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A BRUNN-MINKOWSKI THEORY FOR MINIMAL SURFACES

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Abstract. The aim of this paper is to motivate the development of a Brunn-Minkowski theory for minimal surfaces. In 1988, H. Rosenberg and E. Toubiana studied a sum operation for finite total curvature complete minimal surfaces in $\mathbb{R}^3$ and noticed that minimal hedgehogs of $\mathbb{R}^3$ constitute a real vector space [14]. In 1996, the author noticed that the square root of the area of minimal hedgehogs of $\mathbb{R}^3$ that are modelled on the closure of a connected open subset of $\mathbb{S}^2$ is a convex function of the support function [5]. In this paper, the author (i) gives new geometric inequalities for minimal surfaces of $\mathbb{R}^3$; (ii) studies the relation between support functions and Enneper-Weierstrass representations; (iii) introduces and studies a new type of addition for minimal surfaces; (iv) extends notions and techniques from the classical Brunn-Minkowski theory to minimal surfaces. Two characterizations of the catenoid among minimal hedgehogs are given.

1. Introduction and statement of results

The set $\mathcal{K}^{n+1}$ of convex bodies of the $(n+1)$-Euclidean vector space $\mathbb{R}^{n+1}$ is usually equipped with Minkowski addition and multiplication by nonnegative real numbers. The theory of hedgehogs consists of considering $\mathcal{K}^{n+1}$ as a convex cone of the vector space $\left(\mathcal{H}^{n+1},+,\cdot\right)$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$. More precisely, it consists of:

1. considering each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a hypersurface of $\mathbb{R}^{n+1}$ (possibly with singularities and self-intersections), called a 'hedgehog';
2. extending the mixed volume $V:\left(\mathcal{K}^{n+1}\right)^{n+1}\to\mathbb{R}$ to a symmetric $(n+1)-$linear form on $\mathcal{H}^{n+1}$;
3. considering the Brunn-Minkowski theory in $\mathcal{H}^{n+1}$.

The relevance of this theory can be illustrated by the following two principles:

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1. to study convex bodies by splitting them into a sum of hedgehogs to reveal their structure;
2. to convert analytical problems into geometrical ones by considering certain real functions on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ as support functions of a hedgehog (or of a 'multi-hedgehog', see below).

The first principle permitted the author to disprove an old conjectured characterization of the 2−sphere [9] and the second one to give a geometrical proof of the Sturm-Hurwitz theorem [11]. The reader will find a short introduction of the theory in [12]. For an elementary survey of hedgehogs with a smooth support function, see [8].

The idea of defining geometrical differences of convex bodies goes back to H. Geppert who gave a first study of hedgehogs in $\mathbb{R}^2$ and $\mathbb{R}^3$ (under the German names 'stützbare Bereiche' and 'stützbare Flächen') [1]. The name 'hedgehog' came from a paper by R. Langevin, G. Levitt and H. Rosenberg [3] who implicitly considered differences of convex bodies of class C^2_+ (i.e., of convex bodies whose boundary is a C^2-hypersurface with positive Gauss curvature) as envelopes parametrized by their Gauss map. Let us recall the main points of their approach.

The boundary of a convex body $K \subset \mathbb{R}^{n+1}$ of class C^2_+ is determined by its support function $h : \mathbb{S}^n \to \mathbb{R}$, $u \mapsto \sup\{\langle x, u\rangle \mid x \in K\}$ (which must be of class C^2) as the envelope $\mathcal{H}_h$ of the family of hyperplanes given by

$$\langle x, u\rangle = h(u).$$

Now, this envelope $\mathcal{H}_h$ is well defined for any $h \in C^2(\mathbb{S}^n; \mathbb{R})$ (which is not necessarily the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$, can be interpreted as the inverse of its Gauss map in the sense that, at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$. This envelope $\mathcal{H}_h$ is called the *hedgehog* with support function h.

The notion of hedgehog of $\mathbb{R}^3$ can be extended by considering hedgehogs whose support function is only defined (and C^2) on some spherical domain $\Omega \subset \mathbb{S}^2$. Among hedgehogs defined on the unit sphere $\mathbb{S}^2$ punctured at a finite number of points, we can consider those that are minimal, that is, those whose mean curvature H is zero at all the smooth points. The condition that a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is minimal means simply that its support function h satisfies the equation

$$\triangle_S h + 2h = 0,$$

where $\triangle_S$ is the spherical Laplace operator on $\mathbb{S}^2$ (see [4]). In other words, a minimal hedgehog $\mathcal{H}_h$ (modelled on $\mathbb{S}^2$ punctured at a finite number of points) is a trivial hedgehog (i.e., a point) or a (possibly branched) minimal surface with total curvature -4π that is parametrized by the inverse of its Gauss map.

A study of minimal hedgehogs has been given by H. Rosenberg and E. Toubiana [14]. Concerning linear structures on the collections of minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$, the reader is also referred to the paper by A. Small [17].

Geometric inequalities for minimal hedgehogs (resp. N-hedgehogs) in $\mathbb{R}^3$. In this paper, we are interested in the extension to minimal surfaces of notions and techniques from the Brunn-Minkowski theory. The idea of developing a Brunn-Minkowski theory for minimal surfaces of $\mathbb{R}^3$ arises naturally from the fact that a (reversed) Brunn-Minkowski type inequality holds for minimal hedgehogs.

Let K be the closure of a (nonempty) connected open subset of $\mathbb{S}^2$ and let $\mathcal{H}_k$ be a minimal hedgehog modelled on K. Then the area of $x_k(K)$ is finite and given by

$$\operatorname{Area}[x_k(K)] = -\int_K R_k \, d\sigma,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$ and R_k the 'curvature function' of $\mathcal{H}_k$, that is, $1/K_k$, where K_k is the Gauss curvature of $\mathcal{H}_k$ (regarded as a function of the normal). Now, if $\mathcal{H}_l$ is another minimal hedgehog modelled on K, then

$$\sqrt{A(k+l)} \leq \sqrt{A(k)} + \sqrt{A(l)}, \tag{1.1}$$

where $A(h) = \operatorname{Area}[x_h(K)]$. In fact, we can regard the set of hedgehogs modelled (up to a translation) on K as a real vector space endowed with a prehilbertian structure for which the norm is given by the square root of the area. Consider the set of support functions (of a minimal hedgehog) modelled on K and identify two such functions k and l when $x_k(K)$ and $x_l(K)$ are translates of each other. Then the quotient set $\mathcal{H}(K)$ inherits a real vector space structure and we have the following result.

