}^2 \,|\, 0 \le y \le 1\}$ as above by copies of $\boldsymbol{R}$ so that the boundary components are leaves, the leaves in the interior have both their ends at $x = +\infty$, and the foliation is invariant by translation in the x coordinate. Take the product of this foliation with $\boldsymbol{R}$ to obtain a foliation of $\{(x,y,z) \in \boldsymbol{R}^3 \,|\, 0 \le y \le 1\}$. Now identify (x,y,z) with $(x+1,y,z)$ and with $(x+\alpha, y, z+1)$, where $\alpha \in (0,1)$ and is irrational. As above, both types of foliations of S^3 have the same coarse cohomology.

For $x \notin \partial(T^2 \times [0,1])$, we have:

Proposition 3.17. *If $x \in S^3 - (T^2 \times [0,1])$, then for all q, $\mathcal{H}^q(\mathcal{L})_x = 0$. If $x \in T^2 \times (0,1)$, then $\mathcal{H}^2(\mathcal{L})_x = \boldsymbol{R}$; otherwise $\mathcal{H}^q(\mathcal{L})_x = 0$.*

Proof. As above, the first statement follows from the proof in the Reeb case. For the second, note that any point $x \in T^2 \times (0,1)$ has a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the product metric family $\mathcal{F}_x = \{U \times \boldsymbol{R}^2, d^U, \pi, U\}$, where the metric on each $\{u\} \times \boldsymbol{R}^2$ is the usual metric. The spectral sequence of [HH] gives that the coarse cohomology of the metric family $\mathcal{F}_x$ is the same as the coarse cohomology of $\boldsymbol{R}^2$, namely $\boldsymbol{R}$ in dimension 2 and 0 otherwise, so for $x \in T^2 \times (0,1)$, $\mathcal{H}^2(\mathcal{L})_x = \boldsymbol{R}$ and $\mathcal{H}^2(\mathcal{L})_x = 0$ otherwise. □

We now compute $\mathcal{H}^q(\mathcal{L})_x$ for $x \in T^2 \times \{0,1\}$. The result here does not depend on which gluing maps we use. Each such x has a neighborhood $U \simeq D^3$ so that the metric family $\{s^{-1}(U), d, s, U\}$ is coarsely equivalent to the metric family $\mathcal{F}_x = \{G, d^U, \pi, U\}$, $G = \widehat{G}/\boldsymbol{Z}$, and $\widehat{G} \subset D^3 \times \boldsymbol{R}^2$ is the set

$$\widehat{G} = \left\{(u_1,u_2,u_3,t_1,t_2) \,|\, \text{if } u_3 > 0,\ t_1 > -u_3^{-2}\right\}.$$

The action of $\boldsymbol{Z}$ is generated by:

$$\text{if } u_3 > 0, \text{ then } (u_1,u_2,u_3,t_1,t_2) \to (u_1,u_2,u_3,t_1,t_2+1).$$

The map $\pi : G \to U$ is the natural projection. For each $u \in U$ with $u_3 > 0$, $\pi^{-1}(u) \simeq S^1 \times (-u_3^{-2}, \infty)$ with the usual metric, and for each $u \in U$ with $u_3 \le 0$, $\pi^{-1}(u) \simeq \boldsymbol{R}^2$ with the usual metric.

As G_1 is not compact, $HX^0(\mathcal{F}_x) = 0$ as usual, which implies $\mathcal{H}^0(\mathcal{L}^*)_x = 0$.

To compute the higher coarse cohomology of the metric family $\mathcal{F}_x$, first note that $G_{\ell+1} \subset U \times (\boldsymbol{R}^2)^{\ell+1}$ is the set

$$G_{\ell+1} = \{(u_1, u_2, u_3, t_1, \ldots, t_{2\ell+2}) \,|\, \text{if } u_3 > 0,\, t_{2m+1} > -u_3^{-2}\}/\boldsymbol{Z}^{\ell+1},$$

where the $\boldsymbol{Z}^{\ell+1}$ action is given by:

$$\begin{aligned}&\text{if } u_3 > 0, \text{ then } (a_1, \ldots, a_{\ell+1})((u_1, u_2, u_3, t_1, t_2, t_3, \ldots, t_{2\ell}, t_{2\ell+1}, t_{2\ell+2}))\\ &\quad = (u_1, u_2, u_3, t_1, t_2 + a_1, t_3, \ldots, t_{2\ell} + a_\ell, t_{2\ell+1}, t_{2\ell+2} + a_{\ell+1}).\end{aligned}$$

