# SOME PROPERTIES OF GLOBAL SEMIANALYTIC SUBSETS OF COHERENT SURFACES

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ABSTRACT. Let $X \subset \mathbb{R}^n$ be a coherent analytic surface. We show that the connected components of global analytic subsets of $X$ are global and we compute the stability index and Bröcker's $t$-invariant of $X$. We also state a real Nullstellensatz for normal surfaces.

## Introduction

In this paper we study some properties of non-compact singular analytic surfaces. There are several reasons for considering such surfaces. First, proper analytic subsets of $X$ are 1-dimensional or discrete sets, which are known to have good properties. Second, the Artin-Lang property holds for coherent surfaces.

The paper is organized as follows. In the first section, we recall some definitions and results which will be used later, such as the Hörmander-Łojasiewicz inequality, cf. [Ac-Br-Sh], the Artin-Lang property, cf. [An-DC-Rz], and some properties of global analytic sets of dimension 1, cf. [An-Be] and [Ca-An].

Section 2 is devoted to the connected components of a coherent surface, which are shown to be global semianalytic subsets, a result which was known for analytic 2-dimensional manifolds, cf. [Ca-An].

In Section 3 we study the stability index and Bröcker's $t$-invariant, which are related with minimal descriptions of global semianalytic subsets. Again, our results are valid for coherent surfaces generalizing those of [DC-An].

Finally, in Section 4 we obtain a real Nullstellensatz. We use strong properties of the multiplicities along a divisor, so the singular locus must be a discrete set. This result can be regarded as a generalization of the real Nullstellensatz for 2-dimensional analytic manifolds, cf. [Bo-Rs], but we only assume the surface to be normal.

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## 1. Preliminaries

Let $X$ be a *global analytic set* of $\mathbb{R}^n$, that is, $X$ is the zero set of a finite number of analytic functions on $\mathbb{R}^n$. The ideal of analytic functions vanishing on $X$ is $I(X) := \{ f \in \mathcal{O}(\mathbb{R}^n) \mid f = 0 \text{ on } X \}$ and generates the coherent sheaf of ideals $I_X$ whose stalk at $x$ is $I_{X,x} = I(X)\mathcal{O}(\mathbb{R}^n_x)$. If $X$ is coherent we have the equality $I_{X,x} = I(X_x) := \{ f \in \mathcal{O}(\mathbb{R}^n_x) \mid f = 0 \text{ on } X_x \}$.

The *analytic functions on* $X$ are the global sections of the coherent sheaf $\mathcal{O}_X := \mathcal{O}_{\mathbb{R}^n}/I_X$ and they form the ring $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/I(X)$. We also define $\mathcal{M}(X)$ as the total ring of fractions of $\mathcal{O}(X)$.

The *global semianalytic subsets* of $X$ are those which can be written as

$$\bigcup_{i=1}^{p} \{ x \in X \mid f_i(x) = 0, g_{i1}(x) > 0, \ldots, g_{ij_i}(x) > 0 \},$$

where $f_i, g_{ij} \in \mathcal{O}(X)$.

In general, this definition is more restrictive than the classical definition of a semianalytic subset, which is of local nature. Nevertheless, in dimension one they coincide. More precisely, we have the following result, cf. [Ca-An], Lemma 3.1 and Corollary 3.3.

LEMMA 1.1. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let $C \subset X$ be a 1-dimensional analytic subset. Then any semianalytic subset of $C$ is global. In particular, $C$ is a global analytic set.*

*Moreover, if $S$ is a global semianalytic subset of dimension 1, then any semianalytic subset of $S$ is global.*

The next lemma allows us to write a meromorphic function $f$ as a fraction with a denominator whose zero set coincides with the set of points at which $f$ is not analytic if it is a discrete set.

LEMMA 1.2. *Let $X \subset \mathbb{R}^n$ be a global analytic set and let $f \in \mathcal{M}(X)$ be a meromorphic function which is analytic up to a discrete set $D$. Then there is $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) = D$ and $fh \in \mathcal{O}(X)$.*

*Proof.* We have that $f = f_1/f_2$ for some $f_1, f_2 \in \mathcal{O}(X)$. The sheaf $\mathcal{J} := (f_2 : f_1)$ of denominators of $f$ is a coherent sheaf whose stalk at $x$ is $\mathcal{J}_x = \{ g_x \in \mathcal{O}_{X,x} \mid g_x f_{1,x} \in f_{2,x} \mathcal{O}_{X,x} \}$.

In a neighbourhood $U_p$ of $p \in D$ this sheaf is generated by finitely many sections $g_1, \ldots, g_r$, and $p$ is an isolated zero of $G := \sum g_i^2$, since for any $x$ close enough to $p$, $f_x \in \mathcal{O}_{X,x}$, so that $\mathcal{J}_x = (g_1, \ldots, g_r)\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ and $g_i(x) \neq 0$ for some $i = 1, \ldots, r$.

Hence, $f_{1,p}/f_{2,p} = g'_{1,p}/g_{1,p} = \cdots = g'_{r,p}/g_{r,p}$, for some $g'_{1,p}, \ldots, g'_{r,p} \in \mathcal{O}_{X,p}$ and then $f_{1,p}/f_{2,p} = (g'_{1,p}g_{1,p} + \cdots + g'_{r,p}g_{r,p})/G_p$. That is, for any $p \in D$, $f_p = g_p/G_p$, where $g_p, G_p \in \mathcal{O}_{X,p}$ and $G_p$ is an elliptic germ (i.e., $\{G_p > 0\} = X_p \setminus \{p\}$).