Theorem 1.1 ([5]). *The map $\sqrt{A} : \mathcal{H}(K) \to \mathbb{R}_+, h \longmapsto \sqrt{\operatorname{Area}[x_h(K)]}$, is a norm associated with a scalar product $A : \mathcal{H}(K)^2 \to \mathbb{R}$, which may be interpreted as an algebraic mixed area:*

$$\forall (k,l) \in \mathcal{H}(K)^2, (\text{Mixed Area})[x_k(K), x_l(K)] := A(k,l).$$

By the Cauchy-Schwarz inequality we have

$$A(k,l)^2 \leq A(k) \cdot A(l). \tag{1.2}$$

Corollary 1.2. *As a consequence, the area $A : \mathcal{H}(K) \to \mathbb{R}_+$, $h \longmapsto \operatorname{Area}[x_h(K)]$, is a strictly convex map, and thus, for any nonempty convex subset $\mathcal{K}$ of $\mathcal{H}(K)$, the problem of minimizing A over $\mathcal{K}$ has at most one optimal solution.*

Remark 1.1. Inequality (1.1) (resp. (1.2)) has to be compared with the following Brunn-Minkowski inequality (resp. Minkowski inequality). For any pair (K, L) of convex bodies of $\mathbb{R}^3$, we have (see, for instance, [15])

$$\sqrt{A(K+L)} \geq \sqrt{A(K)} + \sqrt{A(L)}$$

and

$$A(K,L)^2 \geq A(K) \cdot A(L),$$

where $A(H)$ (resp. $A(K,L)$) is the surface area (resp. the mixed surface area) of the convex body $H \subset \mathbb{R}^3$ (resp. of the pair (K,L)).

The author has obtained similar inequalities for various classes of hedgehogs as a consequence of an extension of the Alexandrov-Fenchel inequality [6].

Remark 1.2. Let $\mathcal{H}\left(\mathbb{S}^2\right)$ be the real vector space of support functions of minimal hedgehogs defined (up to a translation) on the unit sphere punctured at a finite number of points. To each $h \in \mathcal{H}\left(\mathbb{S}^2\right)$ let us assign the positive Borel measure μ_h defined on $\mathbb{S}^2$ by

$$\forall \Omega \in \mathcal{B}\left(\mathbb{S}^2\right), \mu_h(\Omega) = -\int_\Omega R_h \, d\sigma,$$

where $\mathcal{B}\left(\mathbb{S}^2\right)$ denotes the σ-algebra of Borel subsets of $\mathbb{S}^2$. Then we notice that the map

$$m : \mathcal{H}\left(\mathbb{S}^2\right) \to \left\{\sqrt{\mu} \,|\, \mu \text{ is a positive Borel measure on } \mathbb{S}^2\right\}, \ h \longmapsto \sqrt{\mu_h},$$

satisfies the following properties:

(i) $\forall h \in \mathcal{H}\left(\mathbb{S}^2\right), m(h) = 0 \Longleftrightarrow h = 0_{\mathcal{H}(\mathbb{S}^2)}$;
(ii) $\forall \lambda \in \mathbb{R}, \forall h \in \mathcal{H}\left(\mathbb{S}^2\right), m(\lambda h) = |\lambda| \, m(h)$;
(iii) $\forall (k,l) \in \mathcal{H}\left(\mathbb{S}^2\right)^2, m(k+l) \leq m(k) + m(l)$.

Remark 1.3. Let $\mathcal{H}_k$ and $\mathcal{H}_l$ be two hedgehogs whose support function is defined (and C^2) on some spherical domain $\Omega \subset \mathbb{S}^2$. On this domain, we can define their mixed curvature function by

$$R_{(k,l)} := \frac{1}{2}\left(R_{k+l} - R_k - R_l\right).$$

The symmetric map $(\alpha, \beta) \mapsto R_{(\alpha,\beta)}$ is bilinear on the vector space of hedgehogs modelled on Ω [10]. Given any $u \in \Omega$, the polynomial function $P_u(t) = R_{k+tl}(u)$ thus satisfies $P_u(t) = R_k(u) + 2tR_{(k,l)}(u) + t^2 R_l(u)$ for all $t \in \mathbb{R}$.

When k and l are the support functions of two convex bodies of class C^2_+, $P_u(t)$ must have a zero, so that

$$R_{(k,l)}(u)^2 \geq R_k(u) \cdot R_l(u)$$

and hence

$$\sqrt{R_{k+l}(u)} \geq \sqrt{R_k(u)} + \sqrt{R_l(u)},$$

by noticing that $R_{(k,l)} > 0$.

When $\mathcal{H}_k$ and $\mathcal{H}_l$ are minimal hedgehogs, $P_u(t)$ is nonpositive on $\mathbb{R}$, so that

$$R_{(k,l)}(u)^2 \leq R_k(u) \cdot R_l(u)$$

and hence

$$\sqrt{-R_{k+l}(u)} \leq \sqrt{-R_k(u)} + \sqrt{-R_l(u)}.$$

Note that $A(k,l) = \int_K R_{(k,l)} d\sigma$ for all $(k,l) \in \mathcal{H}(K)^2$ and that inequality (1.2) can be deduced from the inequality ${R_{(k,l)}}^2 \leq R_k \cdot R_l$.

Inequality (1.1) can be extended to some asymptotic areas of embedded ends in $\mathbb{R}^3$. The (possibly branched) complete minimal surfaces of finite nonzero total curvature in $\mathbb{R}^3$ can be regarded as 'multi-hedgehogs' provided they have only a finite number of branch points [14]: the (possibly singular) envelope of a family of cooriented planes of $\mathbb{R}^3$ is called an *N-hedgehog* if, for an open dense set of $u \in \mathbb{S}^2$, it has exactly N cooriented support planes with normal vector u. Hedgehogs with a C^2 support function are merely 1-hedgehogs.

We know that embedded ends of a minimal surface of $\mathbb{R}^3$ are flat or of catenoid type (i.e., asymptotic to a planar or catenoid end). More precisely (see [16]), each embedded end is the graph (over the exterior of a bounded region in an (x_1, x_2)-plane orthogonal to the limiting normal at the end) of a function of the form

$$u(x_1, x_2) = a \ln(r) + b + \frac{cx_1 + dx_2}{r^2} + O\left(\frac{1}{r^2}\right), \, r = \sqrt{x_1^2 + x_2^2},$$

with $a = 0$ when the end is flat.

