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FROM A FORMULA OF KOVARIK TO THE PARAMETRIZATION OF IDEMPOTENTS IN BANACH ALGEBRAS

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ABSTRACT. If p, q are idempotents in a Banach algebra A and if $p+q-1$ is invertible, then the Kovarik formula provides an idempotent $k(p,q)$ such that $pA = k(p,q)A$ and $Aq = Ak(p,q)$. We study the existence of such an element in a more general situation. We first show that $p+q-1$ is invertible if and only if $k(p,q)$ and $k(q,p)$ both exist. Then we deduce a local parametrization of the set of idempotents from this equivalence. Finally, we consider a polynomial parametrization first introduced by Holmes and we answer a question raised at the end of his paper.

1. Introduction

Let X be a Banach space and let p, q be idempotents (i.e., $p^2 = p$ and $q^2 = q$) in the algebra $\mathcal{L}(X)$ of bounded linear operators on X. If the element $p + q - 1$ is invertible, then the formula

$$k := p(p + q - 1)^{-2} q \tag{1}$$

defines an idempotent in $\mathcal{L}(X)$. We call (1) the *Kovarik formula* since it first appeared in the proof of a theorem of Kovarik [5, Theorem 1, (ii)]. Moreover, k is the unique idempotent which shares its range with p and its nullspace with q (i.e., $\operatorname{Im} k = \operatorname{Im} p$ and $\operatorname{Ker} k = \operatorname{Ker} q$). More generally, if X is equal to the topological direct sum $\operatorname{Im} p \oplus \operatorname{Ker} q$, then we denote by $k(p,q)$ the idempotent k that is determined by the latter conditions. Thanks to the Kovarik formula, the invertibility of $p + q - 1$ is a sufficient condition for $k(p,q)$ to exist. The following example in $\mathcal{L}(\mathbb{R}^2)$,

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad k(p,q) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

shows that $k(p,q)$ may exist although $p + q - 1$ is not invertible.

The first aim of this paper is to give a necessary and sufficient condition for the element $p + q - 1$ to be invertible, with respect to the *interpolating*

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function $k(p,q)$. In fact, we will do so in the general context of a real Banach algebra A with unit.

Definition 1.1. Given two idempotents p and q in A, there is at most one idempotent k in A that satisfies both conditions $kA = pA$ and $Ak = Aq$. If it exists, then we denote it by $k(p,q)$.

This definition will be justified in Section 2; the reader may check that it generalizes the case $A = \mathcal{L}(X)$ treated in the above paragraph. Section 3 is devoted to the proof of the following equivalence:

$$p + q - 1 \text{ invertible} \iff k(p,q) \text{ and } k(q,p) \text{ exist.} \tag{2}$$

As observed by Esterle in [2], the Kovarik formula (1) yields an immediate proof of the implication "$\Longrightarrow$" in this context; we only prove it here for the sake of completeness (Proposition 3.1). The proof of the converse relies on a second formula (Proposition 3.2) that may be derived, for instance, from the study of the particular case $q = p^*$ in a C^*-algebra. We also give an illuminating interpretation of the equivalence (2) through a diagram which may inspire further applications.

Let p be an idempotent in A and let $\mathcal{I}_p(A)$ denote the connected component of p in the set of idempotents in A with respect to the topology inherited from the norm $\| \ \|$ of A. It is a well-known fact that $\mathcal{I}_p(A)$ is a submanifold of A which is modeled on the Banach space

$$T_p := \{h \in A \,|\, ph + hp = h\}. \tag{3}$$

We refer to another article of Kovarik [6, Proposition 2] for a proof of this claim. In fact, one has to adapt the latter from involutions ($\tau^2 = 1$) to idempotents through the application $\tau \mapsto (1+\tau)/2$. Now it is an easy exercise to check that the tangent space T_p is complemented in A by the commutant of p. As a consequence, we can see that $\mathcal{I}_p(A)$ is arcwise connected and that p is isolated in the set of idempotents if and only if it is central (i.e., $pa = ap$ for every $a \in A$). These properties have been proved in the complex case by Zemánek [8] and in the general case by Aupetit [1], independently from this geometric viewpoint.

After this brief account intended to motivate the study of the manifold $\mathcal{I}_p(A)$, we come to the main purpose of this paper, which is to parametrize a certain neighborhood of p with the help of the Kovarik formula (1). This is accomplished in Section 4, where the following result is proved:

Theorem 1.2. *Let U_p denote the set of idempotents q in A such that $p + q - 1$ is invertible and let ϕ_p be the map defined on U_p by*

$$\phi_p(q) := k(p,q) + k(q,p) - 2p.$$

Then ϕ_p is a homeomorphism from U_p onto the following open subset of T_p:

$$\Omega_p := \{h \in T_p \,|\, 2p - 1 + h \text{ invertible}\}.$$

Moreover, for every $h \in \Omega_p$ we have

$$\phi_p^{-1}(h) = (1+h)p(1+h^2)^{-1}p(1+h).$$

It should be noticed at this stage that U_p is an open neighborhood of p in $\mathcal{I}_p(A)$, which is not necessarily connected. The so-called *rational parametrization* of U_p given by the inverse formula $\phi_p^{-1}(h) = (1+h)p(1+h^2)^{-1}p(1+h)$ turns out to be strikingly easy to compute in many situations. For example, let us consider the algebra $\mathcal{M}_2(K)$ with $K = \mathbb{R}$ or $\mathbb{C}$,

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T_p \simeq K^2.$$

Then the above map ϕ_p^{-1} is nothing but the function

$$(s,t) \longmapsto \frac{1}{1+st} \begin{pmatrix} 1 & s \\ t & st \end{pmatrix}.$$

The remainder of this paper is motivated by the following result of Holmes [4, Theorem 7].

