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THE SCHUR-HORN THEOREM FOR OPERATORS AND FRAMES WITH PRESCRIBED NORMS AND FRAME OPERATOR

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ABSTRACT. Let $\mathcal{H}$ be a Hilbert space. Given a bounded positive definite operator S on $\mathcal{H}$, and a bounded sequence $\mathbf{c} = \{c_k\}_{k\in\mathbb{N}}$ of nonnegative real numbers, the pair $(S, \mathbf{c})$ is frame admissible, if there exists a frame $\{f_k\}_{k\in\mathbb{N}}$ on $\mathcal{H}$ with frame operator S, such that $\|f_k\|^2 = c_k$, $k \in \mathbb{N}$. We relate the existence of such frames with the Schur-Horn theorem of majorization, and give a reformulation of the extended version of Schur-Horn theorem, due to A. Neumann. We use this to get necessary conditions (and to generalize known sufficient conditions) for a pair $(S, \mathbf{c})$ to be frame admissible.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let S be a bounded selfadjoint operator on $\mathcal{H}$. In the first part of this note, we give a complete characterization of the closure in $\ell^\infty(\mathbb{N})$ of the set of possible "diagonals" of S, i.e., the set $\mathcal{C}[\mathcal{U}_\mathcal{H}(S)]$ of real sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ such that

$$\langle Se_n, e_n\rangle = c_n, \quad n \in \mathbb{N}, \tag{1}$$

for some orthonormal basis $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$. Note that, if $\dim \mathcal{H} = m < \infty$, this can be made in terms of majorization theory. More precisely, the Schur-Horn theorem ensures that $\mathbf{c} \in \mathbb{R}^m$ satisfies Eq. (1) for some orthonormal basis if and only if $\mathbf{c}$ is majorized by the vector of eigenvalues of S (see Theorem 2.2 for a precise formulation). In the general case, we define an analogous form of "the sum of the greatest k eigenvalues" in the following way: given S, a selfadjoint operator on $\mathcal{H}$, and $k \in \mathbb{N}$, we denote

$$U_k(S) = \sup\{\operatorname{tr} SP : P \in L(\mathcal{H}) \text{ is an orthogonal projection with } \operatorname{tr} P = k\},$$

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and $L_k(S) = -U_k(-S)$. We prove, based on the results obtained by A. Neumann in [17], that $\mathbf{c}$ belongs to the $\ell^\infty(\mathbb{N})$-closure of $\mathcal{C}[\mathcal{U}_\mathcal{H}(S)]$ if and only if

$$U_k(\mathbf{c}) \leq U_k(S) \quad \text{and} \quad L_k(S) \leq L_k(\mathbf{c}), \quad k \in \mathbb{N}, \tag{2}$$

where

$$U_k(\mathbf{c}) = \sup_{|F|=k} \sum_{i\in F} c_i \,,$$

and

$$L_k(\mathbf{c}) = \inf_{|F|=k} \sum_{i\in F} c_i = -U_k(-\mathbf{c}).$$

Similarly, if S is a trace class operator, we show that $\mathbf{c}$ belongs to the $\ell^1(\mathbb{N})$-closure of $\mathcal{C}[\mathcal{U}_\mathcal{H}(S)]$ if and only if $\mathbf{c}$ satisfies formulas (2) and $\sum_{n\in\mathbb{N}} c_n = \operatorname{tr} S$. On the other hand, a somewhat technical characterization of the maps U_k and L_k is obtained (see Proposition 3.5), which is used to compute these quantities and to prove their basic properties. Related results can be found in R. Kadison [14], [15], and Arveson and Kadison [2] (which appeared during the revision process of this work).

In the second part of this note, these extended Schur-Horn theorems are used to give conditions for the existence of frames with prescribed norms and frame operator. First we recall some basic definitions. Let $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \{1, 2, \ldots, m\} := \mathbb{I}_m$, for some $m \in \mathbb{N}$. A sequence $\{f_k\}_{k\in\mathbb{M}}$ in $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants A, $B > 0$ such that

$$A\|x\|^2 \leq \sum_{k\in\mathbb{M}} |\langle x, f_k\rangle|^2 \leq B\|x\|^2, \qquad \text{for every} \qquad x \in \mathcal{H}.$$

For complete descriptions of frame theory and its applications, the reader is referred to [8], [11], [12], [3], or the books by Young [20] and Christensen [7]. Let $\mathcal{F} = \{f_k\}_{k\in\mathbb{M}}$, be a frame for $\mathcal{H}$. The operator

$$S : \mathcal{H} \to \mathcal{H}, \quad \text{given by} \quad S(x) = \sum_{k\in\mathbb{M}} \langle x, f_k\rangle f_k, \quad x \in \mathcal{H}, \tag{3}$$

is called the *frame operator* of $\mathcal{F}$. It is always bounded, positive and invertible (we use the notation $S \in \mathcal{G}l(\mathcal{H})^+$).

In recent papers by Casazza and Leon [5], [6], Casazza, Fickus, Leon and Tremain [4], Dykema, Freeman, Korleson, Larson, Ordower and Weber [10], Kornelson and Larson [16], and Tropp, Dhillon, Heath Jr. and Strohmer [19], the problem of existence and (algorithmic) construction of frames with prescribed norms and frame operator has been considered. Following [5], [6], we say that the pair $(S, \mathbf{c}) \in \mathcal{G}l(\mathcal{H})^+ \times \ell^\infty(\mathbb{M})^+$ is *frame admissible* if there exists a frame $\mathcal{F} = \{f_k\}_{k\in\mathbb{M}}$ on $\mathcal{H}$ such that

(1) $\mathcal{F}$ has frame operator S, and
(2) $\|f_k\|^2 = c_k$ for every $k \in \mathbb{M}$.

In this case, we say that $\mathcal{F}$ is a $(S, \mathbf{c})$-frame. We denote by $F(S, \mathbf{c})$ the set of all $(S, \mathbf{c})$-frames on $\mathcal{H}$. Hence the pair $(S, \mathbf{c})$ is frame admissible if $F(S, \mathbf{c}) \neq \emptyset$.

It is known (see [5], [19]) that, in the finite dimensional case, there is a connection between frame admissibility and the theory of majorization, in particular, the Schur-Horn theorem. We make this connection explicit both in the finite and infinite dimensional context. We use the classical Schur-Horn theorem in the finite dimensional case and its extension, developed in the first part of the paper, for the infinite dimensional case.

This presentation of the problem allows us to get equivalent conditions for the frame admissibility of a pair $(S, \mathbf{c}) \in \mathcal{G}l_n(\mathbb{C})^+ \times \ell^\infty(\mathbb{N})^+$, and necessary conditions for the frame admissibility of a pair $(S, \mathbf{c}) \in \mathcal{G}l(\mathcal{H})^+ \times \ell^\infty(\mathbb{N})^+$. We show that, if the pair $(S, \mathbf{c})$ is frame admissible, then $\sum_{k\in\mathbb{N}} c_k = \infty$, and $U_k(\mathbf{c}) \leq U_k(S)$ for every $k \in \mathbb{N}$. In particular, $\limsup \mathbf{c} \leq \|S\|_e$, the essential norm of S (see Theorem 5.1). Then, by strengthening these conditions we get sufficient conditions for the frame admissibility of pairs $(S, \mathbf{c}) \in \mathcal{G}l(\mathcal{H})^+ \times \ell^\infty(\mathbb{N})^+$ (Theorem 5.4). These conditions are less restrictive than those found by Kornelson and Larson in [16].

We briefly describe the contents of the paper. In Section 2 we fix our notation, and we state the classical Schur-Horn theorem. In Section 3 we prove the extension of the Schur-Horn theorem for general selfadjoint operators. In Section 4 we give some reformulations of the notion of frame admissibility which allow us to apply majorization theory to this problem, and we show equivalent conditions for frame admissibility in the finite dimensional case (both for finite or infinite sequences $\mathbf{c}$). In Section 5 we study the infinite dimensional case, showing separately necessary and sufficient conditions for frame admissibility. In Section 6 we give several examples for the boundary cases of the conditions studied before. These examples show that, in general, the conditions can not be relaxed further. We also study different types of frames in $F(S, \mathbf{c})$, in terms of their excesses.

2. Notations and preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, and $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We denote $L_0(\mathcal{H})$ the ideal of compact operators, $\mathcal{G}l(\mathcal{H})$ the group of invertible operators, $L(\mathcal{H})_h$ the set of hermitian operators, $L(\mathcal{H})^+$ the set of nonnegative definite operators, $\mathcal{U}(\mathcal{H})$ the group of unitary operators, and $\mathcal{G}l(\mathcal{H})^+$ the set of invertible positive definite operators. We denote by $L^1(\mathcal{H})$ the ideal of trace class operators in $L(\mathcal{H})$. We set $L^1(\mathcal{H})_h = L^1(\mathcal{H}) \cap L(\mathcal{H})_h$ and $L^1(\mathcal{H})^+ = L^1(\mathcal{H}) \cap L(\mathcal{H})^+$. We denote by $\ell^1(\mathbb{N})$ the Banach space of complex absolutely summable sequences. By $\ell^1_{\mathbb{R}}(\mathbb{N})$ (resp. $\ell^1(\mathbb{N})^+$) we denote the subsets of real (resp. nonnegative) sequences. Similarly, we use the notations $\ell^\infty(\mathbb{N})$, $\ell^\infty_{\mathbb{R}}(\mathbb{N})$ and $\ell^\infty(\mathbb{N})^+$ for bounded sequences.

Given an operator $A \in L(\mathcal{H})$, $R(A)$ denotes the range of A, $\ker A$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $\rho(A)$ the spectral radius of A, and $\|A\|$ the spectral norm of A. We say that A is an isometry (resp. coisometry) if $A^*A = I$ (resp. $AA^* = I$).