We will view an element $\omega_{i,j} \in A_c^{i,j}(\mathcal{F}_x)$ as a differential i form on

$$\widehat{G}_{\ell+1} = \left\{(u_1, u_2, u_3, t_1, \ldots, t_{2\ell+2}) \in U \times (\boldsymbol{R}^2)^{\ell+1} \,|\, \text{if } u_3 > 0,\, t_{2m+1} > -u_3^{-2}\right\},$$

which is invariant under the action of $\boldsymbol{Z}^{\ell+1}$, that is, which is periodic in the even t variables of period 1 for all u with $u_3 > 0$. Note that this condition is preserved by the construction given in the Controlled Poincaré Lemma.

Now let $\alpha = [\sum_{i+j=k} \omega_{i,j}]$ be a $k > 0$ dimensional coarse class for $\mathcal{F}_x$, where $\omega_{i,j} \in A_c^{i,j}(\mathcal{F}_x)$. Proceeding just as we did for the Reeb foliation, we may assume that $\sum_{i+j=k} \omega_{i,j}$ is completely independent of u. Thus we may view $\omega_{i,j}$ as being an element of $A_c^{i,j}(\boldsymbol{R}^2)$ (i.e., an extended coarse cochain for $\boldsymbol{R}^2$; see [HH]) because this is what it is for $u_3 = 0$. It must also be invariant under the action of $\boldsymbol{Z}^{j+1}$, because it is invariant under this action when $u_3 > 0$. An argument similar to that given in the proof of Lemma 3.4 shows that the only form satisfying these conditions is the zero form. Thus $\alpha = 0$ and we have:

Proposition 3.18. *For $x \in T^2 \times \{0,1\}$, $\mathcal{H}^q(\mathcal{L})_x = 0$ for all q.*

And so we also have:

Proposition 3.19. *For any $\boldsymbol{R}^2$ Reeb foliation, $\mathcal{H}^2(\mathcal{L}) = \mathcal{R}_{T^2\times(0,1)}$; otherwise $\mathcal{H}^q(\mathcal{L}) = \mathbf{0}$.*

The spectral sequence applied to this example has $E_2^{p,2} = H^p(S^3; \mathcal{R}_{T^2\times(0,1)}) \simeq H^{p-1}(T^2; \boldsymbol{R})$, and all other E_2 terms are zero. Thus we have:

Theorem 3.20. *For any $\boldsymbol{R}^2$ Reeb foliation of S^3, $HX^{*+3}(F) \simeq H^*(T^2; \boldsymbol{R})$.*

Representatives of the coarse classes are given as follows. The coarse cohomology of $\boldsymbol{R}^2$ is non-zero only in dimension 2, where it is isomorphic to $\boldsymbol{R}$. A generator (in the extended coarse cohomology of $\boldsymbol{R}^2$) is given by $[\omega_{2,0} + \omega_{1,1}]$, where $\omega_{i,j} \in A_c^{i,j}(\boldsymbol{R}^2)$, and

$$\omega_{2,0}(t_1, t_2) = \psi(t_1, t_2) dt_1 \wedge dt_2,$$

$$\omega_{1,1}(t_1, t_2, t_3, t_4) = \left[\int_{t_1}^{\infty} \psi(z, t_2) dz\right] dt_2 - \left[\int_{t_3}^{\infty} \psi(z, t_4) dz\right] dt_4,$$

where $\psi(t_1, t_2)$ is any non-negative smooth function on $\boldsymbol{R}^2$ with compact support and positive total integral. Let ξ represent a class in $H^k(T^2; \boldsymbol{R})$, and let ϕ and dx be as in the T^2 Reeb foliation. Then, just as in the non-Hausdorff $S^1 \times \boldsymbol{R}$ case, $s^*(\phi\xi dx)\omega_{2,0}$ defines an element of $A_c^{k+3,0}$ and $s^*(\phi\xi dx)\omega_{1,1}$ defines an element of $A_c^{k+2,1}$. The coarse class corresponding to the class of ξ is then the class of $s^*(\phi\xi dx)\omega_{2,0} + s^*(\phi\xi dx)\omega_{1,1}$.

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