Let E be an embedded flat end of a minimal surface of $\mathbb{R}^3$ and let P be its asymptotic plane. Define the asymptotic area of E by

$$A_s[E] = \iint_\Delta \left(\sqrt{1 + u_{x_1}(x_1, x_2)^2 + u_{x_2}(x_1, x_2)^2} - 1\right) dx_1 dx_2 \in [0, +\infty],$$

where $u : \Delta \to \mathbb{R}, (x_1, x_2) \mapsto u(x_1, x_2)$ is the function whose graph is equal to E. Given any increasing sequence (K_n) of compact subsets of P such that $K_n \to \Delta$, $A_s[E]$ may be interpreted as the limit of

$$\text{Area}\left[\pi^{-1}(K_n) \cap E\right] - \text{Area}[K_n],$$

where π denotes the orthogonal projection onto the asymptotic plane.

THEOREM 1.3 ([5]). *The asymptotic area of every embedded flat end of a minimal surface $S \subset \mathbb{R}^3$ is finite.*

Note that hedgehogs never have flat ends: if an end is flat, then the limiting normal at the end is a branch point of the Gauss map so that the surface cannot be a hedgehog (see, for instance, [4]). Let E be an embedded flat end of a minimal N-hedgehog, where $N \geq 2$. After a rotation, we may assume the limiting normal at the end is $n = (0, 0, -1)$. Then E admits a Weierstrass representation $(g\,(z)\,, f(z)dz)$ of the form

$$g(z) = z^N \text{ and } f(z) = \frac{\alpha}{z^2} + \sum_{k=0}^{+\infty} c_k z^k,$$

where α is nonzero [4]. (In the next subsection, the reader will find an introduction and some remarks on the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$.) Given $r \in \,]0,1[$, the pieces of minimal N–hedgehogs defined (up to a translation) by a parametrization of the form

$$\begin{aligned} X_f : D = \{z \in \mathbb{C}\,|0 < |z| \leq r\} \to \mathbb{R}^3, \\ z = x + iy \mapsto \mathrm{Re}\Bigg(\int \frac{1}{2} f(z)\left(1 - z^{2N}\right) dz, \\ \int \frac{i}{2} f(z)\left(1 + z^{2N}\right) dz, \int f(z) z^N dz \Bigg), \end{aligned}$$

where $f\,(z) = (\alpha/z^2) + \sum_{k=0}^{+\infty} c_k z^k$ (α may be 0), constitute a real vector space $(E_N, +, \cdot)$, where addition is defined by $X_{f_1} + X_{f_2} = X_{f_1+f_2}$ and scalar multiplication by $\lambda \cdot X_f = X_{\lambda f}$. Let us denote by S_f the surface parametrized by $X_f : D \to \mathbb{R}^3$.

Theorem 1.4. *For every $S_f \in E_N$, define $A_s\,(f)$ by*

$$A_s\,(f) := \iint_D \left(1 - \langle N(z), n\rangle\right) \left\| \left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y} \right)(z) \right\| dxdy,$$

where

$$N(z) = \frac{2}{|z|^{2N} + 1} \left(\mathrm{Re}\left(z^N\right), \mathrm{Im}\left(z^N\right), \frac{|z|^{2N} - 1}{2} \right)$$

is the unit normal at $X_f\,(z)$ if $\left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y}\right)(z) \neq 0$ and where D is identified with

$$\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, 0 < \sqrt{x^2 + y^2} \leq r \right\}.$$

(i) *If S_f is an embedded flat end, then $A_s\,(f)$ is its asymptotic area $A_s\,[S_f]$.*
(ii) *The map $\sqrt{A_s} : E_N \to \mathbb{R}_+$, $S_f \longmapsto \sqrt{A_s\,(f)}$, is a norm associated with a scalar product (which may be interpreted as a mixed algebraic asymptotic area).*

Addition of minimal surfaces and Enneper-Weierstrass representation. It is well known that any minimal surface $S \subset \mathbb{R}^3$ (possibly with isolated branch points) can be locally represented in the form

$$\left\{\begin{aligned} X_1(x,y) &= \frac{1}{2}\operatorname{Re}\left[\int_{z_0}^{z}\left(1-g(\zeta)^2\right)f(\zeta)\,d\zeta\right]+c_1,\\ X_2(x,y) &= \frac{1}{2}\operatorname{Re}\left[\int_{z_0}^{z} i\left(1+g(\zeta)^2\right)f(\zeta)\,d\zeta\right]+c_2,\\ X_3(x,y) &= \operatorname{Re}\left[\int_{z_0}^{z} g(\zeta)\,f(\zeta)\,d\zeta\right]+c_3, \end{aligned}\right. \tag{1.3}$$

where $f(z)$ is an arbitrary holomorphic function on an open simply connected neighbourhood $\mathcal{U}$ of $z_0 \in \mathbb{C}$ and $g(z)$ an arbitrary meromorphic function on $\mathcal{U}$ such that, at each pole of order n of $g(z)$, $f(z)$ has a zero of order at least $2n$, the integral being taken along any path connecting z_0 to $z = x+iy \in \mathbb{C}$ in $\mathcal{U}$, and naturally, c_1, c_2 and c_3 denote real constants. Recall that (see, e.g., [13])

$$\begin{aligned} N(z) &:= \frac{\left(\frac{\partial X}{\partial x}\times\frac{\partial X}{\partial y}\right)(x,y)}{\left\|\left(\frac{\partial X}{\partial x}\times\frac{\partial X}{\partial y}\right)(x,y)\right\|}\\ &= \frac{2}{|g(z)|^2+1}\left(\operatorname{Re}[g(z)],\operatorname{Im}[g(z)],\frac{|g(z)|^2-1}{2}\right), \end{aligned}$$

is the (unit) normal to the surface at $X(x,y) = (X_1(x,y), X_2(x,y), X_3(x,y))$ and $g(z)$ its image under the stereographic projection $\sigma : \mathbb{S}^2 - \{(0,0,1)\} \to \mathbb{C}$, $(x,y,t) \mapsto \frac{x+iy}{1-t}$. Thus, $X : \mathcal{U} \to \mathbb{R}^3, z = x+iy \mapsto (X_1(x,y), X_2(x,y), X_3(x,y))$, is a hedgehog (that is, X can be interpreted as the inverse of the stereographic projection of its Gauss map) if and only if $g(z) = z$. The simplest choice of 'Weierstrass data' $(g(z), f(z)\,dz) = (z, dz)$ gives Enneper's surface. Recall that this surface and the catenoid, which is given by $(g(z), f(z)\,dz) = \left(z, dz/z^2\right)$, are the only two complete regular minimal surfaces that are hedgehogs (see, e.g., [13]).