We also consider the quotient $\mathcal{A}(\mathcal{H}) = L(\mathcal{H})/L_0(\mathcal{H})$, which is a unital C^*-algebra, known as the Calkin algebra. Given $T \in L(\mathcal{H})$, the *essential spectrum* of T, denoted by $\sigma_e(T)$, is the spectrum of the class $T + L_0(\mathcal{H})$ in the algebra $\mathcal{A}(\mathcal{H})$. The essential norm $\|T\|_e = \inf\{\|T + K\| : K \in L_0(\mathcal{H})\}$ of T is the (quotient) norm of $T + L_0(\mathcal{H})$, also in $\mathcal{A}(\mathcal{H})$. Given $S \in L(\mathcal{H})_h$, we define

$$\alpha^+(S) = \max \sigma_e(S) = \|S\|_e \quad \text{and} \quad \alpha_-(S) = \min \sigma_e(S). \tag{4}$$

If $S = \int_{\sigma(S)} t \, dE(t)$ is the spectral representation of S with respect to the spectral measure E, we shall often consider the following compact operators:

$$\begin{aligned} S^+ &= \int_{[\alpha^+(S),\ \|S\|]} (t - \alpha^+(S)) dE(t), \quad \text{and} \\ S_- &= \int_{[-\|S\|, \alpha_-(S)]} (t - \alpha_-(S)) dE(t). \end{aligned} \tag{5}$$

Note that $S_- \leq 0 \leq S^+$.

Given a subset $\mathcal{M}$ of a Banach space $(\mathcal{X}, \|\cdot\|)$, its closure is denoted by $\overline{\mathcal{M}}$ or $\mathrm{cl}_{\|\cdot\|}(\mathcal{M})$, and the convex hull of $\mathcal{M}$ is denoted by $\mathrm{conv}(\mathcal{M})$. Also, given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal (i.e., selfadjoint) projection onto $\mathcal{S}$. If $B \in L(\mathcal{H})$ satisfies $P_{\mathcal{S}} B P_{\mathcal{S}} = B$, in some cases we shall use the compression of B to $\mathcal{S}$, (i.e., the restriction of B to $\mathcal{S}$ as a linear transformation from $\mathcal{S}$ to $\mathcal{S}$), and we say that *we consider B as acting on $\mathcal{S}$*.

Finally, when $\dim \mathcal{H} = n < \infty$, we shall identify $\mathcal{H}$ with $\mathbb{C}^n$, $L(\mathcal{H})$ with $\mathcal{M}_n(\mathbb{C})$, and we use the following notations: $\mathcal{M}_n(\mathbb{C})_h$ for $L(\mathcal{H})_h$, $\mathcal{M}_n(\mathbb{C})^+$ for $L(\mathcal{H})^+$, $\mathcal{U}(n)$ for $\mathcal{U}(\mathcal{H})$, and $\mathcal{G}l_n(\mathbb{C})$ for $\mathcal{G}l(\mathcal{H})$.

Majorization. In this subsection we present some basic aspects of majorization theory. For a more detailed treatment of this notion see [13]. Given $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, denote by $\mathbf{b}^\downarrow \in \mathbb{R}^n$ the vector obtained by rearranging the coordinates of $\mathbf{b}$ in nonincreasing order. If $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^n$ then we say that $\mathbf{c}$ is majorized by $\mathbf{b}$, and write $\mathbf{c} \prec \mathbf{b}$, if

$$\sum_{i=1}^k b_i^\downarrow \geq \sum_{i=1}^k c_i^\downarrow \quad k = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^n b_i = \sum_{i=1}^n c_i.$$

Majorization is a preorder relation in $\mathbb{R}^n$ that occurs naturally in matrix analysis.

Definition 2.1. Let $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \{1, 2, \ldots, m\} := \mathbb{I}_m$, for some $m \in \mathbb{N}$. Let $\mathcal{K}$ be a Hilbert space with $\dim \mathcal{K} = |\mathbb{M}|$ and let $\mathcal{B} = \{e_n\}_{n \in \mathbb{M}}$ be an orthonormal basis of $\mathcal{K}$.

(1) For any $\mathbf{a} = (a_n)_{n\in\mathbb{M}} \in \ell^\infty(\mathbb{M})$, denote by $M_{\mathcal{B},\mathbf{a}} \in L(\mathcal{K})$ the diagonal operator given by $M_{\mathcal{B},\mathbf{a}} e_n = a_n e_n$, $n \in \mathbb{M}$. When it is clear which basis we are using, we abbreviate $M_{\mathcal{B},\mathbf{a}} = M_{\mathbf{a}}$.
(2) In particular, for $\mathbf{a} \in \mathbb{C}^n$, we denote by $M_{\mathbf{a}} \in \mathcal{M}_n(\mathbb{C})$ the diagonal matrix (with respect to the canonical basis of $\mathbb{C}^n$) which has the entries of $\mathbf{a}$ on its diagonal.
(3) The diagonal pinching $\mathcal{C}_{\mathcal{B}} : L(\mathcal{K}) \to L(\mathcal{K})$ associated to the basis $\mathcal{B}$, is defined by $\mathcal{C}_{\mathcal{B}}(T) = M_{\mathcal{B},\mathbf{a}}$, where $\mathbf{a} = (\langle Te_n, e_n\rangle)_{n\in\mathbb{M}}$.

THEOREM 2.2 (Schur-Horn). *Let* $\mathbf{b},\ \mathbf{c} \in \mathbb{R}^n$. *Then* $\mathbf{c} \prec \mathbf{b}$ *if and only if there exists* $U \in \mathcal{U}(n)$ *such that*

$$\mathcal{C}_{\mathcal{E}}(U^* M_{\mathbf{b}} U) = M_{\mathbf{c}},$$

where $\mathcal{E}$ *is the canonical basis of* $\mathbb{C}^n$.

3. Schur-Horn theorem for selfadjoint operators

In this section we present a different version of the "infinite dimensional Schur-Horn theorem" given by A. Neumann in [17]. Our approach avoids the somewhat technical distinction between the diagonalizable and nondiagonalizable case. On the other hand, this version can be applied more easily to the problem of frame admissibility in the infinite dimensional case. The main tools we use are the Weyl–von Neumann theorem and the known properties of approximately unitarily equivalent operators.

Given a sequence $\mathbf{a} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$, Neumann [17] defines

$$U_k(\mathbf{a}) = \sup_{|F|=k} \sum_{i\in F} a_i \qquad \text{and} \qquad L_k(\mathbf{a}) = \inf_{|F|=k} \sum_{i\in F} a_i.$$

This generalizes the partial sums which appear in the definition of majorization. In the first part of this section we shall extend this definition to arbitrary selfadjoint operators on a Hilbert space $\mathcal{H}$. Denote by $\mathcal{P}_k$ the set of orthogonal projections onto k-dimensional subspaces of $\mathcal{H}$.

DEFINITION 3.1. Given $S \in L(\mathcal{H})_h$, we define, for any $k \in \mathbb{N}$,

$$U_k(S) = \sup_{P\in\mathcal{P}_k} \operatorname{tr}(SP) \quad \text{and} \quad L_k(S) = \inf_{P\in\mathcal{P}_k} \operatorname{tr}(SP) = -U_k(-S).$$

REMARK 3.2. It is easy to see that U_k and L_k satisfy the following properties:

(1) For every $k \in \mathbb{N}$, U_k is a convex map, and L_k is a concave map.
(2) The maps U_k and L_k are unitarily invariant, for every $k \in \mathbb{N}$, i.e., $U_k(S) = U_k(U^*SU)$, for every $U \in \mathcal{U}(\mathcal{H})$ and $S \in L(\mathcal{H})_h$.

The following result asserts that Definition 3.1 extends the natural extrapolation of Neumann's definition for diagonalizable operators.

PROPOSITION 3.3. *Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. If $\mathbf{a} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$, then, for every $k \in \mathbb{N}$,*

$$U_k(M_{\mathcal{B},\mathbf{a}}) = U_k(\mathbf{a}).$$

In order to prove this proposition we need the following technical results.

LEMMA 3.4. *Let $S \in L_0(\mathcal{H})^+$, and denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$ the positive eigenvalues of S, counted with multiplicity (if $\dim R(S) < \infty$, we complete this sequence with zeros). Then, for every $k \in \mathbb{N}$,*

$$U_k(S) = \sum_{i=1}^{k} \lambda_i.$$

Moreover, if $P \in \mathcal{P}_k$ is the projection onto the subspace spanned by an orthonormal set of eigenvectors of $\lambda_1, \ldots, \lambda_k$, then $U_k(S) = \mathrm{tr}(SP)$.

Proof. Fix $k \in \mathbb{N}$. It suffices to show that $\mathrm{tr}(SQ) \leq \mathrm{tr}(SP) = \sum_{i=1}^{k} \lambda_i$ for every $Q \in \mathcal{P}_k$. This follows from Schur's theorem (the diagonal is majorized by the sequence of eigenvalues), which also holds in this setting (see Chapter 1 of Simon's book [18]). □

In [17], Neumann proved the following result (Lemma 2.17): if $\mathbf{a} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$,

$$a_i^+ = \max\{a_i - \limsup\, \mathbf{a}\,,\, 0\}, \quad a_i^- = \min\{a_i - \liminf\, \mathbf{a}\,,\, 0\}, \quad i \in \mathbb{N}, \tag{6}$$

then, for every $k \in \mathbb{N}$,

$$U_k(\mathbf{a}) = U_k(\mathbf{a}^+) + k\ \limsup\, \mathbf{a} \quad \text{and} \quad L_k(\mathbf{a}) = L_k(\mathbf{a}^-) + k\ \liminf\, \mathbf{a}. \tag{7}$$

The next result extends Eq. (7) to selfadjoint operators. This fact is necessary for the proof of Proposition 3.3, but it is also a basic tool in order to deal with the maps U_k and L_k.

PROPOSITION 3.5. *Let $S \in L(\mathcal{H})_h$. Then, for every $k \in \mathbb{N}$,*

1. $U_k(S) = U_k(S^+) + k\ \alpha^+(S)$,
2. $L_k(S) = L_k(S_-) + k\ \alpha_-(S)$.

where $\alpha^+(S)$, $\alpha_-(S)$, S^+, S_- are defined in (4) and (5). In particular,

$$\lim_{k\to\infty} \frac{U_k(S)}{k} = \alpha^+(S) = \|S\|_e \qquad \text{and} \qquad \lim_{k\to\infty} \frac{L_k(S)}{k} = \alpha_-(S). \tag{8}$$

Proof. Denote $\alpha^+ = \alpha^+(S)$, and

$$P_2 = P_2(S) = E[\,\|S\|_e, \|S\|\,] = E[\,\alpha^+, \|S\|\,], \tag{9}$$

where E is the spectral measure of S. Recall that

$$S^+ = \int_{[\alpha^+,\ \|S\|]} (t - \alpha^+)\, dE(t) = (S - \alpha^+)P_2.$$

Then $S - S^+ = S(I - P_2) + \alpha^+ P_2 \leq \alpha^+ I$. Therefore, for every $k \in \mathbb{N}$ and $Q \in \mathcal{P}_k$,

$$\operatorname{tr}(SQ) = \operatorname{tr}(S^+Q) + \operatorname{tr}((S - S^+)Q) \leq U_k(S^+) + k\alpha^+, \tag{10}$$

which shows that $U_k(S) \leq U_k(S^+) + k\alpha^+$ for every $k \in \mathbb{N}$.