Then by [DC-An], Proposition 3.1, we can find a global analytic function $h \in \mathcal{O}(X)$ such that $\mathcal{Z}(h) = D$ and $h_p \mathcal{O}_{X,p} = G_p \mathcal{O}_{X,p}$, $\forall p \in D$. Thus $fh \in \mathcal{O}(X)$ and we are done.

We remark that although Proposition 3.1 of [DC-An] is stated for a coherent analytic set $X \subset \mathbb{R}^n$, the same proof works for any global analytic set equipped with the coherent sheaf $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{R}^n,x}/I(X)\mathcal{O}_{\mathbb{R}^n,x}$.                 $\square$

Another lemma, which will be used later, is the following. It shows a kind of stability of open semianalytic set germs under small perturbations of the functions which define them.

LEMMA 1.3.   *Let* $X_p \subset \mathbb{R}^n$ *be an analytic set germ,* $F \subset X_p$ *a closed semianalytic germ,* $G = \bigcup_{i=1}^{q}\{f_{i1} > 0, \ldots, f_{is_i} > 0\}$, *where* $f_{i1}, \ldots, f_{is_i} \in \mathcal{O}(X_p)$, *and suppose that* $F \subset G \cup \{p\}$.
*There is some* $\nu \in \mathbb{N}$ *such that if* $f'_{ij} \in \mathcal{O}(X_p)$ *and* $f'_{ij} \equiv f_{ij} \mod \mathfrak{m}_p^\nu$, *where* $\mathfrak{m}_p$ *stands for the maximal ideal of* $\mathcal{O}(X_p)$, *then*

$$F \subset \bigcup_{i=1}^{q}\{f'_{i1} > 0, \ldots, f'_{is_i} > 0\} \cup \{p\}.$$

*Proof.* We will first show the result for a principal set $G = \{f > 0\}$. In this case, we have $\{f = 0\} \cap F \subset \{p\} = \{\| x - p \|^2 = 0\}$. Hence by the Hörmander-Łojasiewicz inequality, cf. [An-Br-Rz], Proposition II.1.16, there is an even integer $m$ and $a > 0$ such that $|f| \geq a \| x - p \|^m$ holds on $F$. Now, if $f' \in \mathcal{O}(X_p)$ verifies $f' - f \in \mathfrak{m}_p^{m+2}$, then $f'$ and $f$ must have the same sign on $F$, since otherwise $|f' - f| \geq a \| x - p \|^m$, which implies $f' - f \notin \mathfrak{m}_p^{m+2}$, in contradiction to the assumption. Thus we can take $\nu = m + 2$.

If $G$ is a basic set, $G = \{f_1 > 0, \ldots, f_v > 0\}$, then $F \subset \{f_i > 0\} \cup \{p\}$ and we can take $\nu$ as the maximum of the $\nu_i$'s obtained for each principal set $\{f_i > 0\}$.

In the general case $G = B_1 \cup \cdots \cup B_q$, where each $B_i = \{f_{i1} > 0, \ldots, f_{is_i} > 0\}$ is basic, we can decompose $F$ as a union $F = F_1 \cup \cdots \cup F_q$, where each $F_i$ is closed and $F_i \subset B_i \cup \{p\}$.

For instance, if $F \subset G \cup \{p\}$, where $G = B_1 \cup B_2$, then set $T_1 := \mathrm{Bd}\, B_1 \cap (\overline{B_2 \cap F}) \subset B_2 \cup \{p\}$ and $T_2 := \mathrm{Bd}\, B_2 \cap (\overline{B_1 \cap F}) \subset B_1 \cup \{p\}$. Take neighbourhoods $V_1$ and $V_2$ of $T_1 \setminus \{p\}$ and $T_2 \setminus \{p\}$, respectively, such that $\overline{V}_1 \subset B_2 \cup \{p\}$, $\overline{V}_2 \subset B_1 \cup \{p\}$ and $\overline{V}_1 \cap \overline{V}_2 = \{p\}$. Now, it is straightforward to check that $F_1 := F \cap (\overline{B_1 \setminus V_1})$ and $F_2 := F \cap (\overline{B_2 \setminus V_2})$ verify $F = F_1 \cup F_2$, $F_1 \subset B_1$ and $F_2 \subset B_2$. Moreover, this remains true if $B_1$ and $B_2$ are just open set germs, not necessarily basic.

In general, if $F \subset G \cup \{p\}$, where $G = B_1 \cup \cdots \cup B_q$, then defining $B'_2 := B_2 \cup \cdots \cup B_q$ we can find $F_1 \subset B_1 \cup \{p\}$ and $F'_2 \subset B'_2 \cup \{p\}$ such that $F = F_1 \cup F'_2$ and by the induction hypothesis we can write $F'_2 = F_2 \cup \cdots \cup F_q$ such that $F_i \subset B_i \cup \{p\}$ for all $i = 2, \ldots, q$.

By what has been seen above there are integers $\nu_1, \ldots, \nu_q$ such that if $f'_{ij} \equiv f_{ij} \mod \mathfrak{m}_p^{\nu_i}$, then $F_i \subset B'_i \cup \{p\}$, where $B'_i := \{f'_{i1} > 0, \ldots, f'_{is_i} > 0\}$. Now, it is enough to take $\nu = \max\{\nu_1, \ldots, \nu_q\}$. $\hspace{2cm}\square$

We recall that a *basic open set* (resp. *basic closed set*) is a global semi-analytic subset which can be written as $\{f_1 > 0, \ldots, f_r > 0\}$ (resp., $\{f_1 \geq 0, \ldots, f_r \geq 0\}$). In the sequel closures, interiors, etc., refer always to the topology induced by the Euclidean topology in $\mathbb{R}^n$. The notation $\overline{Y}^z$ denotes the Zariski closure of $Y$, i.e., the minimal global analytic set containing $Y$.