Theorem 1.3 (Holmes). *The polynomial map defined on the tangent space T_p by*

$$f_p(h) := p + h + hph - ph^2p - ph^2ph$$

is idempotent-valued. Moreover, it is a local homeomorphism from a certain neighborhood of 0 in T_p onto a neighborhood of p in $\mathcal{I}_p(A)$.

In particular, the map f_p is such that, for every $h \in T_p$, the polynomial path $t \in [0,1] \mapsto f_p(th)$ connects p and $f_p(h)$ in the set of idempotents. Moreover, the degree of the latter polynomial does not exceed 3. This has to be compared with the following result of Esterle [2]: if p and q lie in the same connected component of the set of idempotents, or more briefly if p and q are homotopic, then there exists a polynomial idempotent-valued path which connects p and q. Thus we may consider the minimal degree $d(p,q)$ of such polynomials. Following earlier work of Trémon who had treated the matrix case, Esterle and the author proved recently in [3] that the estimate $d(p,q) \leq 3$ holds for every pair of homotopic idempotents in a finite-dimensional real algebra. With a view towards a possible extension of this result to a larger class of Banach algebras, it might be of interest to note that for every $q \in f_p(T_p)$ we are provided with an explicit proof of the estimate $d(p,q) \leq 3$. Hence it would be desirable to have a simple characterization of the range of f_p.

The major drawback of Theorem 1.3 is that the proof given in [4] does not yield explicit neighborhoods. Therefore Holmes raises two questions at the end of his paper.

- *Must the functions f_p be 1-1?*
- *Must these functions be homeomorphisms?*

We answer these questions in Section 6, where we prove the following:

Theorem 1.4. *Let V_p denote the set of idempotents q in A such that $k(p,q)$ exists. Then the polynomial map f_p is a homeomorphism from T_p onto V_p. Moreover, for every $q \in V_p$ we have*

$$f_p^{-1}(q) = k(p,q) - p + (1-p)qp.$$

Thanks to the introduction of the function $k(p,q)$, our proof reduces to simple algebraic computations. The topological part of the proof, namely the continuity of $q \mapsto k(p,q)$ (Corollary 5.3), follows from the characterization of the idempotents that lie in V_p among those which are similar to p (Theorem 5.1).

Another direct consequence of Theorem 5.1 is that V_p is an open subset of $\mathcal{I}_p(A)$ whose closure is equal to $\mathcal{I}_p(A)$ if the set of invertible elements is everywhere dense in the subalgebra $(1-p)A(1-p)$ (Corollary 5.2). In particular, if there exists an increasing sequence $A_1 \subset A_2 \subset \cdots$ of finite-dimensional subalgebras in A such that $A = \overline{\bigcup_{n\geq 1} A_n}$, then the set of idempotents q which satisfy the estimate $d(p,q) \leq 3$ is everywhere dense in the connected component $\mathcal{I}_p(A)$. In fact, we know how to prove that the estimate $d(p,q) \leq 5$ holds for every pair of homotopic idempotents in such an algebra (i.e., an AF-algebra). However, the above observation seems to indicate that the optimal bound should be 3. The precise determination of this bound will be achieved in a forthcoming paper.

Final remark. If the algebra A has no unit, then we can consider its unitization $\widetilde{A} := A \oplus \mathbb{R}1$ and observe that $\mathcal{I}_p(A) = \mathcal{I}_p(\widetilde{A})$ for every idempotent p in A. Hence we may assume without loss of generality that A has a unit.

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2. Definition and first properties of $k(p,q)$

We shall assume throughout the whole paper that A is a real Banach algebra with unit denoted by 1. The letters p and q will always stand for idempotents in A, i.e., elements $p, q \in A$ such that $p^2 = p$ and $q^2 = q$.

Lemma 2.1. *The following conditions are equivalent:*

(i) *$pA = qA$ (respectively $Ap = Aq$).*

(ii) *$pq = q$ and $qp = p$ (respectively $pq = p$ and $qp = q$).*

Proof. Assume (i) first and observe that p and q both belong to $pA = qA$, so that $p = qx$ and $q = py$ for some $x, y \in A$. Then it is easily seen that the condition (ii) is satisfied. Assume conversely that $pq = q$ and $qp = p$. Then we have $qA = pqA \subset pA$ and $pA = qpA \subset qA$, so $pA = qA$ and we get (i)$\Longleftrightarrow$(ii). The equivalence of the respective conditions, i.e., (i)' $Ap = Aq$ and (ii)' $pq = p$ and $qp = q$, may be established in a similar manner. □

Remark 2.2. If p is an idempotent and if r is an element of A which satisfies $pr = r$ and $rp = p$, then $r^2 = (pr)^2 = p(rp)r = p^2r = pr = r$, so r is an idempotent. Hence condition (ii) of Lemma 2.1 implies that q is an idempotent, whereas condition (i) does not.