To see the converse inequality, suppose first that $\operatorname{tr} P_2 = +\infty$. Denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$ the eigenvalues of S^+, chosen as in Lemma 3.4.

Let $Q_k \in \mathcal{P}_k$ be the projection onto the subspace spanned by an orthonormal set of eigenvectors of $\lambda_1, \ldots, \lambda_k$. Then $Q_k \leq P_2$. By Lemma 3.4,

$$\operatorname{tr}(SQ_k) = \operatorname{tr}(S^+Q_k) + \operatorname{tr}((S - S^+)Q_k) = \sum_{i=1}^{k} \lambda_i + k\alpha^+ = U_k(S^+) + k\alpha^+ \ .$$

Hence, $U_k(S) = U_k(S^+) + k\alpha^+$. Now, assume that $\operatorname{tr} P_2 = r < \infty$. If $k \leq r$, the same argument as before shows that $U_k(S) = U_k(S^+) + k\alpha^+$. So, let $k > r$ and take $\varepsilon > 0$. Since $P_\varepsilon = E[\,\alpha^+ - \varepsilon\,,\,\alpha^+\,)$ has infinite rank (otherwise $\|S\|_e \leq \alpha^+ - \varepsilon$), we can take $Q \leq P_\varepsilon$, a projection of rank $k - r$. If $Q_k = Q + P_2$, then

$$\begin{aligned} U_k(S) \geq \operatorname{tr}(SQ_k) &= \operatorname{tr}(SP_2) + \operatorname{tr}(SQ) \\ &= \operatorname{tr}(S^+) + r\alpha^+ + \operatorname{tr}(SP_\varepsilon Q) \\ &\geq \operatorname{tr}(S^+) + r\alpha^+ + (k - r)(\alpha^+ - \varepsilon) \\ &= U_k(S^+) + k\alpha^+ - \varepsilon(k - r). \end{aligned}$$

Since ε is arbitrary, $U_k(S) = U_k(S^+) + k\alpha^+$. The formula for $L_k(S)$ follows by applying item 1 to $-S$. Finally, as $S^+ \in L_0(\mathcal{H})^+$, its eigenvalues converge to zero. Hence, by Lemma 3.4, we get that

$$\lim_{k \to \infty} \frac{U_k(S^+)}{k} = 0$$

and similarly for $L_k(S_-)$. Therefore, Eq. (8) follows. □

Proof of Proposition 3.3. The result follows using Lemma 3.4, Proposition 3.5, Eq. (7) and the following obvious identities: if $S = M_{\mathcal{B},\mathbf{a}}$, then

(1) $\alpha^+(S) = \limsup\ \mathbf{a}$, and $\alpha_-(S) = \liminf\ \mathbf{a}$,
(2) $S^+ = M_{\mathcal{B},\mathbf{a}^+}$ and $S_- = M_{\mathcal{B},\mathbf{a}^-}$,

where $\mathbf{a}^+$ and $\mathbf{a}^-$ are defined as in Eq. (6). □

Definition 3.6. Let $\mathcal{H}$ be a Hilbert space, $S \in L(\mathcal{H})$ and $\mathcal{B}$ an orthonormal basis of $\mathcal{H}$. Then:

(1) $\mathcal{U}_{\mathcal{H}}(S) = \{U^*SU \,:\, U \in \mathcal{U}(\mathcal{H})\}$.
(2) $\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)] = \left\{\mathbf{c} \in \ell^\infty(\mathbb{N}) \,:\, M_{\mathcal{B},\,\mathbf{c}} \in \mathcal{C}_{\mathcal{B}}(\mathcal{U}_{\mathcal{H}}(S))\right\}$.

REMARK 3.7. Given $S \in L(\mathcal{H})$, the definition of $\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]$ does not depend on the orthonormal basis $\mathcal{B}$. In fact, if $\mathcal{B}'$ is another orthonormal basis of $\mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$ maps $\mathcal{B}$ onto $\mathcal{B}'$, and $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$ satisfies $M_{\mathcal{B},\,\mathbf{c}} = \mathcal{C}_{\mathcal{B}}(T)$ for some $T \in \mathcal{U}_{\mathcal{H}}(S)$, then

$$M_{\mathcal{B}',\,\mathbf{c}} = U M_{\mathcal{B},\,\mathbf{c}} U^* = U\mathcal{C}_{\mathcal{B}}(T)U^* = \mathcal{C}_{\mathcal{B}'}(UTU^*) \in \mathcal{C}_{\mathcal{B}'}(\mathcal{U}_{\mathcal{H}}(S)).$$

Therefore $\left\{\mathbf{c} \in \ell^\infty(\mathbb{N}) \ : \ M_{\mathcal{B}',\,\mathbf{c}} \in \mathcal{C}_{\mathcal{B}'}(\mathcal{U}_{\mathcal{H}}(S))\right\} = \mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]$.

Given a diagonal operator $M_{\mathbf{a}} \in L(\mathcal{H})_h$, Neumann showed that, if $\mathbf{c} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$, the following statements are equivalent (Corollary 2.18 and Theorem 3.13 of [17]):

(1) $\mathbf{c} \in \overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(M_{\mathbf{a}})]}$.
(2) $U_k(\mathbf{a}) \geq U_k(\mathbf{c})$ and $L_k(\mathbf{a}) \leq L_k(\mathbf{c})$, $k \in \mathbb{N}$.

Now, our objective is to generalize this equivalence to every operator $S \in L(\mathcal{H})_h$ (via a reduction to the diagonalizable case). We need first the following result about approximately unitarily equivalent operators.

LEMMA 3.8. *Let $S, T \in L(\mathcal{H})_h$. Then $S \in \mathrm{cl}_{\|\cdot\|}\left(\mathcal{U}_{\mathcal{H}}(T)\right)$ if and only if*

$$\mathrm{cl}_{\|\cdot\|}\left(\mathcal{U}_{\mathcal{H}}(S)\right) = \mathrm{cl}_{\|\cdot\|}\left(\mathcal{U}_{\mathcal{H}}(T)\right).$$

In this case $U_k(S) = U_k(T)$ and $L_k(S) = L_k(T)$ for every $k \in \mathbb{N}$.

Proof. If $\{V_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}(\mathcal{H})$ such that $\|V_n T V_n^* - S\| \xrightarrow[n\to\infty]{} 0$, then

$$\|V_n^* S V_n - T\| = \|V_n^*(S - V_n T V_n^*)V_n\| = \|V_n T V_n^* - S\| \xrightarrow[n\to\infty]{} 0.$$

Hence $\mathrm{cl}_{\|\cdot\|}\left(\mathcal{U}_{\mathcal{H}}(S)\right) = \mathrm{cl}_{\|\cdot\|}\left(\mathcal{U}_{\mathcal{H}}(T)\right)$. By Remark 3.2, $U_k(V_n T V_n^*) = U_k(T)$ and $L_k(V_n T V_n^*) = L_k(T)$, for n, $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and take $P \in \mathcal{P}_k$. Then

$$\operatorname{tr} SP = \lim_{n\to\infty} \operatorname{tr} V_n T V_n^* P \leq \lim_{n\to\infty} U_k(V_n T V_n^*) = U_k(T).$$

Hence $U_k(S) \leq U_k(T)$. Similarly $L_k(S) \geq L_k(T)$. The reverse inequalities follow from the fact that $V_n^* S V_n \xrightarrow[n\to\infty]{} T$. □

REMARK 3.9. Two operators $S, T \in L(\mathcal{H})_h$ satisfying the conditions of Lemma 3.8 are called *approximately unitarily equivalent*. This equivalence relation has been extensively studied in the theory of operator algebras. For example, as a consequence of the Weyl von Neuman theorem, it is proved in Davidson's book [9] (II.4.4) that $S, T \in L(\mathcal{H})_h$ are approximately unitarily equivalent if and only if $\sigma_e(S) = \sigma_e(T)$ and $\dim\ker(S-\lambda I) = \dim\ker(T-\lambda I)$ for every $\lambda \notin \sigma_e(S)$. From this fact it can be deduced (see the proof of II.4.4 in [9]) that, for every $S \in L(\mathcal{H})_h$, there exists a diagonalizable $D \in L(\mathcal{H})_h$ which is approximately unitarily equivalent to S.

THEOREM 3.10. *Let $S \in L(\mathcal{H})_h$ and $\mathbf{c} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$. Then the following conditions are equivalent:*

(1) $\mathbf{c} \in \overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]}$.
(2) $U_k(S) \geq U_k(\mathbf{c})$ *and* $L_k(S) \leq L_k(\mathbf{c})$ *for every* $k \in \mathbb{N}$.

If one of these conditions holds, then $\max \sigma_e(S) \geq \limsup \mathbf{c}$ *and* $\min \sigma_e(S) \leq \liminf \mathbf{c}$.

Proof. The diagonalizable case was proved by Neumann as mentioned before. Note that, in order to deduce our formulation from Neumann's result, we need Proposition 3.3. If S is not diagonalizable, by Remark 3.9, there exists a diagonalizable operator $D \in \mathrm{cl}_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S))$. By Lemma 3.8, $U_k(D) = U_k(S)$ and $L_k(D) = L_k(S)$ for every $k \in \mathbb{N}$, and $\mathrm{cl}_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(D)) = \mathrm{cl}_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S))$. This implies that

$$\mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(D)]) = \mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]),$$

because the map $T \mapsto \mathcal{C}_{\mathcal{B}}(T)$ is continuous for every orthonormal basis $\mathcal{B}$. Hence, the general case reduces to the diagonalizable case. The final remark follows from the fact that

$$\limsup \mathbf{c} = \lim_{k\to\infty} \frac{U_k(\mathbf{c})}{k} \quad \text{and} \quad \liminf \mathbf{c} = \lim_{k\to\infty} \frac{L_k(\mathbf{c})}{k}, \tag{11}$$

and Eq. (8). □

A similar result can be stated for hermitian operators in $L^1(\mathcal{H})$ and sequences in $\ell^1_{\mathbb{R}}(\mathbb{N})$. In this case our result is a slight generalization, using our maps U_k and L_k, of some results due to Neumann.