A useful tool when dealing with global semianalytic sets is the so-called Hörmander-Łojasiewicz inequality, cf. [Ac-Br-Sh].

THEOREM 1.4 (Hörmander-Łojasiewicz inequality). *Let $X \subset \mathbb{R}^n$ be a global analytic set and let $T \subset X$ be a global closed semianalytic subset. Given $f, g \in \mathcal{O}(X)$ there exist $p, q \in \mathcal{O}(X)$ such that*

(a) *$p > 0$, $q \geq 0$ on $X$,*
(b) *$\operatorname{sign}(pf + qg) = \operatorname{sign} f$ over $T$, and*
(c) *$\{q = 0\} = \overline{\{f = 0\} \cap T}^z$.*

As a consequence of the Hörmander-Łojasiewicz inequality we get the pasting lemmas as in [An-Br-Rz], Lemmas V.2.8 and V.2.13.

LEMMA 1.5. *Let $X$ be a global analytic set, $Y \subset X$ a global analytic subset and $B \subset X$ a closed global semianalytic set. Assume that*

$$B \setminus Y = \{a_1 \geq 0, \ldots, a_k \geq 0\} \setminus Y \quad and \quad B \cap Y = \{b_1 \geq 0, \ldots, b_l \geq 0\} \cap Y$$

*for suitable $a_i, b_i \in \mathcal{O}(X)$. Then there exist $c_1, \ldots, c_m \in \mathcal{O}(X)$, $m \leq k + l$, such that $B = \{c_1 \geq 0, \ldots, c_m \geq 0\}$.*

LEMMA 1.6. *Let $X$ be a global analytic set and $S \subset X$ an open global semianalytic set. If $Y \subset X$ is a global analytic subset and $B_1, \ldots, B_l$ are basic open sets such that*

$$S \setminus Y = (B_1 \cup \cdots \cup B_m) \setminus Y \quad and \quad S \cap Y = (B_{m+1} \cup \cdots \cup B_l) \cap Y,$$

*then $S = B'_1 \cup \cdots \cup B'_l$ for some basic open sets $B'_i \subset X$.*

We also recall that two subsets $S_1, S_2 \subset X$ are called *generically equal* (and write $S_1 \overset{g}{=} S_2$), if there is a codimension 1 global analytic subset $Y \subset X$ such that $S_1 \setminus Y = S_2 \setminus Y$. A subset $S \subset X$ is called *generically basic* if it is generically equal to some basic open (or closed) subset of $X$.

As we are interested in surfaces, *from now on we will suppose that* $\dim X = 2$.

REMARK 1.7. In the case of surfaces the first pasting lemma has two interesting consequences. First, it implies that any global closed semianalytic

set that is generically basic is a basic closed set since in dimension 1 everything is basic, cf. [DC-An], Theorem 4.4. In particular, the closure of any basic open set is basic closed.

Second, as any global semianalytic subset $S \subset X$ is a finite union of basic open sets and a global semianalytic subset of dimension 1, it is immediate that the closure, $\overline{S}$, of such a set is also a global semianalytic subset.

Now, let $Y \subset X$ be an irreducible analytic curve not contained in the singular locus of $X$ and let $f \in \mathcal{O}(X)$. For any point $x \in Y \setminus \mathrm{Sing}\, X$ the ring of germs $\mathcal{O}_{X,x}$ is a unique factorization domain and the ideal $I_{Y,x} = I(Y)\mathcal{O}_{X,x}$ is principal, say $I_{Y,x} = h_x \mathcal{O}_{X,x}$. Then the germ of $f$ at $x$ is $f_x = u_x h_x^m$, where $u_x \in \mathcal{O}_{X,x} \setminus I_{Y,x}$ and $m$ is a non-negative integer. In [An-DC-Rz] it is shown that the integer $m_Y(f) := m$ does not depend on the point $x$ and $m_Y(f)$ is called *the multiplicity of $f$ along $Y$*.

We will use some results about multiplicities. Let $Y \subset X$ be an analytic set of dimension one without isolated points. Then it is a union, possibly infinite, of irreducible curves, say, $Y = \bigcup_{i \in I} Y_i$. If we fix positive integers $m_i > 0$, $i \in I$, then it is possible to find a function $f \in \mathcal{O}(X)$ with these multiplicities, that is, $m_{Y_i}(f) = m_i, \forall i \in I$, cf. [An-DC-Rz], Proposition 2.3. In general, we only have $Y \subset \mathcal{Z}(f)$, but if all the $m_i$'s are even, then we can get the equality $Y = \mathcal{Z}(f)$, cf. [An-DC-Rz], Proposition 2.1.

In [An-DC-Rz] the Artin-Lang property has been proved for coherent analytic surfaces with affine normalization. From this property an ultrafilter theorem can be derived (as in [Ca-An] for the case of smooth analytic surfaces). The real spectrum of a commutative ring $A$ will be denoted by $\mathrm{Spec}_{\mathrm{r}}\, A$. If $A$ is a field, then $\mathrm{Spec}_{\mathrm{r}}\, A$ is a space of orderings. We refer to [An-Br-Rz], [Be] and [Bo-Co-Ro] for generalities on the real spectrum.