Definition 2.3. The relations $pA = qA$ and $Ap = Aq$ define two equivalence relations on the set of idempotents. We denote the equivalence classes by

$$\mathcal{F}_p := \{q \in A \,|\, q^2 = q,\, pA = qA\} \quad \text{and} \quad \mathcal{G}_p := \{q \in A \,|\, q^2 = q,\, Ap = Aq\}.$$

Lemma 2.4. *The subset $\mathcal{F}_p \cap \mathcal{G}_q$ is either empty or equal to a singleton. In particular, if $p = q$, then we have $\mathcal{F}_p \cap \mathcal{G}_p = \{p\}$.*

Proof. Assume $\mathcal{F}_p \cap \mathcal{G}_q$ is not empty and take an element k in it. In particular, k belongs to the equivalence class $\mathcal{F}_p$, so $\mathcal{F}_k = \mathcal{F}_p$. Since $k \in \mathcal{G}_q$, we also have $\mathcal{G}_k = \mathcal{G}_q$. Therefore $\mathcal{F}_p \cap \mathcal{G}_q = \mathcal{F}_k \cap \mathcal{G}_k$ and it follows immediately from Lemma 2.1 that $k = k'$ for every $k' \in \mathcal{F}_k \cap \mathcal{G}_k$. □

Definition 2.5. If the subset $\mathcal{F}_p \cap \mathcal{G}_q$ is not empty then we denote by $k(p, q)$ its unique element. In other words, we have

$$\mathcal{F}_p \cap \mathcal{G}_q = \emptyset \quad \text{or} \quad \mathcal{F}_p \cap \mathcal{G}_q = \{k(p, q)\}.$$

Thus Definition 1.1 is justified. We now give some obvious consequences of these algebraic definitions.

Proposition 2.6. *The following properties hold for every pair of idempotents.*

(1) *The element $k(p, q)$ exists if and only $k(1 - q, 1 - p)$ exists. In this case we have*

$$k(p, q) = 1 - k(1 - q, 1 - p).$$

(2) *If $k(p, q)$ and $k(q, p)$ both exist, then so does $k(k(q, p), k(p, q))$ and we have*

$$k(k(q, p), k(p, q)) = q.$$

Proof. First we notice that the equivalences $pA = qA \iff A(1 - p) = A(1-q)$ and $Ap = Aq \iff (1-p)A = (1-q)A$ follow easily from Lemma 2.1.

So we get the subset equalities $\mathcal{F}_p = 1 - \mathcal{G}_{1-p} = \{1 - k \,|\, k \in \mathcal{G}_{1-p}\}$ and $\mathcal{G}_q = 1 - \mathcal{F}_{1-q} = \{1 - k \,|\, k \in \mathcal{F}_{1-q}\}$, which yield the first property.

To prove the second property, it suffices to observe that q lies in both $\mathcal{F}_{k(q,p)}$ and $\mathcal{G}_{k(p,q)}$, by the definitions of $k(q,p)$ and $k(p,q)$. □

We conclude these preliminaries with two observations which illustrate the deep link between the existence of $k(p,q)$ and particular forms of arcwise connectedness in the set of idempotents. The first is just a generalization of the so-called *poor man's path* in the paper of Kovarik [5]; it involves affine segments $[a,b] := \{(1-t)a + tb \,|\, t \in [0,1]\}$ of A which are actually contained in the set of idempotents. The second goes back to Esterle [2].

PROPOSITION 2.7. *Assume p and q are such that $k(p,q)$ exists. Then the following properties hold.*

(1) *The segments $[p, k(p,q)]$ and $[k(p,q), q]$ are both contained in the set of idempotents. Moreover, the functions $r \mapsto k(p,r)$ and $r \mapsto k(r,q)$ are well-defined on each of these segments.*
(2) *If we set $u := q - k(p,q)$ and $v := p - k(p,q)$, then we have $u^2 = v^2 = 0$ and $q = (1+u)(1+v)p(1-v)(1-u)$. In particular, the element $\sigma := (1+u)(1+v)$ is invertible with inverse $\sigma^{-1} = (1-v)(1-u)$ and the idempotents p and q are similar.*

Proof. Take an element $r = (1-t)p + tk(p,q)$ in $[p, k(p,q)]$. It follows from the definition of $k(p,q)$ and from Lemma 2.1 that we have $pr = (1-t)p^2 + tpk(p,q) = (1-t)p + tk(p,q) = r$ and $rp = (1-t)p^2 + tk(p,q)p = (1-t)p + tp = p$. So r is an idempotent by Remark 2.2 and we deduce again from Lemma 2.1 that r lies in the equivalence class $\mathcal{F}_p = \mathcal{F}_{k(p,q)}$. Then it is obvious that $k(p,r)$ and $k(r,q)$ exist, for we have $k(p,r) = r$ and $k(r,q) = k(p,q)$. We can verify in a similar manner that every element $s \in [k(p,q), q]$ is an idempotent such that $k(p,s) = k(p,q)$ and $k(s,q) = s$ exist. This completes the proof of the first property.