DEFINITION 3.11. Let Π be the set of all bijective maps on $\mathbb{N}$ and, for any $k \in \mathbb{N}$, denote by $\Pi_k \subseteq \Pi$ the set of permutations σ such that $\sigma(n) = n$ for every $n > k$. Given $\mathbf{a} \in \ell^\infty(\mathbb{N})$ and $\sigma \in \Pi$, we define:

(1) $\mathbf{a}_\sigma = (a_{\sigma(1)}, a_{\sigma(2)}, \dots)$.
(2) $\Pi \cdot \mathbf{a} = \{\mathbf{a}_\sigma,\ \sigma \in \Pi\}$, the orbit of $\mathbf{a}$, under the action of Π.
(3) $\mathrm{conv}(\Pi \cdot \mathbf{a})$, the convex hull of the orbit of $\mathbf{a}$.

3.12. If $\mathbf{b}, \mathbf{a}$ are sequences in $\ell^1_{\mathbb{R}}(\mathbb{N})$, Neumann [17] proved that the following statements are equivalent:

(1) $\mathbf{b} \in \mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi \cdot \mathbf{a}))$.
(2) $\sum_{k=1}^\infty b_k = \sum_{k=1}^\infty a_k$ and $U_k(\mathbf{a}) \geq U_k(\mathbf{b})$, $L_k(\mathbf{a}) \leq L_k(\mathbf{b})$, $k \in \mathbb{N}$.

PROPOSITION 3.13. *Let $S \in L^1(\mathcal{H})_h$, and $\mathbf{b} \in \ell^1_{\mathbb{R}}(\mathbb{N})$. Then the following statements are equivalent:*

(1) $\mathbf{b} \in \mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$.
(2) $U_k(S) \geq U_k(\mathbf{b})$, $L_k(S) \leq L_k(\mathbf{b})$ *for every* $k \in \mathbb{N}$, *and* $\sum_{k=1}^\infty b_k = \operatorname{tr} S$.

Proof. $1 \Rightarrow 2$. Note that $\mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]) \subseteq \mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$. Hence, by Proposition 3.10, $U_k(S) \geq U_k(\mathbf{b})$ and $L_k(S) \leq L_k(\mathbf{b})$ for every $k \in \mathbb{N}$. The equality $\sum_{k=1}^{\infty} b_k = \operatorname{tr} S$ clearly holds if $\mathbf{b} \in \mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]$. The general case follows from the $\ell^1(\mathbb{N})$-continuity of the map $\mathbf{b} \mapsto \sum_{k=1}^{\infty} b_k$.

$2 \Rightarrow 1$. Let $\mathbf{a} \in \ell^1_{\mathbb{R}}(\mathbb{N})$ and $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$ such that $S = M_{\mathcal{B},\mathbf{a}}$. By 3.12 and Prop. 3.3, it suffices to show that $\mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi \cdot \mathbf{a})) \subseteq \mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$.

Claim. $\mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi \cdot \mathbf{a})) = \mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi_0 \cdot \mathbf{a}))$, *where* $\Pi_0 = \bigcup_{k\in\mathbb{N}} \Pi_k$.

Indeed, it is sufficient to prove that $\Pi \cdot \mathbf{a} \subseteq \mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi_0 \cdot \mathbf{a}))$. Given $\sigma \in \Pi$, $\mathbf{a}_\sigma \in \Pi \cdot \mathbf{a}$, and $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\sum_{k>N} |a_k| < \varepsilon/2$ and $N_0 \in \mathbb{N}$ such that $\sigma^{-1}(\mathbb{I}_N) \subseteq \mathbb{I}_{N_0}$. There exists $\sigma_0 \in \Pi_{N_0}$ such that $\sigma(k) = \sigma_0(k)$ for every $k \in \mathbb{I}_{N_0}$ such that $\sigma(k) \in \mathbb{I}_N$. Therefore,

$$\begin{aligned}\|\mathbf{a}_\sigma - \mathbf{a}_{\sigma_0}\|_1 &= \sum_{\sigma(k)\notin\mathbb{I}_N} |a_{\sigma(k)} - a_{\sigma_0(k)}| \\ &\leq \sum_{\sigma(k)\notin\mathbb{I}_N} |a_{\sigma(k)}| + \sum_{\sigma_0(k)\notin\mathbb{I}_N} |a_{\sigma_0(k)}| < \varepsilon.\end{aligned}$$

Consider $\mathbf{b} \in \mathrm{conv}(\Pi_0 \cdot \mathbf{a})$. Then there exists $n \in \mathbb{N}$ such that $\mathbf{b} \in \mathrm{conv}(\Pi_n \mathbf{a})$. This means that the first n entries of $\mathbf{b}$ form a convex combination of permutations of the first n entries of $\mathbf{a}$, and $b_k = a_k$ for every $k > n$. Hence $(b_1, \ldots, b_n) \prec (a_1, \ldots, a_n)$. Denote $\mathcal{B}_n = \{e_k : k \leq n\}$ and $\mathcal{H}_n = \operatorname{span}\{\mathcal{B}_n\}$. Then, by the Schur-Horn Theorem 2.2, there exists a unitary $U_0 \in L(\mathcal{H}_n)$ such that

$$M_{\mathcal{B},\mathbf{b}}|_{\mathcal{H}_n} = \mathcal{C}_{\mathcal{B}_n}(U_0^* M_{\mathcal{B},\mathbf{a}}|_{\mathcal{H}_n} U_0).$$

Letting

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} \begin{matrix} \mathcal{H}_n \\ \mathcal{H}_n^\perp \end{matrix} \in \mathcal{U}(\mathcal{H}),$$

we get that $M_{\mathcal{B},\mathbf{b}} = \mathcal{C}_{\mathcal{B}}(U^* M_{\mathcal{B},\mathbf{a}} U)$, and $\mathbf{b} \in \mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]$. Therefore

$$\mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi \cdot \mathbf{a})) = \mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi_0 \cdot \mathbf{a})) \subseteq \mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]),$$

which completes the proof. □

Remark 3.14. Comparing 3.12 with Proposition 3.13, it follows that, if $S = M_{\mathcal{B},\mathbf{a}}$ for some $\mathbf{a} \in \ell^1_{\mathbb{R}}(\mathbb{N})$ and some orthonormal basis $\mathcal{B}$ of $\mathcal{H}$, then

$$\mathrm{cl}_{\|\cdot\|_1}(\mathrm{conv}(\Pi \cdot \mathbf{a})) = \mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]).$$

In particular, $\mathrm{cl}_{\|\cdot\|_1}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$ is a convex set. On the other hand, since the maps U_k are convex and the maps L_k are concave for all $k \in \mathbb{N}$, it can be deduced from Theorem 3.10 that $\mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$ is convex, for every $S \in L(\mathcal{H})_h$. Actually, this fact is known, and can also be deduced from the following results of Neumann [17]:

1. If $S = M_{\mathcal{B},\mathbf{a}}$ for some $\mathbf{a} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$ and some orthonormal basis $\mathcal{B}$, then $\mathrm{cl}_{\|\cdot\|_\infty}(\mathrm{conv}(\Pi \cdot \mathbf{a})) = \mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)])$.
2. If S is not diagonalizable, then

$$\overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]} = \overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S^+)]} + [\alpha_-(S), \alpha^+(S)]^{\mathbb{N}} + \overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S_-)]}, \tag{12}$$

where $\alpha^+(S)$, $\alpha_-(S)$, S^+, S_- are defined in (4) and (5).

Note that formula (12), which holds also for diagonalizable operators, gives another complete characterization of $\overline{\mathcal{C}[\mathcal{U}_{\mathcal{H}}(S)]}$. It can be used to give an alternative proof of Theorem 3.10, but it can also be deduced from the statement of this theorem, and Proposition 3.5.

4. Frames with prescribed norms and frame operator

Preliminaries on frames. We introduce some basic facts about frames in Hilbert spaces. For a complete description of frame theory and its applications, the reader is referred to Daubechies, Grossmann and Meyer [8], Aldroubi [1], the review by Heil and Walnut [11] or the books by Young [20] and Christensen [7].

DEFINITION 4.1. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ a sequence in a Hilbert space $\mathcal{H}$.

1. $\mathcal{F}$ is called a *frame* if there exist numbers $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad \text{for every} \qquad f \in \mathcal{H}. \tag{13}$$

2. The optimal constants A, B for Eq. (13) are called the *frame bounds* for $\mathcal{F}$. The frame $\mathcal{F}$ is called *tight* if $A = B$, and *Parseval* if $A = B = 1$. Parseval frames are also called normalized tight frames.

DEFINITION 4.2. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$. Let $\mathcal{K}$ be a separable Hilbert space. Let $\mathcal{B} = \{\varphi_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{K}$. From Eq. (13), it follows that there exists a unique $T \in L(\mathcal{K}, \mathcal{H})$ such that

$$T(\varphi_n) = f_n, \quad n \in \mathbb{N}.$$

We shall say that the triple $(T, \mathcal{K}, \mathcal{B})$ is a *synthesis (or preframe) operator* for $\mathcal{F}$. Another consequence of Eq. (13) is that T is surjective.

REMARK 4.3. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$ and $(T, \mathcal{K}, \mathcal{B})$ a synthesis operator for $\mathcal{F}$, with $\mathcal{B} = \{\varphi_n : n \in \mathbb{N}\}$.

1. The adjoint $T^* \in L(\mathcal{H}, \mathcal{K})$ of T is given by

$$T^*(x) = \sum_{n\in\mathbb{N}} \langle x, f_n\rangle \varphi_n, \quad x \in \mathcal{H}.$$

It is called an *analysis operator* for $\mathcal{F}$.

2. By the previous remarks, the operator $S = TT^* \in L(\mathcal{H})^+$, called the *frame operator* of $\mathcal{F}$, satisfies

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle\, f_n, \quad \text{for every} \quad f \in \mathcal{H}. \tag{14}$$

It follows from (13) that $AI \leq S \leq BI$. So that $S \in \mathcal{G}l(\mathcal{H})^+$. Note that, by formula (14), the frame operator of $\mathcal{F}$ does not depend on the chosen synthesis operator.