THEOREM 1.8 (Ultrafilter theorem). *Let $X$ be a coherent analytic surface with affine normalization. Then there is a one-to-one correspondence between orderings of $\mathcal{M}(X)$ and the ultrafilters of the lattice $\mathcal{S}$ of all open global semianalytic subsets of $X$. More precisely, if $\beta \in \mathrm{Spec}_{\mathrm{r}}\, \mathcal{M}(X)$, its associated ultrafilter is*

$$\mathcal{V}_\beta := \{S \in \mathcal{S} \,|\, \{f_1 > 0, \ldots, f_r > 0\} \subset S \ \text{for some } f_1, \ldots, f_r \in \beta\}.$$

Given any semianalytic subset (not necessarily global) $S \subset X$, we define its "tilde" image in $\mathrm{Spec}_{\mathrm{r}}\, \mathcal{M}(X)$ as

$$\tilde{S} = \{\beta \in \mathrm{Spec}_{\mathrm{r}}\, \mathcal{M}(X) \,|\, S \supset V \ \text{for some } V \in \mathcal{V}_\beta\}.$$

This tilde map has some useful properties, cf. [Ca-An], Proposition 2.4. For example, two global semianalytic sets that are generically equal have the same

tilde image. Moreover, if $S \subset$ is a global semianalytic subset,

$$S = \bigcup_{i=1}^{p} \{x \in X \mid f_i(x) = 0, g_{i1}(x) > 0, \ldots, g_{ij_i}(x) > 0\},$$

then its tilde image is given by the same formula,

$$\tilde{S} = \bigcup_{i=1}^{p} \{\beta \in \mathrm{Spec}_\mathrm{r}\, \mathcal{M}(X) \mid f_i(\beta) = 0, g_{i1}(\beta) > 0, \ldots, g_{ij_i}(\beta) > 0\}.$$

Subsets of $\mathrm{Spec}_\mathrm{r}\, \mathcal{M}(X)$ like this one are called *constructible sets* and are the counterpart of the global semianalytic sets. Notice that $\{0 > 0\}$ defines the empty set in $\mathrm{Spec}_\mathrm{r}\, \mathcal{M}(X)$.

## 2. Connected components of global semianalytic sets

In this section we will see that if $X$ is a coherent surface with affine normalization, then the connected components of a global semianalytic subset of $X$ are global. This is a generalization of an analogous result stated in [Ca-An] for 2-dimensional analytic manifolds.

The following lemma will be used to separate a semianalytic set from its complement.

LEMMA 2.1. *Let $X$ be a normal analytic surface, $S \subset X$ an open semianalytic subset whose boundary, $\mathrm{Bd}\, S$, is global and $T := X \setminus \overline{S}$. Then for any $\beta \in \mathrm{Spec}_\mathrm{r}\, \mathcal{M}(X)$ there is a global analytic function $g \in \beta$ such that*

$$\overline{S \cap \{g > 0\}} \cap \overline{T \cap \{g > 0\}}$$

*is a discrete set.*

*Proof.* As $\mathrm{Bd}\, S$ is global, its Zariski closure is also a global analytic set of dimension less than or equal to 1. Therefore $\overline{\mathrm{Bd}\, S}^{\,Z} = (\bigcup Y_i) \cup D$, where the $Y_i$'s are irreducible analytic sets of dimension 1 and $D$ is a discrete set.

By [An-DC-Rz], Lemma 2.2, there is $g \in \mathcal{O}(X)$ with multiplicity one along each $Y_i$, i.e., $m_{Y_i}(g) = 1$. We can suppose $g \in \beta$ since otherwise we can replace $g$ by $-g$. In any case, it is straightforward to check that $\overline{S \cap \{g > 0\}} \cap \overline{T \cap \{g > 0\}}$ is a discrete set. $\qquad\square$

From now on we will suppose that $X$ is an irreducible coherent analytic surface with affine normalization. In this case the normalization $\pi : X^\nu \to X$ is birational and $X^\nu$ is a normal analytic surface which can be embedded in some $\mathbb{R}^p$. The normalization induces an injective homomorphism $\pi^* : \mathcal{O}(X) \hookrightarrow \mathcal{O}(X^\nu)$ which associates to any $f \in \mathcal{O}(X)$ the function $f^* := \pi^*(f) = f \circ \pi \in \mathcal{O}(X^\nu)$.

This homomorphism can be extended to an isomorphism between the fields of fractions of these surfaces, $\mathcal{M}(X)$ and $\mathcal{M}(X^\nu)$, and it induces another

isomorphism between their real spectra. Namely, if $\beta \in \mathrm{Spec}_r \mathcal{M}(X^\nu)$, then $\pi(\beta)$ is defined as follows: a meromorphic function $f \in \mathcal{M}(X)$ is positive in the ordering $\pi(\beta)$ if and only if $f^*$ is positive in $\beta$.

It is known that, since the normalization is a proper map, the image of a semianalytic subset is again semianalytic, cf. [Ga]. Moreover, in dimension two the map $\pi$ sends global semianalytic sets of $X^\nu$ to global semianalytic sets of $X$. More precisely, let $S^\nu \subset X^\nu$ be a global semianalytic set

$$S^\nu = \bigcup \{x \in X^\nu \mid h_i(x) = 0, g_{i1}(x) > 0, \dots, g_{iv_i}(x) > 0\}$$

with $h_i, g_{ij} \in \mathcal{O}(X^\nu)$ and suppose that $\pi^*(h_i'/h_i'') = h_i$, $\pi^*(g_{ij}'/g_{ij}'') = g_{ij}$ for some $h_i', h_i'', g_{ij}', g_{ij}'' \in \mathcal{O}(X)$.