To prove the second property, first observe that the definition of $k(p,q)$ and Lemma 2.1 imply, by direct computations, the relations $u^2 = v^2 = vp = uq = 0$, $pv = v$ and $uq = q$. By expanding and simplifying we then get the identities $(1+v)p(1-v) = k(p,q) = (1-u)q(1+u)$, from which the result follows. □

COROLLARY 2.8. *The set of idempotents q such that $k(p,q)$ exists, which is denoted by V_p, is arcwise connected. Moreover, for every $q \in V_p$, there exists a polynomial idempotent-valued path which connects p and q with degree 3 at most.*

Proof. The first assertion is a direct consequence of Property (1) of Proposition 2.7. Now if q lies in V_p, it follows from Property (2) of this proposition that we can write $q = (1+u)(1+v)p(1-v)(1-u)$ with $u^2 = v^2 = 0$. Following Esterle's construction [2], we then consider the polynomial map $t \mapsto (1+tu)(1+tv)p(1-tv)(1-tu)$ whose values are all similar to p. Thus we obtain a polynomial path which connects p and q in the set of idempotents. Since $vp = 0$, it is easily seen that its degree does not exceed 3 and the proof is complete. □

3. A necessary and sufficient condition for the element $p+q-1$ to be invertible

To begin with, we recall the well-known necessary condition that has already been used, for instance, in [1], [2], [7], [3].

PROPOSITION 3.1 (Kovarik formula). *If the element $p+q-1$ is invertible then the element $k(p,q)$ exists and we have the formula*

$$k(p,q) = p(p+q-1)^{-2}q.$$

Proof. We first note that we have $p(p+q-1) = (p+q-1)q = pq$. So if we set $\omega := (p+q-1)^2$, this yields the relations $p\omega = \omega p = pqp$ and $q\omega = \omega q = qpq$. The element ω is invertible by assumption, so the latter equations imply in particular that ω^{-1} commutes with p and q. Then it follows from a routine verification that the element $k := p\omega^{-1}q$ fulfills the required conditions, namely $k^2 = k$, $kp = p$ and $pk = k$ (i.e., $k \in \mathcal{F}_p$ by Lemma 2.1), $kq = k$ and $qk = q$ (i.e., $k \in \mathcal{G}_q$). So $k(p,q)$ exists and it is equal to k. □

By the symmetry of the assumption in Proposition 3.1, we point out that the invertibility of $p+q-1$ also implies the existence of $k(q,p)$. In fact, it turns out that the simultaneous existence of $k(p,q)$ and $k(q,p)$ implies the invertibility of $p+q-1$. As claimed in the introduction, this converse statement arises quite naturally from the study of the particular case below.

Assume for a moment that A is the algebra $\mathcal{L}(H)$ of bounded linear operators on a Hilbert space H and let p be an idempotent in $\mathcal{L}(H)$, that is, a (possibly oblique) projection onto $\operatorname{Im} p$ along $\operatorname{Ker} p$. Then p^* is the projection onto $\operatorname{Im} p^* = (\operatorname{Ker} p)^\perp$ along $\operatorname{Ker} p^* = (\operatorname{Im} p)^\perp$. So $k(p,p^*)$ and $k(p^*,p)$ both exist since they are equal, respectively, to the orthogonal projections onto $\operatorname{Im} p$ and $(\operatorname{Ker} p)^\perp$. In addition to this first observation, we note that the element $(p+p^*-1)^2 = 1-(p-p^*)^2$ is invertible since it is of the form $1+u^*u$ with $u = p-p^*$. So Proposition 3.1 provides us with the following formulas:

$$k(p,p^*) = p(p+p^*-1)^{-2}p^* \quad \text{and} \quad k(p,p^*) = p^*(p+p^*-1)^{-2}p.$$

Since $(p+p^*-1)^{-2}$ commutes with p and p^*, we therefore obtain

$$\begin{aligned} k(p,p^*)+k(p^*,p)-1 &= (pp^*+p^*p-(p+p^*-1)^2)(p+p^*-1)^{-2} \\ &= (p+p^*-1)(p+p^*-1)^{-2} \\ &= (p+p^*-1)^{-1}. \end{aligned}$$

Returning to the general case of a real Banach algebra, we can generalize the above computation as follows.

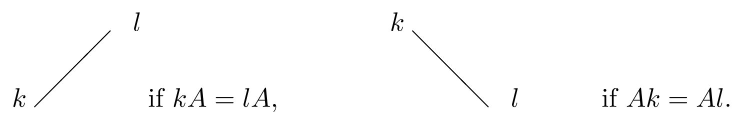
Proposition 3.2. *If the elements $k(p,q)$ and $k(q,p)$ both exist then $p+q-1$ is invertible with inverse given by the formula*

$$(p+q-1)^{-1} = k(p,q)+k(q,p)-1.$$

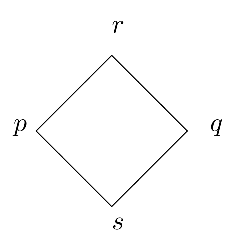
Proof. Set $k := k(p,q) \in \mathcal{F}_p \cap \mathcal{G}_q$ and $k' := k(q,p) \in \mathcal{F}_q \cap \mathcal{G}_p$. We recall that Lemma 2.1 implies the following relations: $kp = p, pk = k, kq = k, qk = q, k'p = k', pk' = p, k'q = q$ and $qk' = k'$. Then it only remains to expand the products $(p+q-1)(k+k'-1)$ and $(k+k'-1)(p+q-1)$. In fact, after some immediate cancellations we get $(p+q-1)(k+k'-1) = (k+k'-1)(p+q-1) = 1$; the details are left to the reader. □

Thus the equivalence (2) announced in the introduction is now established: the element $p+q-1$ is invertible if and only if the elements $k(p,q)$ and $k(q,p)$ both exist.