DEFINITION 4.4. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$. The cardinal number

$$e(\mathcal{F}) = \dim\Big\{(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : \sum_{n\in\mathbb{N}} c_n f_n = 0\Big\},$$

is called the *excess* of the frame. Holub [12] and Balan, Casazza, Heil and Landau [3] proved that

$$e(\mathcal{F}) = \sup\{\ |I| : I \subseteq \mathbb{N} \ \text{ and } \ \{f_n\}_{n\notin I} \ \text{ is still a frame on } \mathcal{H}\}.$$

This characterization justifies the name "excess of $\mathcal{F}$". It is easy to see that, for every synthesis operator $(T, \mathcal{K}, \mathcal{B})$ of $\mathcal{F}$, $e(\mathcal{F}) = \dim\ker T$. The frame $\mathcal{F}$ is called a *Riesz basis* if $e(\mathcal{F}) = 0$, i.e., if the synthesis operators of $\mathcal{F}$ are invertible.

Reformulation of frame admissibility. Recall that, given a sequence $\mathbf{c} = (c_k)_{k\in\mathbb{M}} \in \ell^\infty(\mathbb{M})^+$ and $S \in \mathcal{G}l(\mathcal{H})^+$, we denote by $F(S, \mathbf{c})$ the set of $(S, \mathbf{c})$-frames, i.e., those frames $\mathcal{F} = \{f_k\}_{k\in\mathbb{M}}$ for $\mathcal{H}$, with frame operator S, such that $\|f_k\|^2 = c_k$, for every $k \in \mathbb{M}$, and we say that the pair $(S, \mathbf{c})$ is frame admissible if $F(S, \mathbf{c}) \neq \emptyset$. We shall consider the following equivalent formulation of frame admissibility, which makes clear its relationship with the Schur-Horn theorem of majorization theory.

PROPOSITION 4.5. *Let $\mathbf{c} \in \ell^\infty(\mathbb{M})^+$ and let $S \in \mathcal{G}l(\mathcal{H})^+$. Then the following conditions are equivalent:*

1. *The pair $(S, \mathbf{c})$ is frame admissible.*
2. *There exists a sequence of unit vectors $\{y_k\}_{k\in\mathbb{M}}$ in $\mathcal{H}$ such that*

$$S = \sum_{k\in\mathbb{M}} c_k\, y_k \otimes y_k,$$

where, if $\mathbb{M} = \mathbb{N}$, the sum converges in the strong operator topology.

3. *There exists an extension $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote*

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(\mathcal{K})^+, \tag{15}$$

then $\mathbf{c} \in \mathcal{C}\,[\mathcal{U}_{\mathcal{K}}(S_1)]$.

In this case, there exists a frame $\mathcal{F} \in F(S, \mathbf{c})$ with $e(\mathcal{F}) = \dim\mathcal{H}_d$.

Proof. The equivalence between conditions 1 and 2 is well known (see, for example, [10]). Hence we shall prove $1 \Leftrightarrow 3$. Assume that $\mathcal{F} = \{f_k\}_{k\in\mathbb{M}} \in F(S, \mathbf{c})$. Let $(T_0, \mathcal{K}_0, \mathcal{B}_0)$ be a synthesis operator for $\mathcal{F}$. Consider the polar decomposition $T_0 = U|T_0|$, where $U : \mathcal{K}_0 \to \mathcal{H}$ is a coisometry with initial space $(\ker T_0)^\perp$ and range $\mathcal{H}$. Note that U^* maps isometrically $\mathcal{H}$ onto $\ker T_0^\perp$. Denote $\mathcal{H}_d = \ker T_0$, and $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$. Let $V : \mathcal{K} \to \mathcal{K}_0$ be the unitary operator given by

$$V(\xi_1, \xi_2) = U^*\xi_1 + \xi_2 \ , \qquad \text{for} \qquad (\xi_1, \xi_2) \in \mathcal{H} \oplus \mathcal{H}_d = \mathcal{K}.$$

Consider the orthonormal basis $\mathcal{B} = V^*(\mathcal{B}_0)$ of $\mathcal{K}$, and $T = T_0V \in L(\mathcal{K}, \mathcal{H})$. Then $(T, \mathcal{K}, \mathcal{B})$ is another synthesis operator for $\mathcal{F}$, with $\ker T = \mathcal{H}_d$.

Let $T_1 \in L(\mathcal{K})$ given by $T_1\xi = T\xi \oplus 0_{\mathcal{H}_d}$, $\xi \in \mathcal{K}$. Then $T_1^*T_1 = T^*T$, $|T_1| = |T|$, and

$$T_1T_1^* = \begin{pmatrix} TT^* & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = S_1.$$

If $T_1 = U_1|T_1| = U_1|T|$ is the polar decomposition of T_1, then U_1 acts on $\mathcal{H} = (\ker T_1)^\perp$ as a unitary operator. Hence $W = U_1 + P_{\mathcal{H}_d} \in \mathcal{U}(\mathcal{K})$. Since $T_1 = W|T|$,

$$S_1 = T_1T_1^* = W|T|^2W^* = W(T^*T)W^* \quad \Longrightarrow \quad W^*S_1W = T^*T.$$

On the other hand, if $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$, then $\langle T^*Te_k, e_k\rangle = \langle Te_k, Te_k\rangle = \|f_k\|^2 = c_k$, for every $k \in \mathbb{M}$. Therefore,

$$\mathcal{C}_{\mathcal{B}}\left(W^*S_1W\right) = \mathcal{C}_{\mathcal{B}}(T^*T) = M_{\mathcal{B},\mathbf{c}} \quad \Longrightarrow \quad \mathbf{c} \in \mathcal{C}\left[\mathcal{U}_{\mathcal{K}}(S_1)\right].$$

Conversely, suppose that there exists an extension $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ and $V \in \mathcal{U}(\mathcal{K})$ such that $M_{\mathcal{B},\mathbf{c}} = \mathcal{C}_{\mathcal{B}}(V^*S_1V)$, for some orthonormal basis $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K}$. Let $T = S_1^{1/2}V$. Since S is invertible, we have $R(T) = \mathcal{H}$ and $\dim\ker T = \dim\mathcal{H}_d$. Thus $\mathcal{F} = \{Te_k\}_{k\in\mathbb{M}}$ is a frame for $\mathcal{H}$, with frame operator $TT^*\big|_{\mathcal{H}} = S_1\big|_{\mathcal{H}} = S$. Since $T^*T = V^*S_1V$ and $\mathcal{C}_{\mathcal{B}}(V^*S_1V) = M_{\mathcal{B},\mathbf{c}}$, we have $\|Te_k\|^2 = \langle T^*Te_k, e_k\rangle = c_k$, for every $k \in \mathbb{M}$. Hence $\mathcal{F} \in F(S, \mathbf{c})$ with $e(\mathcal{F}) = \dim\mathcal{H}_d$. □

The finite-dimensional case. In this section we assume that $\mathcal{H}$ is finite dimensional. We shall consider separately the cases of frames of finite or infinite length. Suppose that $S \in \mathcal{M}_n(\mathbb{C})^+$ and $|\mathbb{M}| = m < \infty$. In this case, the classical Schur-Horn Theorem 2.2 gives a complete characterization of frame admissibility for $(S, \mathbf{c})$.

THEOREM 4.6. *Let $\mathbf{c} \in \mathbb{R}^m_{>0}$ and let $S \in \mathcal{G}l_n(\mathbb{C})^+$, with eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then, the pair $(S, \mathbf{c})$ is frame admissible if and only if*

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} c_i \quad \textit{for} \quad 1 \leq k \leq n-1, \qquad \textit{and} \qquad \sum_{i=1}^{n} b_i = \sum_{i=1}^{m} c_i.$$

In other words, if $\mathbf{c} \prec (b_1, \dots, b_n, 0, \dots, 0) \in \mathbb{R}^m$. □

This result was obtained in [5] and [16], from an operator theoretic point of view. Actually the proofs given there can be adapted so as to obtain a proof of the classical Schur-Horn theorem that is quite conceptual and simpler than those in the literature. Now, we consider frame admissibility for infinite sequences in finite dimensional Hilbert spaces. The case $S = I$ of the next result appeared in [4].

THEOREM 4.7. *Let* $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$. *Let* $S \in \mathcal{G}l_n(\mathbb{C})^+$, *with eigenvalues* $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. *Then the following conditions are equivalent:*

(1) *The pair* $(S, \mathbf{c})$ *is frame admissible.*
(2) $\sum_{i=1}^k b_i \geq U_k(\mathbf{c})$, *for every* $1 \leq k \leq n-1$, *and* $\sum_{i=1}^n b_i = \sum_{i\in\mathbb{N}} c_i$.

Proof. Let $\mathbf{b} = (b_1, \dots, b_n, 0, \dots, 0, \dots) \in \ell^\infty(\mathbb{N})^+$.
$2 \Rightarrow 1$: Let $\mathcal{H}$ be a infinite dimensional Hilbert space, and consider

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathbb{C}^n \oplus \mathcal{H}).$$

Then there exists an orthonormal basis $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K} = \mathbb{C}^n \oplus \mathcal{H}$ such that $S_1 = M_{\mathcal{B},\mathbf{b}}$. Hence, by Proposition 3.3,

$$U_k(S_1) = \sum_{i=1}^k b_i \geq U_k(\mathbf{c}), \quad \text{for every } k \in \mathbb{N}.$$

On the other hand, note that $L_k(S_1) = 0 \leq L_k(\mathbf{c})$ for every $k \in \mathbb{N}$ and $\sum_{i=1}^n b_i = \sum_{i\in\mathbb{N}} c_i$. Then, by Proposition 3.13, there exists a sequence $\{V_m\}_{m\in\mathbb{N}}$ in $\mathcal{U}(\mathcal{K})$ such that

$$\mathcal{C}_{\mathcal{B}}\left(V_m^* S_1 V_m\right) \xrightarrow[m\to\infty]{\|\ \|_1} M_{\mathbf{c}},$$

where $\|A\|_1 = \operatorname{tr}|A|$. Therefore, by Proposition 4.5, there exists a norm bounded sequence of epimorphisms $T_m : \mathcal{K} \to \mathbb{C}^n$ such that $T_m T_m^* = S$ for all $m \in \mathbb{N}$, and $(\|T_m(e_i)\|^2)_{i\in\mathbb{N}} \xrightarrow[m\to\infty]{\ell^1(\mathbb{N})} \mathbf{c}$. Then, by a standard diagonal argument, we can ensure the existence of a subsequence, which we still call $\{T_m\}_{m\in\mathbb{N}}$, such that

$$T_m(e_i) \xrightarrow[m\to\infty]{} f_i \in \mathbb{C}^n, \quad \text{with } \|f_i\|^2 = c_i \ \text{ for every } i \in \mathbb{N}.$$

Let $T_0 : \operatorname{span}\{\mathcal{B}\} \to \mathbb{C}^n$ be the unique (densely defined) operator such that $T_0(e_i) = f_i$ for every $i \in \mathbb{N}$. Note that T_0 is bounded because, if $x = \sum_{i=1}^r \alpha_i e_i$

and $C = \sum_{i\in\mathbb{N}} c_i = \operatorname{tr} S$, then

$$\|T_0(x)\| = \left\|\sum_{i=1}^{r} \alpha_i\, f_i\right\| \leq \sum_{i=1}^{r} |\alpha_i|\|f_i\| \leq \left(\sum_{i=1}^{r} c_i\right)^{1/2} \left(\sum_{i=1}^{r} |\alpha_i|^2\right)^{1/2} \leq C^{1/2}\|x\|.$$

The bounded extension of T_0 to $\mathcal{K}$ is denoted T.