Representation (1.3) can be generalized to generate all minimal surfaces of $\mathbb{R}^3$: if $S \subset \mathbb{R}^3$ is a minimal surface (possibly with isolated branch points), M its Riemann surface and $g = \sigma \circ N : M \to \mathbb{C}\cup\{\infty\}$ the stereographic projection (from the north pole) of its Gauss map, then S can be represented in the form (1.3) for some holomorphic function f on M and some fixed $z_0 \in M$.

Given any two (possibly branched) minimal surfaces S_1 and S_2 modelled (up to a translation) by Weierstrass data $(g(z), f_1(z)\,dz)$ and $(g(z), f_2(z)\,dz)$ on a Riemann surface M (and thus sharing the same 'Gauss map' $g(z)$), we can define their sum $S_1 + S_2$ as the (possibly branched) minimal surface given (up to a translation) by $(g(z), (f_1(z) + f_2(z))\,dz)$. For any minimal surface

S modelled (up to a translation) by Weierstrass data $(g(z), f(z)\,dz)$ on M and for any complex number λ, we can define the minimal surface λS as the minimal surface given (up to a translation) by $(g(z), \lambda f(z)\,dz)$. Of course, in order for $z \mapsto \mathrm{Re}\left[\int \phi_\lambda(z)\,dz\right]$ to be well-defined on M, where

$$\phi_\lambda(z) := \lambda f(z)\left(\frac{1}{2}\left(1 - g(z)^2\right), \frac{i}{2}\left(1 + g(z)^2\right), g(z)\right),$$

we need that no component of ϕ_λ has a real period on M, that is,

$$\mathrm{Period}_\gamma[\phi_\lambda] := \mathrm{Re}\oint_\gamma \phi_\lambda(z)dz = 0_{\mathbb{R}^3},$$

for all closed curves γ on M, but in the case when this period condition is not satisfied, we may consider the minimal surface λS modelled on the universal covering space of M (i.e., $\mathbb{C}$ or the open unit disc). By hypothesis, ϕ_1 has no real period on M since S is modelled on M. It follows that for any $\lambda \in \mathbb{R}$ the surface λS is also modelled on M (since ϕ_λ clearly has no real period on M if $\lambda \in \mathbb{R}$). Thus, minimal surfaces modelled (up to a translation) by Weierstrass data $(g(z), f(z)\,dz)$ on a common Riemann surface M and sharing the same 'Gauss map' $g(z)$ constitute a real vector space E_M (which can be identified with the space of all holomorphic functions $f(z)$ having a zero of order at least $2n$ at each pole of order n of $g(z)$ and satisfying

$$\mathrm{Period}_\gamma\left[f\left(\frac{1}{2}\left(1 - g^2\right), \frac{i}{2}\left(1 + g^2\right), g\right)\right] = 0_{\mathbb{R}^3}$$

for all closed curves γ on M).

Recall that (i) the associate surfaces to a minimal surface S modelled (up to a translation) by Weierstrass data $(g(z), f(z)\,dz)$ on a Riemann surface M are the surfaces $S_\theta = e^{i\theta}S$ given (up to a translation) by $\left(g(z), e^{i\theta} f(z)\,dz\right)$, where $\theta \in \left[0, \frac{\pi}{2}\right]$; and (ii) the conjugate surface S^* to S is the associated surface $S_{\pi/2}$. Clearly, S^* and S_θ are (locally) parametrized by $X^*(z) = -\mathrm{Im}\left[\int \phi(z)\,dz\right]$ and $X_\theta = (\cos\theta)X - (\sin\theta)X^*$, where $\phi := f\left(\frac{1}{2}\left(1 - g^2\right), \frac{i}{2}\left(1 + g^2\right), g\right)$ and $X(z) := \mathrm{Re}\left[\int \phi(z)\,dz\right]$. In other words, we have $S_\theta = (\cos\theta)S - (\sin\theta)S^*$, where the surfaces are modelled on the universal covering space of M in the case when ϕ has a real period on M.

Remark 1.4. Every hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ has a unique representation in the form

$$\mathcal{H}_h = \mathcal{H}_c + \mathcal{H}_p, \tag{1.4}$$

where $\mathcal{H}_c$ is centred (i.e., centrally symmetric with centre at the origin) and $\mathcal{H}_p$ projective (i.e., modelled on $\mathbb{P}^n(\mathbb{R}) = \mathbb{S}^n/(\text{antipodal relation})$). This representation is given by

$$h = c + p,$$

where

$$c(u) = \frac{1}{2}(h(u) + h(-u)) \text{ and } p(u) = \frac{1}{2}(h(u) - h(-u)).$$

In the same way, every minimal hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ has a unique representation in the form (1.4). If $\mathcal{H}_h$ is given by Weierstrass data $(z, f(z)\,dz)$, then $\mathcal{H}_c$ and $\mathcal{H}_p$ are given (up to a translation) by the following decomposition of $f(z)$:

$$f(z) = f_c(z) + f_p(z),$$

where

$$f_c(z) = \frac{1}{2}\left(f(z) + \frac{1}{z^4}\overline{f\left(\frac{-1}{\overline{z}}\right)}\right),$$
$$f_p(z) = \frac{1}{2}\left(f(z) - \frac{1}{z^4}\overline{f\left(\frac{-1}{\overline{z}}\right)}\right)$$

(see [18] for the determination of $f_p(z)$). Let us consider the case of Enneper's surface, whose support function is given by

$$h(u) = \frac{\left(x^2 - y^2\right)(2r - t)}{2(r-t)^2},$$

where $r = \sqrt{x^2 + y^2 + t^2}$ and $u = (x, y, t) \in \mathbb{S}^2 \subset \mathbb{R}^3$. In this case, we get

$$c(u) = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \text{ and } p(u) = \frac{t\left(x^2 - y^2\right)\left(2r^2 + x^2 + y^2\right)}{2\left(x^2 + y^2\right)^2}$$

(resp. $f_c(z) = \frac{1}{2}\left(1 + 1/z^4\right)$ and $f_p(z) = \frac{1}{2}\left(1 - 1/z^4\right)$) and we notice that (i) $\mathcal{H}_c$ has 5 planes of symmetry (with equations $x = 0$, $y = 0$, $z = 0$, $x + y = 0$ and $x - y = 0$), 4 curves of double points lying on the plane $z = 0$, and 4 branch points (namely $\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ and the points deduced from it by symmetry); (ii) $\mathcal{H}_p$ is Henneberg's surface (which is thus the 'projective part' of Enneper's surface). Figure 1 below shows the central symmetrization of Enneper's surface.