Defining

$$T = \bigcup_{i \in I} \left\{ x \in X \mid g_{i1}'(x) g_{i1}''(x) > 0, \dots, g_{iv_i}'(x) g_{iv_i}''(x) > 0 \right\},$$

where $I = \{i \mid h_i \text{ is a unit in } \mathcal{O}(X)\}$, it is clear that

$$T \subset \pi(S^\nu) \subset T \cup (\cup\{h_i' h_i'' = 0\}) \cup (\cup\{g_{ij}'' = 0\}).$$

In other words, $\pi(S^\nu)$ is the union of a global semianalytic set, $T$, and a semianalytic subset of a global analytic set of dimension 1. Therefore by Lemma 1.1 it is global.

On the other hand, if $S \subset X$ is a global semianalytic set, then $\pi^{-1}(S)$ is a global semianalytic subset of $X^\nu$. In fact, if $S = \bigcup\{h_i = 0, f_{i1} > 0, \dots, f_{is_i} > 0\}$, where $h_i, f_{ij} \in \mathcal{O}(X)$, then it is easy to check that $\pi^{-1}(S) = \bigcup\{h_i^* = 0, f_{i1}^* > 0, \dots, f_{is_i}^* > 0\}$.

PROPOSITION 2.2.    *Let $S \subset X$ be a semianalytic set. Then $S$ is global if and only if $\mathrm{Bd}\, S$ is global.*

*Proof.* If $S$ is global, then its boundary is a semianalytic set of dimension 1 contained in the set of zeros of the functions describing $S$. Then by Lemma 1.1 it is a global semianalytic set.

Suppose that $\mathrm{Bd}\, S$ is global. In this case we have that $S$ is global if and only if $\tilde{S}$ is constructible. This result has been stated in [Ca-An], Proposition 3.5, but the proof is valid for any surface $X$ verifying the Artin-Lang property. Hence we have to prove that $\tilde{S}$ is constructible.

If $X$ is normal we can follow the proof of [Ca-An], Proposition 3.6, to conclude that $\tilde{S}$ is constructible and consequently that $S$ is global. We just remark that we must use Lemmas 1.3 and 2.1 above instead of Lemmas 4.1 and 4.2 of [Ca-An], which are stated and proved only for regular surfaces.

Now, let $X$ be a coherent surface and consider its normalization $\pi : X^\nu \to X$. We define $S^\nu := \pi^{-1}(S) \subset X^\nu$, which is a semianalytic set with global boundary since $\mathrm{Bd}\, S^\nu \subset \pi^{-1}(\mathrm{Bd}\, S)$ and $\pi^{-1}(\mathrm{Bd}\, S)$ is a global semianalytic

set of dimension 1. As $\operatorname{Bd} S^\nu$ is global, we can apply the result for normal surfaces to conclude that $S^\nu$ is global. But then $S = \pi(S^\nu)$ is also global.    $\square$

COROLLARY 2.3.    *Let $S \subset X$ be a global semianalytic set. Then its connected components are also global semianalytic subsets of $X$.*

*Proof.* If $T$ is a connected component of $S$, then $\operatorname{Bd}(T) \subset \operatorname{Bd}(S)$. But $\operatorname{Bd}(S)$ is a global semianalytic subset of dimension 1, so $\operatorname{Bd}(T)$ is also global. By the previous proposition $T$ is a global semianalytic set.    $\square$

## 3. Stability index

Suppose $X$ is a pure dimensional surface. Given a basic open set $B$, it is clear that $B \subset \operatorname{Int} \overline{B}$ and also that $B' \subset \operatorname{Int} \overline{B}$ for any basic open set $B'$ generically equal to $B$ since they have the same closure.

EXAMPLE 3.1.    In some cases $\operatorname{Int} \overline{B}$ is not a basic open set. For example, let $X \subset \mathbb{R}^3$ be the double cone, with vertex at the origin, defined by $y^2 z^2 = x^2 + y^4$ (see Figure 1) and $B \subset X$ the basic open set $B = \{y^2 z^2 - 2xyz + x^2 - y^2 z - xy > 0\}$. Then $B$ contains the lower half part of $X$, i.e., $X \cap \{z < 0\}$, except the negative $z$-axis.

In the upper half $(z > 0)$ it contains two of the four sheets converging to the positive $z$-axis. Hence $\operatorname{Int} \overline{B}$ contains the negative $z$-axis, but not the positive $z$-axis, which belongs to its Zariski boundary. Therefore $\operatorname{Int} \overline{B}$ cannot be a basic open set, cf. [An-Br-Rz], I.3.3.

To see where the subset $B$ comes from, consider the normalization of $X$ (see Figure 2), which is the cone $X^\nu : z'^2 = x'^2 + y'^2$, where $x = x'y'$, $y = y'$, $z = z'$. The preimage of the singular locus of $X$, which is the $z$-axis, is the pair of lines $\{x' + z' = 0\}$ and $\{x' - z' = 0\}$. The basic set $B^\nu := \{(z' - x')^2 - (x' + z') > 0\}$ contains the lower half-line $\{x' - z' = 0\} \cap \{z' < 0\}$ ($\gamma_2'$ in Figure 2) and the line (except the origin) $\{x' + z' = 0\}$ ($\gamma_1$ and $\gamma_1'$ in Figure 2).
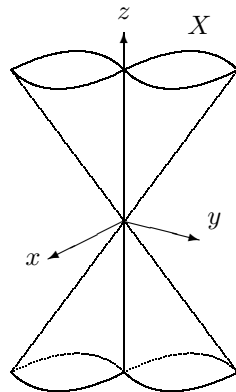

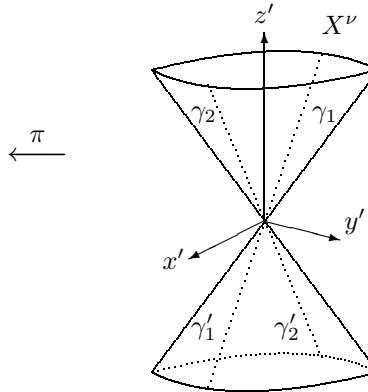
FIGURE 1                    FIGURE 2

We have

$$(z' - x')^2 - (x' + z') = \pi^* \left( \frac{y^2 z^2 - 2xyz + x^2 - y^2 z - xy}{y^2} \right).$$

Hence $\pi(B^\nu)$ and $B$ are generically equal.