CLAIM. $\|T_m - T\| \xrightarrow[m\to\infty]{} 0$.

Indeed, let $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ be such that $\sum_{i=i_0}^{\infty} c_i < \varepsilon$. Then there exists $m_1 \in \mathbb{N}$ such that

$$\sum_{i=i_0}^{\infty} \|T_m(e_i)\|^2 \leq \varepsilon, \text{ for every } m \geq m_1. \tag{16}$$

This is a consequence of the fact that $(\|T_m(e_i)\|^2)_{i=i_0}^{\infty} \xrightarrow[m\to\infty]{\ell^1(\mathbb{N})} (c_i)_{i=i_0}^{\infty}$. On the other hand, there exists $m_2 \geq m_1$ such that

$$\sum_{i=1}^{i_0-1} \|T_m(e_i) - f_i\|^2 \leq \varepsilon, \text{ for every } m \geq m_2. \tag{17}$$

Let $m \geq m_2$ and $x = \sum_{i=1}^{r} \alpha_i e_i \in \operatorname{span}\{\mathcal{B}\}$. By equations (16) and (17),

$$\begin{aligned}\|(T_m - T)(x)\|^2 &\leq \left(\sum_{i=1}^{r} |\alpha_i|^2\right)\left(\sum_{i=1}^{r} \|(T_m - T)(e_i)\|^2\right)\\ &\leq \|x\|^2 \left(\sum_{i=1}^{i_0-1} \|(T_m - T)(e_i)\|^2 + 2\sum_{i=i_0}^{\infty} \|T_m(e_i)\|^2 + \|T(e_i)\|^2\right)\\ &\leq 5\,\varepsilon\,\|x\|^2,\end{aligned}$$

which proves the claim. Therefore

$$TT^* = \lim_{m\to\infty} T_m T_m^* = S.$$

We have proved that the frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}} \in F(S, \mathbf{c})$.

$1 \Rightarrow 2$: This follows from Theorem 3.10, applied to S_1 and $\mathbf{c}$, and Proposition 4.5. □

REMARK 4.8. The statement of Theorem 4.7 can be reformulated in terms of finite rank operators and sequences in $\ell^1(\mathbb{N})$ in the following way: Let $\mathcal{K}$ be a separable, infinite dimensional Hilbert space. Let $S_1 \in L(\mathcal{K})^+$ be such that $\dim R(S_1) < \infty$. Then $\mathcal{C}[\mathcal{U}_{\mathcal{K}}(S_1)]$ is closed, as a subset of $\ell^1(\mathbb{N})$.

Indeed, suppose that $S_1 \neq 0$ (the case $S_1 = 0$ is trivial). Then there exists a sequence $\mathbf{b} = (b_1, \dots, b_m, 0, \dots, 0, \dots) \in \ell^1(\mathbb{N})^+$, with $b_m > 0$, and an orthonormal basis $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{K}$ such that $S_1 = M_{\mathcal{B},\mathbf{b}}$. Let $\mathbf{c} \in \ell^1(\mathbb{N})^+$. By Proposition 3.13, Condition 2 of Theorem 4.7 means that $\mathbf{c} \in \mathrm{cl}_{\|\cdot\|_1}\left(\mathcal{C}[\mathcal{U}_{\mathcal{K}}(S_1)]\right)$. But, by Proposition 4.5, Condition 1 of Theorem 4.7 means that $\mathbf{c} \in \mathcal{C}[\mathcal{U}_{\mathcal{K}}(S_1)]$.

Note that, although $\mathrm{cl}_{\|\cdot\|_1}\left(\mathrm{conv}(\Pi \cdot \mathbf{b})\right) = \mathrm{cl}_{\|\cdot\|_1}\left(\mathcal{C}[\mathcal{U}_{\mathcal{K}}(S_1)]\right) = \mathcal{C}[\mathcal{U}_{\mathcal{K}}(S_1)]$, as shown in Remark 3.14, it is not true that $\mathrm{conv}(\Pi \cdot \mathbf{b})$ is closed, as a subset of $\ell^1(\mathbb{N})^+$. For example, if $\mathbf{b} = (1, 0, 0, \dots)$, then, by Proposition 3.13,

$$\mathbf{c} = \left(\frac{1}{2^n}\right)_{n\in\mathbb{N}} \in \mathrm{cl}_{\|\cdot\|_1}\left(\mathcal{C}[\mathcal{U}_{\mathcal{K}}(e_1 \otimes e_1)]\right) = \mathrm{cl}_{\|\cdot\|_1}\left(\mathrm{conv}(\Pi \cdot \mathbf{b})\right).$$

Nevertheless, $\mathbf{c} \notin \mathrm{conv}(\Pi \cdot \mathbf{b})$, because every sequence in $\mathrm{conv}(\Pi \cdot \mathbf{b})$ has finite nonzero entries. In this case, $\mathbf{c} = \mathcal{C}_{\mathcal{B}}(x \otimes x) \in \mathcal{C}[\mathcal{U}_{\mathcal{K}}(e_1 \otimes e_1)]$, where $x = \sum_{n\in\mathbb{N}} 2^{-\frac{n}{2}} e_n$.

5. The infinite-dimensional case

Throughout this section $\mathcal{H}$ denotes a separable infinite dimensional Hilbert space. The first result gives necessary conditions for frame admissibility:

THEOREM 5.1. *Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$. If the pair $(S, \mathbf{c})$ is frame admissible, then*

$$\sum_{i\in\mathbb{N}} c_i = \infty, \text{ and}$$

$U_k(S) \geq U_k(\mathbf{c})$, for every $k \in \mathbb{N}$. In particular, $\limsup \mathbf{c} \leq \|S\|_e$.

Proof. Suppose that there exists a frame $\mathcal{F} \in F(S, \mathbf{c})$. Then, by Proposition 4.5, there exists an extension $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(\mathcal{K})^+,$$

then $\mathbf{c} \in \mathcal{C}\left[\mathcal{U}_{\mathcal{K}}(S_1)\right]$. Hence, $\sum_{i\in\mathbb{N}} c_i = \mathrm{tr}\, M_{\mathbf{c}} = \mathrm{tr}\, S_1 = \infty$. On the other hand, by Proposition 3.5, $U_k(S) = U_k(S_1)$ for every $k \in \mathbb{N}$. Then, applying Theorem 3.10, the statement follows. □

REMARK 5.2. Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$. Then, by Theorem 3.10 and Proposition 4.5, the following conditions are equivalent :

(1) $U_k(S) \geq U_k(\mathbf{c})$ for every $k \in \mathbb{N}$.
(2) There exists a sequence $\mathcal{F}_k = \{f_{ik}\}_{i\in\mathbb{N}}$, $k \in \mathbb{N}$ of frames on $\mathcal{H}$, such that S is the frame operator of every $\mathcal{F}_k$ and $\|f_{ik}\| \xrightarrow[k\to\infty]{} \sqrt{c_i}$ uniformly for $i \in \mathbb{N}$.

Indeed, note that the inequalities involving the maps L_k, $k \in \mathbb{N}$, can always be fulfilled if we consider a sufficiently large extension $\mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$. In this case, $\limsup \mathbf{c} \leq \|S\|_e$.

At this point we should note that the conditions of Theorem 5.1 are not sufficient to ensure that the pair $(S, \mathbf{c})$ is frame admissible, as Example 6.1 below shows. That is, we can not remove the closures in the equalities of Theorem 3.10, as it was already mentioned in [17], for the diagonalizable case.

In [16] (see also [4]) there is the following result which gives sufficient conditions for a pair $(S, \mathbf{c})$ in order to be frame admissible:

THEOREM 5.3 (Kornelson-Larson). *Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} \in l^\infty(\mathbb{N})^+$. Suppose that $\sum_{i\in\mathbb{N}} c_i = \infty$ and $\|\mathbf{c}\|_\infty < \|S\|_e$. Then the pair $(S, \mathbf{c})$ is frame admissible.* □

The following result, which generalizes Theorem 5.3, strengths slightly the necessary conditions for frame admissibility given by Theorem 5.1, to get sufficient conditions. A tight frame version of this result appeared in R. Kadison [14] and [15]. Recall the notation $P_2(S) = E[\,\|S\|_e, \|S\|\,]$, where E is the spectral measure of $S \in L(\mathcal{H})^+$.

THEOREM 5.4. *Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} \in l^\infty(\mathbb{N})^+$, such that $\sum_{i\in\mathbb{N}} c_i = \infty$. Assume one of the following two conditions:*

1. (a) $\operatorname{tr} P_2(S) = \infty$,
   (b) $U_k(S) \geq U_k(\mathbf{c})$ *for every* $k \in \mathbb{N}$, *and*
   (c) $\|S\|_e > \limsup(\mathbf{c})$.

2. (a) $\operatorname{tr} P_2(S) = r \in \mathbb{N}$,
   (b) $U_k(S) \geq U_k(\mathbf{c})$ *for* $1 \leq k \leq r$,
   (c) $U_k(S) > U_k(\mathbf{c})$, *for* $k > r$, *and*
   (d) $\|S\|_e > \limsup(\mathbf{c})$.