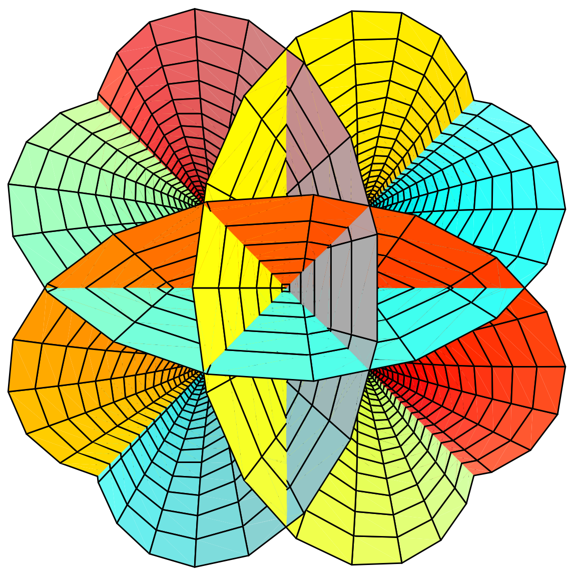

FIGURE 1

Relation between Enneper-Weierstrass representation and support function. We have the following result.

THEOREM 1.5. *Let* $X : \mathcal{U} \ni z_0 \to \mathbb{R}^3, z \mapsto \mathrm{Re}\left[\int_{z_0}^{z} \phi(\zeta)\, d\zeta\right]$, *where*

$$\phi(z) := f(z)\left(\frac{1}{2}\left(1 - g(z)^2\right), \frac{i}{2}\left(1 + g(z)^2\right), g(z)\right),$$

be the Weierstrass representation of a piece of a minimal surface (possibly with isolated branch points) such that

$$N : \mathcal{U} \to N(\mathcal{U}) \subset \mathbb{S}^2,$$
$$z \mapsto N(z) = \frac{2}{|g(z)|^2 + 1}\left(\mathrm{Re}[g(z)], \mathrm{Im}[g(z)], \frac{|g(z)|^2 - 1}{2}\right),$$

is a diffeomorphism of $\mathcal{U}$ *onto* $N(\mathcal{U})$. *Then* $X(\mathcal{U})$ *can be regarded as a hedgehog* $\mathcal{H}_h$ *whose parametrization* $x_h : N(\mathcal{U}) \to \mathcal{H}_h \subset \mathbb{R}^3$ *is given by* $x_h = \nabla\varphi$, *where* $\varphi : v \mapsto \|v\|\, h(v/\|v\|)$ *is the positively* 1*-homogeneous extension of* h *to*

$\left\{tu \,\middle|\, u \in N(\mathcal{U}) \text{ and } t \in \mathbb{R}_+^*\right\}$ [8]. Given $g(z)$, the support function h and the holomorphic function f are related by

$$\phi(z) = \frac{2g'(z)}{1+|g(z)|^2}\,(L_\varphi)_{N(z)}\left(\overline{v_g(z)}\right), \tag{1.5}$$

where $(L_\varphi)_{N(z)}$ is the endomorphism of $\mathbb{C}^3$ that is represented in the standard basis by the Hessian matrix $(\mathrm{Hess}\,\varphi)_{N(z)}$ of φ at $N(z)$ and $v_g(z) = (1, i, g(z))$, so that

$$f(z) = \frac{2g'(z)}{\left(1+|g(z)|^2\right)^2}\left[{}^t\overline{V_g(z)} \cdot (\mathrm{Hess}\,\varphi)_{N(z)} \cdot \overline{V_g(z)}\right],$$

where $V_g(z)$ is the column matrix ${}^t v_g(z)$.

Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a minimal hedgehog defined by Weierstrass data $(z, f(z)\,dz)$ on the sphere $\mathbb{S}^2$ punctured at a finite number of points. From (1.5) it follows that

$$f(z) = \frac{2}{z\left(1+|z|^2\right)}\left[(\nabla\varphi_t)(N(z)).\overline{V_g(z)}\right],$$

where φ_t is the partial derivative of φ with respect to the third coordinate in the standard basis of $\mathbb{R}^3$ and $\nabla\varphi_t = (\varphi_{xt}, \varphi_{yt}, \varphi_{t^2})$ is its gradient. Changing the orientation of the normal, this gives

$$\widetilde{f}(z) = \frac{2}{z\left(1+|z|^2\right)}\left[(\nabla\widetilde{\varphi}_t)(N(z)).\overline{V_g(z)}\right],$$

where $\widetilde{f}(z) = -(1/z^4)\overline{f(-1/\overline{z})}$ and $\widetilde{\varphi}(u) = -\varphi(-u)$. Noting that $N(-1/\overline{z}) = -N(z)$ and comparing $f(-1/\overline{z})$ with $\widetilde{f}(z)$, we get easily

$$\varphi_{t^2}(N(z)) = \mathrm{Re}\left[z^2 f(z)\right].$$

Now, inflection points of level curves of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ (with a support function of class C^∞) are given by

$$\varphi_{t^2}(u) = 0,\ \nabla\varphi_t(u) \neq 0 \text{ and } R_h(u) \neq 0,$$

where $\varphi(u) = \|u\|\,h(u/\|u\|)$. (By 'inflection point' of a level curve $\mathcal{C} \subset \mathcal{H}_h$ we mean a point where $\mathcal{C}$ has a contact of order ≥ 2 with its tangent line.) Therefore we have:

Corollary 1.6. *Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a nontrivial minimal hedgehog defined by Weierstrass data $(z, f(z)\,dz)$ on the unit sphere $\mathbb{S}^2$ punctured at a finite number of points. The inflection points of level curves of $\mathcal{H}_h$ are given by*

$$\mathrm{Re}\left[z^2 f(z)\right] = 0,\ z \neq 0 \text{ and } f(z) \neq 0.$$

It follows easily that the hedgehog $\mathcal{H}_h$ is necessarily a catenoid if it is complete and if no level curve of $\mathcal{H}_h$ has an inflection point.