Note that the function $f := y^2 z^2 - 2xyz + x^2 - y^2 z - xy$ changes sign on the upper $z$-axis, which is not a global analytic set. We remark that in the case of a normal surface the set of points at which an analytic function changes sign is a global analytic set, as can be deduced from [An-DC-Rz], Proposition 3.1.

We will see that by removing the Zariski boundary of $\operatorname{Int} \overline{B}$ we get a basic open set with some remarkable properties.

PROPOSITION 3.2.   *Let $X \subset \mathbb{R}^n$ be a pure dimensional analytic surface, $B = \{f_1 > 0, \ldots, f_m > 0\} \subset X$ a basic open set and $S_B := \operatorname{Int} \overline{B} \backslash \overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z$.*

  (a) *If $B'$ is a basic open set generically equal to $B$, then $B' \subset S_B$.*
  (b) *$S_B$ is a basic open set and can be described by $m$ inequalities, that is, $S_B = \{g_1 > 0, \ldots, g_m > 0\}$ for some $g_1, \ldots, g_m \in \mathcal{O}(X)$.*
  (c) *If $B'$ is a basic open set generically equal to $B$, then $B'$ can be described by $m$ inequalities.*

*Proof.* (a) It is clear that $B' \subset \operatorname{Int} \overline{B}$ and that $B' \stackrel{g}{=} \operatorname{Int} \overline{B}$. Moreover, we have $\operatorname{Bd} B' \supset \operatorname{Bd}(\operatorname{Int} \overline{B})$. Indeed, if $x \in \operatorname{Bd}(\operatorname{Int} \overline{B})$, then for any open neighbourhood $U$ of $x$ the sets $U \cap \operatorname{Int} \overline{B}$ and $U \cap (X \setminus \overline{B})$ are non-empty open sets. Hence $x \in \operatorname{Bd} B'$, since otherwise $B'$ and $\operatorname{Int} \overline{B}$ would not be generically equal.

Therefore $\overline{\operatorname{Bd} B'}^z \supset \overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z$ and $B' \setminus \overline{\operatorname{Bd} B'}^z \subset \operatorname{Int} \overline{B} \setminus \overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z = S_B$. But $B' \setminus \overline{\operatorname{Bd} B'}^z = B'$ since $B'$ is basic open, cf. [An-Br-Rz], I.3.3.

(b) We have $B \subset \operatorname{Int} \overline{B}$ and $\operatorname{Int} \overline{B} \setminus B$ is a global semianalytic set of dimension 1. Therefore $A := \overline{\operatorname{Int} \overline{B} \setminus B}^z$ is an analytic set of dimension 1. We decompose $A$ into irreducible components

$$A = \left( \bigcup Y_i \right) \cup \left( \bigcup Y'_j \right) \cup D,$$

where $Y = \cup Y_i$ collects the 1-dimensional components of $A$ which are not components of $\overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z$, $Y' = \cup Y'_i$ is the collection of components of $A \cap \overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z$ of dimension 1 and $D$ is some discrete set.

We can suppose that the discrete set $D$ is empty. Indeed, if this is not the case, then for any $i = 1, \ldots, m$ the germ of $f_i$ at each point $p \in D$ is elliptic, that is, there is a neighbourhood $U_p$ of $p$ such that $f_i|_{U_p} \geq 0$ and $\{f_i = 0\} \cap U_p = \{p\}$. Then we can find non-negative analytic functions $h_i$ such that $\mathcal{Z}(h_i) = D$ and at each $p \in D$ the germs of $f_i$ and $h_i$ are equal up

to a unit of $\mathcal{O}_{X,p}$, cf. [DC-An], Proposition 3.1. Finally, we can replace every $f_i$ by $f_i/h_i$.

We set $Y_1 := Y \cap \operatorname{Int} \overline{B}$ and $Y_2 := Y \cap (X \setminus \operatorname{Int} \overline{B})$. Then $Y = Y_1 \cup Y_2$ and $S_B = \operatorname{Int} \overline{B} \setminus Y' = B \cup (Y_1 \setminus Y')$.

Now, take $h \in \mathcal{O}(X)$ separating $\operatorname{Int} \overline{B}$ and $Y_2$, cf. [Br-Pi], Proposition 2.7, so that $h > 0$ on $\operatorname{Int} \overline{B}$ and $h < 0$ on $Y_2 \setminus \overline{B}$. We note that since $Y$ and $\overline{\operatorname{Bd}(\operatorname{Int} \overline{B})}^z$ do not share any component, $Y_2 \setminus \overline{B}$ is a discrete set.

Next, we apply the Hörmander-Łojasiewicz inequality to $T := (X \setminus \operatorname{Int} \overline{B}) \cup Y'$, $f_i$, $h$ to find $g_i = p_i f_i + q_i h$ $(p_i > 0,\, q_i \geq 0)$ such that $\operatorname{sign} g_i = \operatorname{sign} f_i$ on $T$ and $\{q_i = 0\} = \overline{\{f_i = 0\} \cap T}^z$.