Then, the pair $(S, \mathbf{c})$ is frame admissible.

Proof. By Proposition 4.5, it suffices to show that there exists a sequence of unit vectors $\{x_k\}_{k\in\mathbb{N}}$ such that $S = \sum_{k\in\mathbb{N}} c_k\, x_k \otimes x_k$. Assume that the first condition holds. Then, since $\|S\|_e > \limsup(\mathbf{c})$, there exist $m_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$c_m \leq \|S\|_e - \varepsilon \quad \text{for } m \geq m_0$$

Let $\mu_1 \geq \mu_2 \cdots \geq \mu_n \geq \cdots$ be the sequence of eigenvalues of S^+, chosen as in Lemma 3.4. Let $\{y_n\}_{n\in\mathbb{N}}$ be an orthonormal system such that $S^+ y_n = \mu_n y_n$. Denote $\lambda_n = \mu_n + \|S\|_e$, $n \in \mathbb{N}$. Note that $\|S\| \geq \lambda_n \geq \|S\|_e$, and $S y_n = \lambda_n y_n$, $n \in \mathbb{N}$. By Proposition 3.5, for every $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} \lambda_i\, y_i \otimes y_i \leq S, \qquad \text{and} \qquad U_k(S) = \sum_{i=1}^{k} \lambda_i.$$

Let n_0 be the first integer such that

$$\sum_{i=1}^{n_0} c_i > \sum_{i=1}^{m_0} \lambda_i.$$

Then $n_0 \geq m_0 + 1$, and

$$h = \sum_{i=1}^{n_0} c_i - \sum_{i=1}^{m_0} \lambda_i \leq c_{n_0} < \|S\|_e \leq \lambda_{m_0+1}.$$

Let $\mathbf{c}_0 = (c_1, \ldots, c_{n_0})$. Since

$$\sum_{i=1}^{k} \lambda_i = U_k(S) \geq U_k(\mathbf{c}) \geq U_k(\mathbf{c}_0), \quad 1 \leq k \leq m_0,$$

we have $\mathbf{c}_0 \prec (\lambda_1, \ldots, \lambda_{m_0}, h, 0, \ldots, 0) \in \mathbb{R}^{n_0}$. Denote

$$S_1 = h\ y_{m_0+1} \otimes y_{m_0+1} + \sum_{i=1}^{m_0} \lambda_i\ y_i \otimes y_i \leq S,$$

and $S_2 = S - S_1$. Then the pair $(S_1, \mathbf{c}_0)$, acting on $\operatorname{span}\{y_1, \ldots, y_{m_0+1}\}$, satisfies the conditions of Theorem 4.6. Hence, there exists a set of unit vectors $\{x_1, \ldots, x_{n_0}\}$ such that $\sum_{i=1}^{n_0} c_i\ x_i \otimes x_i = S_1$. Note that $S_2 \geq 0$, $R(S_2)$ is closed (by Fredholm theory), and $\|S_2\|_e = \|S\|_e$. Then we can apply Theorem 5.3 to the pair $(S_2, \{c_i\}_{i>n_0})$, acting on $R(S_2)$. So, there exist unit vectors x_k, for $k > n_0$, such that

$$S_2 = \sum_{i=n_0+1}^{\infty} c_i\ x_i \otimes x_i.$$

Therefore we obtain the rank-one decomposition $S = \sum_{i\in\mathbb{N}} c_i\ x_i \otimes x_i$.

Assume Condition 2. Note that, by equations (8) and (11), the condition $\|S\|_e > \limsup(\mathbf{c})$ implies that $U_m(S) - U_m(\mathbf{c}) \xrightarrow[m\to\infty]{} \infty$. Therefore, by item (c), we can assume that there exists $\delta > 0$ such that

(1) $U_{r+k}(S) - \delta > U_{r+k}(\mathbf{c})$, for every $k \in \mathbb{N}$.
(2) There exists $m_0 \geq 1$ such that $c_m \leq \|S\|_e - \delta$ for $m \geq m_0$.

Let $m_1 = \max\{m_0, r+1\}$. Let $\mu_1 \geq \cdots \geq \mu_r$ be the greatest eigenvalues of S^+, and let $\{y_1, \ldots, y_r\}$ be an associated orthonormal set of eigenvectors. Denote

$$\lambda_i = \mu_i + \|S\|_e\,,\ 1 \leq i \leq r\,,\ \text{ and }\ \lambda_i = \|S\|_e - \frac{\delta}{2m_1}\,,\ \ r+1 \leq i \leq m_1+1\,.$$

Then, by Proposition 3.5,

(1) $U_k(S) = \sum_{i=1}^{k} \lambda_i$, for $1 \leq k \leq r$, and
(2) $U_k(\mathbf{c}) \leq U_k(S) - \delta \leq \sum_{i=1}^{k} \lambda_i$, for $r+1 \leq k \leq m_1+1$.

On the other hand, since $Q = E([\|S\|_e - \delta/2m_1, \|S\|_e))$ has infinite rank, there exists an orthonormal set $\{y_{r+1}, \dots, y_{m_1+1}\} \subseteq R(Q)$. Therefore

$$\sum_{i=1}^{m_1+1} \lambda_i \; y_i \otimes y_i \leq S.$$

Let n_0 be the first integer such that $\sum_{i=1}^{n_0} c_i > \sum_{i=1}^{m_1} \lambda_i$. Then $n_0 \geq m_1 + 1$ and

$$h = \sum_{i=1}^{n_0} c_i - \sum_{i=1}^{m_1} \lambda_i \leq c_{n_0} \leq \|S\|_e - \delta \leq \lambda_{m_1+1} \; .$$

Let $\mathbf{c}_0 = (c_1, \dots, c_{n_0})$. Since

$$\sum_{i=1}^{k} \lambda_i = U_k(S) \geq U_k(\mathbf{c}) \geq U_k(\mathbf{c}_0), \quad 1 \leq k \leq r, \quad \text{and}$$

$$\sum_{i=1}^{k} \lambda_i \geq U_k(S) - \delta \geq U_k(\mathbf{c}) \geq U_k(\mathbf{c}_0), \quad r+1 \leq k \leq m_1,$$

we have $\mathbf{c}_0 \prec (\lambda_1, \dots, \lambda_{m_1}, h, 0, \dots, 0) \in \mathbb{R}^{n_0}$. So, by Corollary 4.6, there exists a set of unit vectors $\{x_1, \dots, x_{n_0}\} \subseteq \mathcal{H}$ such that

$$S_1 = \sum_{i=1}^{m_1} \lambda_i \; y_i \otimes y_i + h \; y_{m_0+1} \otimes y_{m_0+1} = \sum_{i=1}^{n_0} c_i \; x_i \otimes x_i.$$

Since $S_1 \leq \sum_{i=1}^{m_1+1} \lambda_i \; y_i \otimes y_i$, we have $S_2 = S - S_1 \geq 0$ and $\|S_2\|_e = \|S\|_e$. As before, we apply Theorem 5.3 to the pair $(S_2, \{c_i\}_{i>n_0})$, acting on $R(S_2)$, and we obtain a decomposition

$$S_2 = \sum_{i=n_0+1}^{\infty} c_i \; x_i \otimes x_i.$$

Therefore we obtain the rank-one decomposition $S = \sum_{i\in\mathbb{N}} c_i \; x_i \otimes x_i$. □

Example 6.2 below shows that the Condition 2 (c) of Theorem 5.4 can not be dropped in general.

COROLLARY 5.5. *Let $0 < A \in \mathbb{R}$ and $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$ be such that $0 < c_i \leq A$, $i \in \mathbb{N}$. Denote $J = \{i \in \mathbb{N} : c_i = A\}$. Assume that*

$$\sum_{i\notin J} c_i = \infty, \qquad \text{and} \qquad \limsup_{i\notin J} c_i < A \quad \text{(or, equivalently,} \quad \sup_{i\notin J} c_i < A\text{).}$$

Then the pair $(AI, \mathbf{c})$ is admissible. This means that there exists a tight frame with norms prescribed by $\mathbf{c}$ and frame constant A. □

6. Some examples

In the following example we shall see that

$$U_k(S) > U_k(\mathbf{c}),\ k \in \mathbb{N}, \quad \text{and} \quad \|S\|_e = \limsup(\mathbf{c}) \quad \not\Rightarrow \quad F(S,\mathbf{c}) \neq \emptyset\ .$$

EXAMPLE 6.1. Let $S = I \in L(\mathcal{H})$ and $a \in (0,1)$. Let $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$ be given by $c_1 = p \in (0,1)$ and

$$c_k = \begin{cases} a^k & \text{if } k \neq 1 \text{ is odd,} \\ 1 - a^k & \text{if } k \text{ is even.} \end{cases}$$

Then $0 < c_k < 1$ for $k \in \mathbb{N}$, $\sum_k c_k = \infty = \sum_k (1 - c_k)$, and $\limsup \mathbf{c} = 1 = \|S\|_e$. Suppose that there exists a frame $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}} \in F(S,\mathbf{c})$. Then

$$\|x\|^2 = \sum_{k\in\mathbb{N}} |\langle x, f_k\rangle|^2, \quad \text{for every } x \in \mathcal{H}.$$

In particular, we get, for every $j \in \mathbb{N}$,

$$\|f_j\|^2 = \sum_{k\in\mathbb{N}} |\langle f_j, f_k\rangle|^2 = \|f_j\|^4 + \sum_{k\neq j} |\langle f_j, f_k\rangle|^2.$$

Thus, if $j \neq 1$, we obtain the inequality

$$|\langle f_1, f_j\rangle|^2 = |\langle f_j, f_1\rangle|^2 \leq \sum_{k\neq j} |\langle f_j, f_k\rangle|^2 = \|f_j\|^2 - \|f_j\|^4 = c_j(1 - c_j).$$

Therefore,

$$\begin{aligned} p = \|f_1\|^2 &\leq \|f_1\|^4 + \sum_{j\neq 1} c_j(1 - c_j) \\ &= p^4 + \sum_{j\neq 1} a^j(1 - a^j) = p^4 + \sum_{j\neq 1} a^j - \sum_{j\neq 1} a^{2^j} \\ &= p^2 + \frac{1}{1-a} - \frac{1}{1-a^2} = p^2 + \frac{a}{1-a^2}. \end{aligned} \tag{18}$$

Taking

$$p = \frac{1}{2}$$

and $a \in (0,1)$ such that

$$\frac{a}{1-a^2} < \frac{1}{4},$$

we get that

$$p > p^2 + \frac{a}{1-a^2},$$

contradicting Eq. (18). Hence, in this case, $F(S,\mathbf{c}) = \emptyset$. Note that the pair $(S,\mathbf{c})$ satisfies all of the necessary conditions of Theorem 5.1, because $U_k(S) = k = U_k(\mathbf{c})$ for every $k \in \mathbb{N}$.