Orthogonal-projection techniques. Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with support function $h \in C^2\left(\mathbb{S}^2; \mathbb{R}\right)$. We can get information on $\mathcal{H}_h$ by considering its images under orthogonal projections onto planes. We proceed as follows. For any $u \in \mathbb{S}^2$ we consider the restriction h_u of h to the great circle $\mathbb{S}^1_u = \mathbb{S}^2 \cap u^\perp$, where $u^\perp$ is the linear subspace orthogonal to u. This restriction is the support function of a plane hedgehog $\mathcal{H}_{h_u} \subset u^\perp$, which is merely the image of $x_h\left(\mathbb{S}^1_u\right)$ under the orthogonal projection onto $u^\perp$:

$$\mathcal{H}_{h_u} = \pi_u\left[x_h\left(\mathbb{S}^1_u\right)\right],$$

where π_u is the orthogonal projection onto the plane $u^\perp$. The index of a point $x \in u^\perp - \mathcal{H}_{h_u}$ with respect to $\mathcal{H}_{h_u}$ (i.e., the winding number of $\mathcal{H}_{h_u}$ around x) gives us information on the curvature of $\mathcal{H}_h$ on the line $\{x\} + \mathbb{R}u$:

THEOREM 1.7 ([7]). *Let x be a regular value of the map $x_h^u = \pi_u \circ x_h : \mathbb{S}^2 \to u^\perp$. The index of $x \in u^\perp - \mathcal{H}_{h_u}$ with respect to $\mathcal{H}_{h_u}$ is given by*

$$i_{h_u}(x) = \frac{1}{2}\left(\nu_h(x)^+ - \nu_h(x)^-\right),$$

where $\nu_h(x)^+$ (resp. $\nu_h(x)^-$) is the number of $v \in \mathbb{S}^2$ such that $x_h(v)$ is an elliptic (resp. a hyperbolic) point of $\mathcal{H}_h$ lying on the line $\{x\} + \mathbb{R}u$.

Recall that the index $i_h(x)$ of a point x with respect to a plane hedgehog $\mathcal{H}_h$ can be related to the number of cooriented support lines of $\mathcal{H}_h$ passing through x:

THEOREM 1.8 ([7]). *For any hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ we have*

$$\forall x \in \mathbb{R}^2 - \mathcal{H}_h, i_h(x) = 1 - \frac{1}{2}n_h(x),$$

where $n_h(x)$ is the number of cooriented support lines of $\mathcal{H}_h$ passing through x, i.e., the number of zeros of the map $h_x : \mathbb{S}^1 \to \mathbb{R}$, $u \longmapsto h(u) - \langle x, u\rangle$.

Theorem 1.7 admits an analogue for minimal hedgehogs:

THEOREM 1.9. *Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a complete minimal hedgehog modelled on $\mathbb{S}^2$ punctured at a finite number of points $e_1, \ldots, e_n$ (corresponding to its ends) and let $u \in \mathbb{S}^2$ be such that $\mathbb{S}^1_u \subset \mathbb{S}^2 - \{e_1, \ldots, e_n\}$. Then, for any regular value $x \in u^\perp - \mathcal{H}_{h_u}$ of the map $x_h^u = \pi_u \circ x_h : \mathbb{S}^2 - \{e_1, \ldots, e_n\} \to u^\perp$, we have*

$$i_{h_u}(x) + N_h^u(x)^+ = \sum_{e_k \in \mathbb{S}^+_u} d(e_k),$$

where $\mathbb{S}^+_u \subset \mathbb{S}^2$ is the halfsphere defined by $\langle u, v\rangle > 0$, $N_h^u(x)^+$ the number of $v \in \mathbb{S}^+_u - \{e_j \,|\langle e_j, u\rangle > 0\}$ such that $x_h(v) \in \{x\} + \mathbb{R}u$ and $d(e_k)$ the winding *number of the end with limiting normal* e_k. *Replacing* u *by* $-u$, *it follows that*

$$i_{h_u}(x) + N_h^u(x)^- = \sum_{e_k \in \mathbb{S}_u^-} d(e_k),$$

where $\mathbb{S}_u^- \subset \mathbb{S}^2$ *is the halfsphere defined by* $\langle u, v\rangle < 0$ *and* $N_h^u(x)^-$ *the number of* $v \in \mathbb{S}_u^- - \{e_j \,|\langle e_j, u\rangle < 0\}$ *such that* $x_h(v) \in \{x\} + \mathbb{R}u$. *Consequently,*

$$i_{h_u}(x) = \frac{1}{2}\left(N(h) - N_h^u(x)\right),$$

where $N_h^u(x) = N_h^u(x)^- + N_h^u(x)^+$ *is the number of* $v \in \mathbb{S}^2 - \{e_1, \ldots, e_n\}$ *such that* $x_h(v) \in \{x\} + \mathbb{R}u$ *and* $N(h)$ *the total spinning of* $\mathcal{H}_h$, *that is,* $N(h) = \sum_{k=1}^n d(e_k)$.

Corollary 1.10. *Let* $\mathcal{H}_h \subset \mathbb{R}^3$ *be a complete nontrivial minimal hedgehog. If* $\mathcal{H}_h$ *does not intersect a pencil of lines that fill up a right circular cone, then* $\mathcal{H}_h$ *is a catenoid.*

Theorem 1.9 can be generalized as follows. Consider a minimal multihedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ given by a Weierstrass representation $X : \mathcal{U} \to \mathbb{R}^3$ and let $N : \Omega \to \mathbb{S}^2$ be its Gauss map (regarded as a map defined on the set Ω of regular points of X). The support function h can be regarded as a function of $z \in \Omega$ and defined by: $\forall z \in \Omega, h(z) = \langle X(z), N(z)\rangle$. For any $u \in \mathbb{S}^2$ such that $\mathbb{S}_u^1$ contains no limiting normal at an end of $\mathcal{H}_h$ let h_u be the restriction of h to $N^{-1}\left(\mathbb{S}_u^1\right)$. If $X\left[N^{-1}\left(\mathbb{S}_u^1\right)\right]$ contains no parabolic point of $\mathcal{H}_h$, then h_u can be interpreted as the support function of the family of plane multihedgehogs, say $\mathcal{H}_{h_u}$, that constitute the image of $X\left[N^{-1}\left(\mathbb{S}_u^1\right)\right]$ under the orthogonal projection onto the plane $u^\perp$. The index of a point $x \in u^\perp - \mathcal{H}_{h_u}$ with respect to the family of multihedgehogs $\mathcal{H}_{h_u}$ can be defined as the algebraic intersection number of almost every oriented half-line of $u^\perp$ with origin x with the family of multihedgehogs equipped with their transverse orientation.