We will see that $S_B = \{g_1 > 0, \ldots, g_m > 0\}$. As $\operatorname{sign} g_i = \operatorname{sign} f_i$ on $T = X \setminus S_B$, it follows that $\{g_1 > 0, \ldots, g_m > 0\} \cap T = B \cap T = \emptyset$ (recall that $B \subset S_B$) and therefore $\{g_1 > 0, \ldots, g_m > 0\} \subset S_B$.

For the other inclusion, take $x \in S_B$. Since $h > 0$ on $\operatorname{Int} \overline{B}$ and $S_B \subset \operatorname{Int} \overline{B}$ we have that $h(x) > 0$. If $f_i(x) > 0$, $\forall i$, then clearly $g_i(x) > 0$, $\forall i$. On the other hand, if $f_i(x) = 0$ for some $i$, then $q_i(x) > 0$, since $\overline{\{f_i = 0\} \cap T}^z \cap \operatorname{Int} \overline{B} \subset Y'$ and $S_B \cap Y' = \emptyset$, so that $g_i(x) > 0$. Thus, in any case, $x \in \{g_1 > 0, \ldots, g_m > 0\}$.

(c) We have $B' \subset S_B$. Let $r \in \mathcal{O}(X)$ be a non-negative function such that $\{r = 0\} = \overline{S_B \setminus B'}^z$. Using that $\overline{S_B \setminus B'}^z \subset \overline{\operatorname{Bd} B'}^z$ and $B' \cap \overline{\operatorname{Bd} B'}^z = \emptyset$, cf. [An-Br-Rz], I.3.3, it is easy to check that $B' = \{g_1 r > 0, g_2 > 0, \ldots, g_m > 0\}$. $\qquad \square$

REMARK 3.3. (a) The relation "generically equal" decomposes the family of basic open subsets of $X$ into equivalence classes. In each class there is a maximal element with respect to the inclusion relation, namely $S_B$, where $B$ is any member of the class. Moreover, all the basic open sets in the same class can be described by the same number of inequalities.

(b) If $X$ is not of pure dimension, then we can get a similar result, but we have to add the part of dimension 1. More precisely, if $B$ is a basic open subset of $X$, then we set $B_1 := B \cup (X \setminus X^*)$, where $X^*$ denotes the part of maximal dimension of $X$. Finally, we put $S_B := \operatorname{Int} \overline{B}_1 \setminus \overline{\operatorname{Bd}(\operatorname{Int} \overline{B}_1)}^z$.

The *stability index* (resp. the *closed stability index*) of $X$ is the smallest integer $s(X)$ (resp. $\overline{s}(X)$) such that every basic open (resp. closed) set can be described by $s(X)$ (resp. $\overline{s}(X)$) inequalities.

By the finiteness theorem, cf. [Ac-Br-Sh], any open (resp. closed) global semianalytic set can be written as a union of basic open (resp. closed) sets. The invariant $t(X)$ (resp. $\overline{t}(X)$) is defined as the smallest integer such that every open (resp. closed) global semianalytic subset can be written as a union of $t(X)$ (resp. $\overline{t}(X)$) basic open (resp. closed) sets.

There are similar definitions for the constructible sets of $\mathrm{Spec_r}\,\mathcal{M}(X)$, although in this case there is no difference between open and closed sets. These "generic" invariants will be denoted by $s(\mathcal{M}(X))$ and $t(\mathcal{M}(X))$.

We will follow the pattern of [DC-An], where this problem has been solved for analytic manifolds of dimension two. Given an ordering $\beta \in \mathrm{Spec_r}\,\mathcal{M}(X)$ we define the ring $W_\beta$ as the convex hull of $\mathbb{R}$ in $\mathcal{M}(X)$ with respect to $\beta$, i.e.,

$$W_\beta := \{f \in \mathcal{M}(X)|f^2 <_\beta r^2, \text{ for some } r \in \mathbb{R}\}.$$

Its maximal ideal is $\mathfrak{n}_\beta = \{f \in \mathcal{M}(X)|f^2 <_\beta r^2, \forall r \in \mathbb{R}\}$. We denote by $U_\beta$ the set of units of $W_\beta$; $\Gamma_\beta$ and $w_\beta$ stand, respectively, for the value group and the associated valuation of $W_\beta$. The ring of bounded analytic functions $\mathcal{O}_b(X)$ is contained in $W_\beta$ and $\mathfrak{m}_\beta := \mathfrak{n}_\beta \cap \mathcal{O}_b(X)$ is a maximal ideal of $\mathcal{O}_b(X)$.

We denote by $\mathcal{C}$ the lattice of closed global semianalytic subsets of $X$. The family

$$\mathcal{U}_\beta := \{Y \in \mathcal{C}\,|Y \cap f^{-1}[-1,1] \neq \emptyset, \,\forall f \in \mathfrak{m}_\beta\}$$

is an ultrafilter of $\mathcal{C}$, cf. [Ca] and [Jw].