These properties may be represented by a simple diagram constructed according to the following rule: Given two idempotents k, l in A, we draw



Then the simultaneous existence of $k(p,q)$ and $k(q,p)$ is equivalent to the existence of two idempotents r and s which fulfill the diagram below.



Conversely, if such a diagram makes sense, then the following properties hold:

(i) $(p+q-1)(r+s-1) = (r+s-1)(p+q-1) = 1.$

(ii) $r = k(p, q)$, $s = k(q, p)$, $p = k(r, s)$ and $q = k(s, r)$.

Moreover, the Kovarik formula may be applied to compute each the four idempotents above.

4. Rational parametrization

Let p be an idempotent in A. We recall that the connected component of p in the set of idempotents is denoted by $\mathcal{I}_p(A)$ and we set

$$U_p := \{q \in A \,|\, q^2 = q,\, p + q - 1 \text{ invertible}\},$$
$$T_p := \{h \in A \,|\, ph + hp = h\},$$
$$\Omega_p := \{h \in T_p \,|\, 2p - 1 + h \text{ invertible}\}.$$

It is obvious that T_p is a closed subspace of A, so it is a Banach space itself. Since $(2p-1)^2 = 1$, the element $2p-1$ is invertible, so p lies in U_p and 0 lies in Ω_p. Moreover, the fact that the set of invertible elements is open in a Banach algebra implies that U_p is open in the set of idempotents and that Ω_p is open in T_p. In fact, it follows from Proposition 3.1 and from Proposition 2.7(1) that U_p is contained in $\mathcal{I}_p(A)$.

The purpose of this section is to construct a homeomorphism $\phi_p : U_p \longrightarrow \Omega_p$ from the open neighborhood U_p of p in A onto the open neighborhood Ω_p of 0 in T_p. We begin with an alternate description of T_p.

Lemma 4.1. *The Banach space T_p is equal to the topological direct sum*

$$T_p = pA(1 - p) \oplus (1 - p)Ap$$

of the closed subspaces $pA(1-p)$ and $(1-p)Ap$, which appear in the following descriptions of the equivalence classes $\mathcal{F}_p = \{q \in A \,|\, q^2 = q,\, pA = qA\}$ and $\mathcal{G}_p = \{q \in A \,|\, q^2 = q,\, Ap = Aq\}$ as affine subspaces of A:

$$\mathcal{F}_p = p + pA(1 - p) \quad \text{and} \quad \mathcal{G}_p = p + (1 - p)Ap.$$

The mapping $h \mapsto (p + ph, p + hp)$ is 1-1 and sends T_p onto $\mathcal{F}_p \times \mathcal{G}_p$ with inverse $(q, r) \mapsto q + r - 2p$.

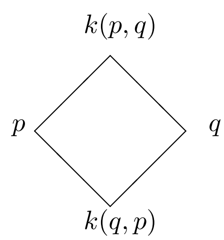
Proof. It is easily seen that the closed subspaces $pA(1-p)$ and $(1-p)Ap$ are contained in T_p with trivial intersection, i.e., $pA(1-p) \oplus (1-p)Ap \subset T_p$. Now assume that $h = ph + hp$ lies in T_p. Then $h(1 - p) = ph(1 - p) + hp(1 - p) = ph(1 - p)$, so $ph = h - hp = h(1 - p) = ph(1 - p)$ lies in $pA(1 - p)$. We can prove similarly that hp lies in $(1 - p)Ap$. Hence $T_p = pA(1 - p) \oplus (1 - p)Ap$. We now prove that $\mathcal{F}_p = p + pA(1 - p)$. Take q in $p + pA(1 - p)$ and write $q = p + px(1 - p)$. By direct computations it follows that the relations $q^2 = q$, $pq = q$ and $qp = p$ hold. So q lies in $\mathcal{F}_p$ by Lemma 2.1 and the inclusion $p + pA(1 - p) \subset \mathcal{F}_p$ is proved. Now assume that q lies in $\mathcal{F}_p$ and set $x := q - p$. Then by Lemma 2.1 we get the relations $px = x$ and $xp = 0$. Hence $x = px1 = px(p + 1 - p) = pxp + px(1 - p) = px(1 - p)$ and so $q = p + px(1 - p)$

lies in $p + pA(1-p)$. Thus we get the first relation, $\mathcal{F}_p = p + pA(1-p)$. The second relation, $\mathcal{G}_p = p + (1-p)Ap$, may be established in a similar manner, or derived from the first using Proposition 2.6(1). The latter follows by direct computations with the given maps; we leave the details to the reader. □

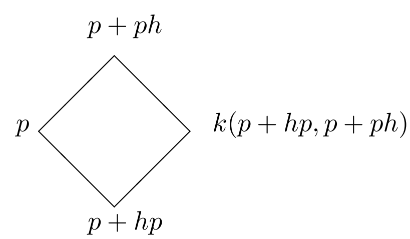
In other words, the tangent space T_p may be identified with the product space $\mathcal{F}_p \times \mathcal{G}_p$. We also point out that the affine structure of $\mathcal{F}_p$ and $\mathcal{G}_p$ implies that these spaces are both contained in the connected component $\mathcal{I}_p(A)$. Thus the first property of Proposition 2.7 becomes obvious.