Theorem 1.11. *Let* $\mathcal{H}_h \subset \mathbb{R}^3$ *be a complete minimal multihedgehog having* n *ends with limiting normals* $e_1, \ldots, e_n$. *Let* $X : \mathcal{U} \to \mathbb{R}^3$ *be a Weierstrass representation of* $\mathcal{H}_h$ *and let* $N : \Omega \to \mathbb{S}^2$ *be its Gauss map (regarded as a map defined on the set* Ω *of regular points of* X*). Let* $u \in \mathbb{S}^2$ *be such that* $\mathbb{S}_u^1 \subset \mathbb{S}^2 - \{e_1, \ldots, e_n\}$ *and such that* $X\left[N^{-1}\left(\mathbb{S}_u^1\right)\right]$ *contains no parabolic point of* $\mathcal{H}_h$. *Then, for any* $x \in u^\perp - \mathcal{H}_{h_u}$ *such that the line* $\{x\} + \mathbb{R}u$ *contains no branch point of* $\mathcal{H}_h$, *we have*

$$i_{h_u}(x) + N_h^u(x)^+ = \sum_{\{k \mid \langle e_k, u\rangle > 0\}} d_k,$$

where $N_h^u(x)^+$ *is the number of* $z \in N^{-1}(\mathbb{S}_u^+)$ *such that* $X(z) \in \{x\} + \mathbb{R}u$ *and* d_k *the winding number of the kth end. Replacing* u *by* $-u$*, it follows that*

$$i_{h_u}(x) + N_h^u(x)^- = \sum_{\{k \mid \langle e_k, u\rangle < 0\}} d_k,$$

where $N_h^u(x)^-$ *is the number of* $z \in N^{-1}(\mathbb{S}_u^-)$ *such that* $X(z) \in \{x\} + \mathbb{R}u$*. Consequently,*

$$i_{h_u}(x) = \frac{1}{2}\left(N(h) - N_h^u(x)\right),$$

where $N_h^u(x) = N_h^u(x)^- + N_h^u(x)^+$ *is the number of* $z \in \mathcal{U}$ *such that* $X(z) \in \{x\} + \mathbb{R}u$ *and* $N(h)$ *the total spinning of* $\mathcal{H}_h$*, that is,* $N(h) = \sum_{k=1}^n d_k$*. In particular, the total spinning of* $\mathcal{H}_h$ *has the same parity as* $N_h^u(x)$*.*

2. Further remarks and proof of results

Proof of Theorem 1.4. (i) If S_f is an embedded flat end, then $A_s(f)$ is its asymptotic area $A_s[S_f]$ for $\langle N(z), n\rangle \left\| \left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y}\right)(z)\right\| dxdy$ is the area of the orthogonal projection, onto the asymptotic plane, of the element of area $\left\| \left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y}\right)(z)\right\| dxdy$ on the end.

(ii) We know that (see, e.g., [13])

$$\forall z = x + iy \in D, \left\| \left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y}\right)(z)\right\| = \left(|f(z)| \frac{\left(1 + |z|^{2N}\right)}{2}\right)^2,$$

so that

$$\begin{aligned} A_s(f) &= \iint_D \left(1 - \langle N(z), n\rangle\right) \left\| \left(\frac{\partial X_f}{\partial x} \times \frac{\partial X_f}{\partial y}\right)(z)\right\| dxdy \\ &= \iint_D |f(z)|^2 |z|^{2N} \frac{1 + |z|^{2N}}{2} dxdy. \end{aligned}$$

Consequently, $\sqrt{A_s} : E_N \to \mathbb{R}_+$ is a norm associated with the scalar product given by

$$A_s(f_1, f_2) = \iint_D \mathrm{Re}\left[f_1(z)\overline{f_2(z)}\right] |z|^{2N} \frac{1 + |z|^{2N}}{2} dxdy. \qquad \square$$

Remark 2.1. Recall that the Gauss curvature of a minimal surface S modelled (up to a translation) by Weierstrass data $(g(z), f(z)\,dz)$ on a Riemann surface M is given by (see, e.g., [13])

$$K_S = -\left[\frac{4|g'|}{|f|\left(1 + |g|^2\right)^2}\right]^2.$$

If the surface S is different from a plane, we define its curvature function by $R_S := 1/K_S$ outside the isolated zeros of K_S. Consequently, it is natural to define the mixed curvature function of two (possibly branched) minimal surfaces S_1 and S_2 modelled (up to a translation) by Weierstrass data $(g(z), f_1(z)\,dz)$ and $(g(z), f_2(z)\,dz)$ on M (and thus sharing the same 'Gauss map' $g(z)$) by

$$R_{(S_1,S_2)}(z) = -\operatorname{Re}\left[f_1(z)\,\overline{f_2(z)}\right]\left[\frac{\left(1+|g(z)|^2\right)^2}{4\,|g'(z)|}\right]^2.$$

Note that $R_{(S_1,S_2)} = 0$ if and only if the surface S_2 is homothetic to the conjugate surface S_1^* to S_1. We have obviously the inequalities $R_{(S_1,S_2)}{}^2 \leq R_{S_1} \cdot R_{S_2}$ and $\sqrt{-R_{S+S_2}} \leq \sqrt{-R_{S_1}} + \sqrt{-R_{S_2}}$, which generalize those of Remark 1.3.

Remark 2.2. For any $h \in \mathcal{H}\left(\mathbb{S}^2\right)$, denote by $r_h(u)$ the common absolute value of the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$. In other words, define r_h by $r_h = \sqrt{-R_h}$, where R_h is the curvature function of $\mathcal{H}_h$.

Let K be the closure of a (nonempty) connected open subset of $\mathbb{S}^2$ and let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog modelled on K. The Cauchy-Schwarz inequality gives

$$\operatorname{Area}[x_h(K)] \geq \frac{M_K(h)^2}{\operatorname{Area}[K]},$$

where $M_K(h) = \int_K r_h d\sigma$. This inequality has to be compared with the Minkowski inequality

$$S \leq \frac{M^2}{4\pi},$$

where S is the surface area and M the integral of mean curvature of a convex body $K \subset \mathbb{R}^3$ (see [15]). Recall that if K is a convex body of class C^2_+, then M is simply given by

$$M = \frac{1}{2}\int_{\mathbb{S}^2} (R_1 + R_2)\,d\sigma,$$

where R_1 and R_2 are the principal radii of curvature of K. The above Minkowski inequality was extended in [6] to any hedgehog whose support function is of class C^2 on $\mathbb{S}^2$.