In the second example we see that, in general,

$$U_k(S) \geq U_k(\mathbf{c}),\ k \in \mathbb{N} \quad \text{and} \quad \|S\|_e > \limsup(\mathbf{c}) \quad \not\Rightarrow \quad F(S,\mathbf{c}) \neq \emptyset\ .$$

EXAMPLE 6.2. Let $S = M_{\mathbf{s}}$ be the diagonal operator, with respect to an orthonormal basis of $\mathcal{H}$, given by $\mathbf{s} = \{1-(i+1)^{-1}\}_{i\in\mathbb{N}}$, and let $(c_i)_{i\in\mathbb{N}}$ be given by $c_1 = 1$ and $c_i = 1/2$ for every $i \geq 2$. Note that

- $1 = \|S\|_e > 1/2 = \limsup(\mathbf{c})$,
- $U_1(S) = U_1(\mathbf{c})$, and
- $U_k(S) = k > 1 + (k-1)/2 = U_k(\mathbf{c})$ for every $k \geq 2$.

Still, we have $F(S,\mathbf{c}) = \emptyset$. Indeed, suppose that there exists $\mathcal{F} \in F(S,\mathbf{c})$. Then, by Proposition 4.5 there exists an extension $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(\mathcal{K})^+,$$

then $\mathbf{c} \in \mathcal{C}\,[\mathcal{U}_{\mathcal{K}}(S_1)]$. Let $V \in \mathcal{U}(\mathcal{K})$ be such that, in a orthonormal basis $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$, $M_{\mathbf{c}} = \mathcal{C}_{\mathcal{B}}(V^*S_1V)$. Take $x = P_{\mathcal{H}}Ve_1$. We have that $\|x\| \leq 1$ and $\langle Sx, x\rangle = \langle M_{\mathbf{c}}e_1, e_1\rangle = c_1 = 1$, while $\|S\| = 1$. Then $Sx = x$, and 1 would be an eigenvalue of S, which is false. In this example, Condition 2 (c) of Theorem 5.4 does not hold, because $\|S\| = \|S\|_e$, which implies that $r = \operatorname{tr} P_2(S) = 0$; but $U_1(S) = 1 = U_1(\mathbf{c})$. Note that $\sum_k c_k = \infty = \sum_k (1-c_k)$, as in the previous example.

The excess of frames in $F(S,\mathbf{c})$. Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} = (c_i)_{i\in\mathbb{M}} \in \ell^\infty(\mathbb{M})^+$ be such that the pair $(S,\mathbf{c})$ is frame admissible. Then there can be many different types of frames $\mathcal{F} \in F(S,\mathbf{c})$. We consider the set

$$\operatorname{Null}(S,\mathbf{c}) = \{\ e(\mathcal{F}) : \mathcal{F} \in F(S,\mathbf{c})\ \}.$$

In the example below, we show that this set can be arbitrarily large. Moreover, this example shows that there exists an admissible pair $(S,\mathbf{a})$, satisfying just the necessary conditions of Theorem 5.1, and in this case $U_k(S) = U_k(\mathbf{a})$, $k \in \mathbb{N}$, and $\limsup \mathbf{a} = \|S\|_e$.

EXAMPLE 6.3. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$. Let

$$\mathbf{a} = \Big(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \dots\Big) \in \ell^\infty(\mathbb{N})^+\ , \qquad \text{and} \qquad S = M_{\mathcal{B},\mathbf{a}} \in \mathcal{G}l(\mathcal{H})^+.$$

Then the frame (Riesz basis) $\mathcal{F}_0 = \{a_n^{1/2}x_n\}_{n\in\mathbb{N}}$ has frame operator S, so that $\mathcal{F}_0 \in F(S,\mathbf{a})$. On the other hand, let

$$\mathcal{F}_1 = \Big\{\frac{1}{\sqrt{2}}\,x_2, x_4, \frac{1}{\sqrt{2}}\,x_2\,, x_6, \frac{1}{\sqrt{2}}\,x_1, x_8, \frac{1}{\sqrt{2}}\,x_3, x_{10}, \dots\Big\}\ .$$

It is easy to see that also $\mathcal{F}_1 \in F(S, \mathbf{a})$, but $e(\mathcal{F}_1) = 1$. Analogously,

$$\mathcal{F}_2 = \left\{ \frac{1}{\sqrt{2}}\, x_2, x_4, \frac{1}{\sqrt{2}}\, x_2\,, x_6, \frac{1}{\sqrt{2}}\, x_8, x_{10}, \frac{1}{\sqrt{2}}\, x_8, x_{12}, \frac{1}{\sqrt{2}}\, x_1, \dots \right\} \in F(S, \mathbf{a}),$$

with $e(\mathcal{F}_2) = 2$. In a similar way, we can construct frames $\mathcal{F}_k \in F(S, \mathbf{a})$ with $e(\mathcal{F}_k) = k$, for every $k \in \mathbb{N} \cup \{\infty\}$. Note that

$$\mathcal{F}_\infty = \left\{ \frac{1}{\sqrt{2}}\, x_1, x_4, \frac{1}{\sqrt{2}}\, x_2, x_8, \frac{1}{\sqrt{2}}\, x_2, x_{12}, \frac{1}{\sqrt{2}}\, x_3, x_{16}, \frac{1}{\sqrt{2}}\, x_6, x_{20}, \frac{1}{\sqrt{2}}\, x_6, \dots \right\}.$$

In other words, $\mathcal{F}_\infty$ is the frame induced by the bounded operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ given by

$$T(e_n) = \begin{cases} x_{4k} & \text{if} \quad n = 2k, \\ \frac{1}{\sqrt{2}}\, x_{2k-1} & \text{if} \quad n = 6k-5, \\ \frac{1}{\sqrt{2}}\, x_{4k-2} & \text{if} \quad n = 6k-3, \\ \frac{1}{\sqrt{2}}\, x_{4k-2} & \text{if} \quad n = 6k-1. \end{cases}$$

Therefore $\mathrm{Null}(S, \mathbf{a}) = \mathbb{N} \cup \{0, \infty\}$.

PROPOSITION 6.4. *Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $\mathbf{c} \in \ell^\infty(\mathbb{N})^+$. Assume that the pair $(S, \mathbf{c})$ is frame admissible and $\liminf \mathbf{c} < \min\ \sigma_e(S)$. Then $\mathrm{Null}(S, \mathbf{c}) = \{\infty\}$.*

Proof. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \in F(S, \mathbf{c})$, with $e(\mathcal{F}) = d$. By Proposition 4.5 there exists an extension $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(\mathcal{K})^+,$$

then $\mathbf{c} \in \mathcal{C}\left[\mathcal{U}_{\mathcal{K}}(S_1)\right]$. By Theorem 3.10, $\min\ \sigma_e(S_1) \leq \liminf\ \mathbf{c}$. But, if $\dim \mathcal{H}_d = e(\mathcal{F}) < \infty$, then $\sigma_e(S_1) = \sigma_e(S)$, which contradicts the fact that $\liminf \mathbf{c} < \min\ \sigma_e(S)$. □

REMARK 6.5. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a Parseval frame for $\mathcal{H}$ (i.e., it has frame operator $S = I_{\mathcal{H}}$). If $\liminf_{n\in\mathbb{N}} \|f_n\| < 1$, then, by Proposition 6.4, $e(\mathcal{F}) = \infty$. This results was proved in [3]

EXAMPLE 6.6. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\mathcal{B} = \{x_i\}_{i\in\mathbb{N}}$. Let

$$\mathbf{a} = (1, 2, 1, 2, \dots), \quad S = M_{\mathcal{B},\mathbf{a}} \in \mathcal{G}l(\mathcal{H})^+ \qquad \text{and} \qquad \mathbf{c} = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \dots\right).$$

We shall show that also $\mathrm{Null}(S, \mathbf{c}) = \mathbb{N} \cup \{0, \infty\}$. Note that, in this case,

$$\alpha_-(S) = 1 < \liminf \mathbf{c} = \frac{3}{2} = \limsup \mathbf{c} < 2 = \|S\|_e.$$

Indeed, take the Riesz basis $\mathcal{F}_0 = \{f_n\}_{n\in\mathbb{N}}$ given by

$$f_n = \begin{cases} \frac{x_n}{\sqrt{2}} + x_{n+1} & \text{if } n \text{ is odd,} \\ \\ \frac{-x_{n-1}}{\sqrt{2}} + x_n & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $\mathcal{F}_0 \in F(S, \mathbf{c})$. Using that

$$\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right) \prec (2, 2, 2, 0) \ ,$$

an arbitrary number of packs of four vectors with norm $\sqrt{3/2}$ associated to packs of three even places of the diagonal of S can be interlaced into the previous construction. Each of these packs adds excess 1 to the whole system. In this way, frames $\mathcal{F}_k \in F(S, \mathbf{c})$ with $e(\mathcal{F}_k) = k$ can be found for every $k \in \mathbb{N} \cup \{\infty\}$.

REMARK 6.7. Let $S \in \mathcal{G}l_n(\mathbb{C})^+$ and $\mathbf{c} \in \ell^\infty(\mathbb{M})^+$. If the pair $(S, \mathbf{c})$ is frame admissible, then $\text{Null}(S, \mathbf{c}) = \{|\mathbb{M}| - n\}$. Nevertheless, if $k > n$, $\mathbf{c} = (1, \ldots, 1) \in \mathbb{C}^k$ and $S = \frac{k}{n} I \in \mathcal{M}_n(\mathbb{C})$, then $F(S, \mathbf{c})$ is the set of spherical tight frames of k elements in $\mathbb{C}^n$. Dykema, Freeman, Korleson, Larson, Ordower and Weber [10] have shown that, in this case, $F(S, \mathbf{c})$ has a rich geometrical structure, with several orbits of qualitatively different elements.

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