We can summarize the principle of the so-called *coordinates map* ϕ_p and that of the *parametrization map* in the following two diagrams (see the end of the preceding section).

Coordinates



Parametrization



By the equivalence (2), the left-hand diagram makes sense if and only if $p + q - 1$ is invertible. So $(k(p,q), k(q,p)) \in \mathcal{F}_p \times \mathcal{G}_p$ is a natural pair of coordinates for every q in U_p. After composition with the map $(q,r) \mapsto q + r - 2p$, which sends $\mathcal{F}_p \times \mathcal{G}_p$ onto T_p, we obtain

$$\phi_p : q \longmapsto k(p,q) + k(q,p) - 2p,$$

which is a well-defined map from U_p into T_p. Since $2p - 1 + \phi_p(q) = k(p,q) + k(q,p) - 1$, Proposition 3.2 implies that ϕ_p actually takes its values in Ω_p. Moreover, by Proposition 3.1, the Kovarik formula may be applied to compute $k(p,q)$ and $k(q,p)$. Hence ϕ_p is easily seen to be continuous on U_p.

As illustrated by the right-hand diagram above, the map

$$\theta : h \longmapsto k(p + hp, p + ph)$$

is well-defined on Ω_p. Indeed, if the pair of idempotents $(p + hp, p + ph)$ is such that $(p+hp)+(p+ph)-1 = 2p-1+h$ is invertible, then Proposition 3.1 applies. Thus θ is continuous on Ω_p and it follows from Proposition 3.2 that $p + \theta(h) - 1$ is invertible with inverse $2p - 1 + h$ for every $h \in \Omega_p$. Hence θ takes its values in U_p and the Kovarik formula yields

$$\theta(h) = (1+h)p(1+h^2)^{-1}p(1+h),$$

after an easy simplification using the identity $(2p-1+h)^2 = 1 + h^2$, which is satisfied by every $h \in \Omega_p$.

It only remains to check that $\phi_p \circ \theta = \mathrm{Id}_{\Omega_p}$ and $\theta \circ \phi_p = \mathrm{Id}_{U_p}$. Let h lie in Ω_p first and observe that $k(p, \theta(h)) = p + ph$ and $k(\theta(h), p) = p + hp$; this is a direct consequence of Definition 2.5, which may also be seen from the right-hand diagram above. Then $\phi_p(\theta(h)) = k(p, \theta(h)) + k(\theta(h), p) - 2p = ph + hp = h$ and the first identity follows. Now if q lies in U_p, it is easy to check that $p + \phi_p(q)p = k(q, p)$ and $p + p\phi_p(q) = k(p, q)$ with the help of Lemma 2.1. Thus $\theta(\phi_p(q)) = k(p + \phi_p(q)p, p + p\phi_p(q)) = k(k(q, p), k(p, q))$, and finally $\theta(\phi_p(q)) = q$ by Proposition 2.6(2). Hence the second identity holds and we see that f is the inverse of ϕ_p. This completes the proof of Theorem 1.2.

5. Characterization of similar idempotents

Let p be an idempotent in A. Having exploited the symmetries of the Kovarik formula in the two preceding sections, we now turn our attention to the asymmetrical properties of the function $k(p, q)$. Consequently, we will no longer consider the open set $U_p = \{q \in A \mid q^2 = q,\ p + q - 1 \text{ invertible}\}$, and instead focus on the larger sets

$$V_p := \{q \in A \mid q^2 = q,\ k(p, q) \text{ exists}\}$$

and

$$W_p := \{q \in A \mid q^2 = q,\ k(q, p) \text{ exists}\}.$$

We point out that the equivalence (2) may be restated as follows:

$$U_p = V_p \cap W_p.$$

The purpose of this section is to show that both V_p and W_p are open connected neighborhoods of p in the component $\mathcal{I}_p(A)$ and that the mappings $q \mapsto k(p, q)$ and $q \mapsto k(q, p)$ are continuous on V_p and W_p, respectively. We note that Proposition 2.6(1) allows us to restrict our attention to V_p.

We have already proved in Corollary 2.8 that the set V_p is arcwise connected, so it is contained in $\mathcal{I}_p(A)$. Hence, as is apparent from the proof of this corollary, every element in V_p is similar to p. This raises the question: Given an idempotent q which is similar to p, how can we determine whether or not q lies in V_p?

The answer is given below. It requires the introduction of the Peirce decomposition of the algebra A with respect to p, i.e., the identification

$$a \leftrightarrow \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}$$

between the elements $a \in A$ and their coefficients in the subspace decomposition $A = pAp \oplus pA(1-p) \oplus (1-p)Ap \oplus (1-p)A(1-p)$. We recall that the Peirce decomposition provides a compatibility between operations on A and matrix block computations.