Proof of Theorem 1.5. For all $z = x + iy \in \mathcal{U}$ we have

$$X(z) = x_h[N(z)] = (\nabla\varphi)[N(z)],$$

and thus

$$X_\xi(z) = (L_\varphi)_{N(z)}(N_\xi(z)),$$

where $N_\xi(z) = \frac{\partial}{\partial \xi}[N(x+iy)]$, $X_\xi(z) = \frac{\partial}{\partial \xi}[X(x+iy)]$ and $\xi = x$ or y. Note that

$$N_\xi(z) = \frac{2}{1+|g(z)|^2}\left[(P_\xi, Q_\xi, PP_\xi + QQ_\xi)(z) - (PP_\xi + QQ_\xi)(z)\, N(z)\right],$$

where $g(z) = P(x,y) + iQ(x,y)$, $P_\xi = \frac{\partial P}{\partial \xi}$ and $Q_\xi = \frac{\partial Q}{\partial \xi}$. As φ is positively 1-homogeneous, we have

$$(L_\varphi)_{N(z)}(N(z)) = 0,$$

and we thus get

$$X_\xi(z) = \frac{2}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left[(P_\xi, Q_\xi, PP_\xi + QQ_\xi)(z)\right].$$

Now, direct calculation gives

$$\begin{aligned}
&\operatorname{Re}\left[\frac{2g'(z)}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left(\overline{v_g(z)}\right)\right] \\
&\qquad = \frac{2}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left[(P_x, Q_x, PP_x + QQ_x)(z)\right], \\
&\operatorname{Im}\left[\frac{2g'(z)}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left(\overline{v_g(z)}\right)\right] \\
&\qquad = -\frac{2}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left[(P_y, Q_y, PP_y + QQ_y)(z)\right],
\end{aligned}$$

so that

$$\phi(z) = X_x(z) - iX_y(z) = \frac{2g'(z)}{1+|g(z)|^2}(L_\varphi)_{N(z)}\left(\overline{v_g(z)}\right). \qquad \square$$

Proof of Theorem 1.9. It suffices to prove the relation

$$i_{h_u}(x) + N_h^u(x)^+ = \sum_{e_k \in \mathbb{S}_u^+} d(e_k),$$

for any regular value $x \in u^\perp - \mathcal{H}_{h_u}$ of $x_h^u = \pi_u \circ x_h : \mathbb{S}^2 - \{e_1, \ldots, e_n\} \to u^\perp$.

Let (x_1, x_2, x_3) be the standard coordinates in $\mathbb{R}^3$. Without loss of generality, we can identify $u^\perp$ with the plane given by the equation $x_3 = 0$ (and thus with the Euclidean vector plane $\mathbb{R}^2$) and assume that x is its origin $0_{\mathbb{R}^2}$. The index $i_{h_u}(x)$ is the winding number of $\mathcal{H}_{h_u}$ around $x \in u^\perp - \mathcal{H}_{h_u}$. It is given by

$$i_{h_u}(x) = \frac{1}{2\pi}\int_{\mathcal{H}_{h_u}} \omega,$$

where ω is the closed 1-form defined by

$$\omega_{(x_1,x_2)} = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$$

on $\mathbb{R}^2 - \{0_{\mathbb{R}^2}\}$. This index $i_{h_u}(x)$ can also be regarded as the winding number of $x_h\left(\mathbb{S}^1_u\right)$ around the oriented line, say $D_x(u)$, passing through x and directed by u. In other words, $i_{h_u}(x)$ is given by

$$i_{h_u}(x) = \frac{1}{2\pi}\int_{x_h\left(\mathbb{S}^1_u\right)} \omega,$$

which can be checked by an easy calculation. Writing $\Sigma^+_u = \mathbb{S}^+_u - \{e_j \,| e_j \in \mathbb{S}^+_u\}$, we thus have

$$i_{h_u}(x) = \frac{1}{2\pi}\int_{\partial S} \omega,$$

where S denotes the surface $x_h\left[\Sigma^+_u\right]$ equipped with its transverse orientation. Let $\{f_1, \ldots, f_L\}$ be the set consisting of all e_j such that $\langle e_j, u\rangle > 0$, i.e., $e_j \in \mathbb{S}^+_u$. Since x is a regular value of the map $x^u_h = \pi_u \circ x_h : \mathbb{S}^2 - \{e_1, \ldots, e_n\} \to u^\perp$, there exists a small closed disc, say D, centred at x whose inverse image under $\left(x^u_h\right)^+ : \mathbb{S}^+_u - \{f_1, \ldots, f_L\} \to u^\perp, v \mapsto x^u_h(v)$, is empty or admits a partition of the form

$$\left[\left(x^u_h\right)^+\right]^{-1}(D) = \bigcup_{k=1}^{K} D_k,$$

where $K = N^u_h(x)^+$ and D_k is such that the map $\pi_u \circ x_h$ defines a diffeomorphism from D_k onto D for all $k \in \{1, \ldots, K\}$. As $f_1, \ldots, f_L$ are limiting normals at ends of the complete minimal hedgehog $\mathcal{H}_h$, there exist small disjoint spherical discs $\triangle_1, \ldots, \triangle_L$ punctured at $f_1, \ldots, f_L$ that are disjoint from $\mathbb{S}^1_u$ and from each D_k $(1 \leq k \leq K)$. Now, Stokes's formula gives

$$\int_{\partial S} \omega = \sum_{k=1}^{K} \int_{\partial S_k} \omega + \sum_{l=1}^{L} \int_{\partial \Sigma_l} \omega,$$

where S_k (resp. Σ_l) denotes the surface $x_h(D_k)$ (resp. $x_h(\triangle_l)$) equipped with its transverse orientation. As $\mathcal{H}_h$ is a (possibly branched) minimal surface, the maps $x_h : D_k \to S_k$ are orientation reversing and thus the orthogonal projections of the oriented curves ∂S_k into the (x_1, x_2)-plane have winding number -1 around x. Consequently,

$$\sum_{k=1}^{K} \int_{\partial S_k} \omega = -N^u_h(x)^+ .$$

To complete the proof, it suffices to notice that we have also

$$\sum_{l=1}^{L} \int_{\partial \Sigma_l} \omega = \sum_{l=1}^{L} d(f_l) = \sum_{e_k \in \mathbb{S}^+_u} d(e_k),$$

from the definition of the winding number of an end. □

The proof of Theorem 1.9 can be easily adapted to obtain a proof of Theorem 1.11; the details are left to the reader.

Proof of Corollary 1.10. By assumption, there exists a line D that does not intersect $\mathcal{H}_h$ and that is such that no limiting normal at an end of $\mathcal{H}_h$ belongs to the vector plane that is orthogonal to D. Let $u \in \mathbb{S}^2$ be a unit vector parallel to the line D and define x by $\{x\} = D \cap u^{\perp}$. According to Theorem 1.9 we have

$$i_{h_u}(x) = \frac{1}{2}\left(N(h) - N_h^u(x)\right) = \frac{N(h)}{2} > 0.$$

Theorem 1.8 now implies $i_{h_u}(x) = 1$ and thus $N(h) = 2$. The proof is completed by showing that $\mathcal{H}_h$ must be a catenoid if $N(h) = 2$. This was proved by Hoffman and Karcher (see [2, Corollary 3.2]) for a connected complete minimal immersed surface $M \subset \mathbb{R}^3$ with finite total curvature and their proof remains valid if we drop the assumption that M has no branch points. □

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