THEOREM 5.1. *Let σ be an invertible element and set $q := \sigma p \sigma^{-1}$. If σ has the form*

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in the Peirce decomposition of A with respect to p, then the element $k(p,q)$ exists if and only if the coefficient d is invertible in the subalgebra $(1-p)A(1-p)$. In this case we have

$$k(p,q) = \begin{pmatrix} p & -bd^{-1} \\ 0 & 0 \end{pmatrix}.$$

Proof. Assume that d is invertible in $(1-p)A(1-p)$ and set $k := p - bd^{-1}$. Observe that k belongs to $p+pA(1-p) = \mathcal{F}_p$ (cf. Lemma 4.1) by construction. Thus, in order to establish the existence of $k(p,q)$ and the required formula, it only remains to show that k lies in $\mathcal{G}_q$.

According to Lemma 2.1, we have $k \in \mathcal{G}_q$ if and only if $kq = k$ and $qk = q$. We compute

$$k\sigma = \begin{pmatrix} p & -bd^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & b - bd^{-1}d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & 0 \end{pmatrix}.$$

Then it is easily seen that $k\sigma(1-p) = 0$, so $k - kq = k\sigma(1-p)\sigma^{-1} = 0$, so the first relation is established.

In order to prove the second relation, namely $qk = q \Longleftrightarrow p\sigma^{-1}k = p\sigma^{-1}$, we introduce the coefficients of σ^{-1}:

$$\sigma^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

All we need to know about these coefficients is that they satisfy $\alpha b + \beta d = 0$. To see this, we compute the (1,2)-coefficient in the product $\sigma^{-1}\sigma$:

$$\sigma^{-1}\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & \alpha b + \beta d \\ * & * \end{pmatrix}.$$

Since $\sigma^{-1}\sigma = 1$, this coefficient must be equal to 0 and our claim is proved. Moreover, we get

$$p\sigma^{-1}k = \begin{pmatrix} \alpha & -\alpha bd^{-1} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p\sigma^{-1} = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$$

by direct computations. The relation $p\sigma^{-1}k = p\sigma^{-1}$ follows from this, since we have $(-\alpha b)d^{-1} = (\beta d)d^{-1} = \beta$.

We now prove that the existence of $k(p,q)$ implies the invertibility of d. So assume that $k := k(p,q)$ exists. Since $k \in \mathcal{F}_p = p + pA(1-p)$, we can write

$$k = \begin{pmatrix} p & x \\ 0 & 0 \end{pmatrix}.$$

Moreover, $k \in \mathcal{G}_q$ so $kq = k$ and $qk = q$. Hence we get the relations $k\sigma(1-p) = 0$ and $p\sigma^{-1}(1-k) = 0$, which imply

$$b + xd = 0 \quad \text{and} \quad \beta - \alpha x = 0$$

by matrix computations. We also need the three relations

$$\gamma b + \delta d = 1 - p, \quad c\beta + d\delta = 1 - p \quad \text{and} \quad c\alpha + d\gamma = 0,$$

which follow directly from the matrix computation of the identities $\sigma^{-1}\sigma = 1$ and $\sigma\sigma^{-1} = 1$. Then we get $(\delta - \gamma x)d = \delta d + \gamma(-xd) = \delta d + \gamma b = 1 - p$ and $d(\delta - \gamma x) = d\delta + (-d\gamma)x = d\delta + c\alpha x = d\delta + c\beta = 1 - p$. Thus d is invertible in $(1-p)A(1-p)$ with inverse

$$d^{-1} = \delta - \gamma x,$$

and the proof is complete. □

Corollary 5.2. *The set V_p is an open connected neighborhood of p in $\mathcal{I}_p(A)$.*

Proof. Let $q_0 \in V_p$ and let σ_{q_0} be an invertible element such that $q_0 = \sigma_{q_0} p \sigma_{q_0}^{-1}$. Now if q is any idempotent in A, it is a well-known trick to introduce the element $\tau_q := 1 - q_0 - q + 2qq_0$ in order to prove that q and q_0 are similar if they are close enough from each other. As a matter of fact, we have $\tau_q q_0 = q\tau_q (= qq_0)$ and we can write $\tau_q = 1 - (q_0 - q)(2q_0 - 1)$. So if $\|q - q_0\| < \|2q_0 - 1\|^{-1}$, then τ_q is invertible and we get $q = \tau_q q_0 \tau_q^{-1}$. Thus we can set $\sigma_q := \tau_q \sigma_{q_0}$, so that we have

$$q = \sigma_q p \sigma_q^{-1} \quad \text{with} \lim_{q \to q_0} \sigma_q = \sigma_{q_0}$$

for $\|q - q_0\| < \|2q_0 - 1\|^{-1}$. Now write the Peirce decomposition of σ_q with respect to p as

$$\sigma_q = \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix}.$$

By Theorem 5.1 the coefficient d_{q_0} is invertible in $(1-p)A(1-p)$. Moreover, $d_q = (1-p)\sigma_q(1-p) \to d_{q_0}$ when $q \to q_0$ and the set of invertible elements is open in the Banach algebra $(1-p)A(1-p)$. Hence the coefficient d_q is also invertible in an open neighborhood of q_0, say Ω, in the set of idempotents. Using Theorem 5.1 again, it follows that Ω is contained in V_p. Thus V_p is open in $\mathcal{I}_p(A)$. Since it has already been shown in Corollary 2.8 that V_p is arcwise connected, the proof is complete. □

Corollary 5.3. *The mapping $q \longmapsto k(p, q)$ is continuous on V_p.*

Proof. Consider the neighborhood Ω of q_0 exhibited in the proof of Corollary 5.2 above. For every $q \in \Omega$ we have $k(p, q) = p - b_q d_q^{-1}$ by Theorem 5.1. The asserted continuity is now obvious. □

6. Polynomial parametrization

This final section is devoted to the proof of Theorem 1.4. Let p be a fixed idempotent in A and let $f_p : T_p \longrightarrow A$ denote the polynomial function exhibited by Holmes and defined for every $h \in T_p$ by the formula

$$f_p(h) := p + h + hph - ph^2p - ph^2ph.$$

To begin with, we interpret this function in the Peirce decomposition of A with respect to p. Since $T_p = pA(1-p) \oplus (1-p)Ap$ (cf. Lemma 4.1), an element $h \in A$ lies in T_p if and only if it has the form

$$h = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$$

in this decomposition. Observe that $x = ph$ and $y = hp$. By direct matrix computations it follows that $f_p(h)$ is decomposed as

$$f_p(h) = \begin{pmatrix} p - xy & x - xyx \\ y & yx \end{pmatrix}. \tag{4}$$

Let us consider an element $\sigma_h \in A$, defined by $\sigma_h := 1 - ph + hp - ph^2p$ or, equivalently, by

$$\sigma_h := \begin{pmatrix} p - xy & -x \\ y & 1-p \end{pmatrix}. \tag{5}$$

It is easy to verify that σ_h is invertible with inverse

$$\sigma_h^{-1} = \begin{pmatrix} p & x \\ -y & 1 - p - yx \end{pmatrix},$$

i.e., $\sigma_h^{-1} = 1 + ph - hp - (1-p)h^2(1-p)$. Another matrix computation gives the similarity relation

$$f_p(h) = \sigma_h p \sigma_h^{-1};$$

the reader can check the details. Since $(1-p)\sigma_h(1-p) = 1-p$ is obviously invertible in $(1-p)A(1-p)$, it follows from Theorem 5.1 that $\sigma_h p \sigma_h^{-1}$ lies in V_p. Thus we see that $f_p : T_p \longrightarrow V_p$ is a continuous map from T_p into V_p.

We now consider the map $\psi : V_p \longrightarrow T_p$ defined for every $q \in V_p$ by

$$\psi(q) := k(p, q) - p + (1-p)qp.$$

It follows from Corollary 5.3 that ψ is continuous on V_p. Hence it only remains to show that ψ is the inverse of f_p, i.e., $\psi \circ f_p = \mathrm{Id}_{T_p}$ and $f_p \circ \psi = \mathrm{Id}_{V_p}$.

Let $h = x \oplus y \in T_p = pA(1-p) \oplus (1-p)Ap$ and recall that $f_p(h) = \sigma_h p \sigma_h^{-1}$, where σ_h has the matrix form (5). Then Theorem 5.1 implies

$$k(p, f_p(h)) = \begin{pmatrix} p & x \\ 0 & 0 \end{pmatrix}.$$

Moreover, we derive from (4) the coefficient $(1-p)f_p(h)p = y$ and so $\psi(f_p(h)) = k(p, f_p(h)) - p + (1-p)f_p(h)p = p + x - p + y = x + y = h$. Thus we have established the first identity, namely $\psi \circ f_p = \mathrm{Id}_{T_p}$.

The second identity is a little bit more difficult to prove. Take $q \in V_p$ and consider the Peirce decompositions of q and $k(p,q) \in \mathcal{F}_p = p \oplus pA(1-p)$ with respect to p,

$$q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad k(p,q) = \begin{pmatrix} p & \beta \\ 0 & 0 \end{pmatrix}.$$

Then it is easily seen that $\psi(q) = k(p,q) - p + (1-p)qp$ has the following matrix form

$$\psi(q) = \begin{pmatrix} 0 & \beta \\ c & 0 \end{pmatrix}.$$

So by formula (4) we get

$$f_p(\psi(q)) = \begin{pmatrix} p - \beta c & \beta - \beta c \beta \\ c & c\beta \end{pmatrix}. \tag{6}$$

Now recall that the condition $k(p,q) \in \mathcal{G}_q$ is equivalent, by Lemma 2.1, to the relations $k(p,q)q = k(p,q)$ and $qk(p,q) = q$. By matrix computations, the latter relations become, respectively,

$$\begin{pmatrix} a + \beta c & b + \beta d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & \beta \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & a\beta \\ c & c\beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In particular, we get the equations $p - \beta c = a$, $c\beta = d$ and $a\beta = b$. Also $\beta - \beta c \beta = (p - \beta c)\beta = a\beta = b$, and after substitution in (6), we finally obtain $f_p(\psi(q)) = q$. Thus $f_p \circ \psi = \mathrm{Id}_{V_p}$, and the proof is complete.

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