To compute the stability index of $\mathcal{M}(X)$ we will use the *Stability Formula*, due to L. Bröcker (we refer to [An-Br-Rz] for the concept of fan)

$$s(\mathcal{M}(X)) = \sup\{s \in \mathbb{Z}|\text{ there is a fan } F \subset \mathrm{Spec_r}\,\mathcal{M}(X) \text{ with } \#F = 2^s\}.$$

A remarkable property of fans, which will be used later, is the following, cf. [An-Br-Rz], Proposition III.3.8: If $F$ is a finite fan with $2^n$ elements, then $\#(\{f > 0\} \cap F) = 0, 2^{n-1}$ or $2^n$, for any $f \in \mathcal{O}(X)$. As $\{f > 0\} \cap F$ is also a fan, the possibilities for the number of elements of $\{f > 0, g > 0\} \cap F$ are $0$, $2^{n-2}$, $2^{n-1}$ or $2^n$.

It is known that the orderings of $\mathrm{Spec_r}\,\mathcal{M}(X)$ with the same valuation ring $W$ form the fan

$$F_W := \{\alpha \in \mathrm{Spec_r}\,\mathcal{M}(X)|\,W_\alpha = W\}.$$

Moreover, the number of elements of these fans is given by the formula $\#F_W = \#(\Gamma_W/2\Gamma_W)$ and every fan in $\mathrm{Spec_r}\,\mathcal{M}(X)$ is contained in the union of two fans of this type, cf. [Ma]. The key result is the following proposition, which bounds the number of elements of these fans.

PROPOSITION 3.4. *Let $X$ be a normal analytic surface and let $\beta \in \mathrm{Spec_r}\,\mathcal{M}(X)$. Then $\#F_{W_\beta} \leq 4$.*

*Proof.* We distinguish three cases according to the dimension of $\mathcal{U}_\beta$, which is defined as the minimum of the dimensions of the sets in $\mathcal{U}_\beta$.

(a) If $\dim \mathcal{U}_\beta = 2$, then $\#F_{W_\beta} = 1$. Let $f \in \mathcal{O}_b(X)$, $f \neq 0$. Since $\dim \mathcal{U}_\beta = 2$ we must have $\mathcal{Z}(f) \notin \mathcal{U}_\beta$, so $f = ug^2$ for some $u \in U_\beta$, cf. [DC-An], Remark 1.2. This implies that $w(f) = 2w(g) \in 2\Gamma_\beta$. As any

$f \in \mathcal{M}(X)$ can be written as a quotient of bounded analytic functions, we have $w(f) \in 2\Gamma_\beta, \forall f \in \mathcal{M}(X)$, that is, $\Gamma_\beta = 2\Gamma_\beta$ and $\#F_\beta = \#(\Gamma_\beta/2\Gamma_\beta) = 1$.

(b) If $\dim \mathcal{U}_\beta = 1$, then $\#F_{W_\beta} \le 2$. Let $f \in \mathcal{O}(X)$ and denote by $\mathcal{Z}(f)^c$ the set of points at which $f$ changes sign. Suppose that $\mathcal{Z}(f)^c \notin \mathcal{U}_\beta$. By Proposition 3.1 of [An-DC-Rz] there are $h_0, \ldots, h_r, \tilde{f} \in \mathcal{O}(X)$ such that $h_0^2 f = (\Sigma_1^r h_i^2)\tilde{f}$ and $\mathcal{Z}(\tilde{f}) = \mathcal{Z}(f)^c$. Thus $\mathcal{Z}(\tilde{f}) \notin \mathcal{U}_\beta$ and, by [DC-An], Remark 1.2, we have $\tilde{f} = uh^2$, for some $u \in U_\beta$. As $h_0^2 f = (\Sigma_1^r h_i^2)uh^2$, we have $w(f) \in 2\Gamma_\beta$.

Take now $f, g \in \mathcal{O}(X)$ such that $f, g \notin 2\Gamma_\beta$. As seen above, this implies $\mathcal{Z}(f)^c, \mathcal{Z}(g)^c \in \mathcal{U}_\beta$, so $Y := \mathcal{Z}(f)^c \cap \mathcal{Z}(g)^c \in \mathcal{U}_\beta$. As $X$ is a normal surface, the multiplicity of the product $fg$ is well defined on $Y \setminus D$ for some discrete set $D$ and then the multiplicity is even, so $fg$ does not change sign on $Y \setminus D$. But $\dim \mathcal{U}_\beta = 1$, so $Y \setminus D \in \mathcal{U}_\beta$. It follows that $\mathcal{Z}(fg)^c \notin \mathcal{U}_\beta$ and then as above that $w(fg) \in 2\Gamma_\beta$. That is, $w(fg) = w(f) + w(g) \in 2\Gamma_\beta$ whenever $f, g \notin 2\Gamma_\beta$. This shows that $\Gamma_\beta/2\Gamma_\beta$ is a subgroup of $\mathbb{Z}/2\mathbb{Z}$.

(c) If $\dim \mathcal{U}_\beta = 0$, then $\#F_{W_\beta} \le 4$. We will see that there are no 8-element subfans of $F_{W_\beta}$ containing $\beta$.

Suppose that $F := \{\beta = \beta_1, \beta_2, \ldots, \beta_8\}$ is an 8-element fan and $F \subset F_{W_\beta}$, that is, $W_{\beta_i} = W_\beta$, for all $i = 1, \ldots, 8$. Let $\nu_{\beta_1}, \ldots, \nu_{\beta_8}$ be the maximal filters of $\mathcal{S}$ corresponding to these orderings, cf. Theorem 1.8. Since $\beta_i \ne \beta_j$, for $i \ne j$, there are basic open sets $B_i \in \nu_{\beta_i}$ such that $B_i \cap B_j = \emptyset$, for $i \ne j$.

We can reduce to a multilocal problem by choosing a discrete set $D \in \mathcal{U}_\beta$ and for each $p \in D$ taking a neighbourhood $U(p, \epsilon_p)$ of $p$ of radius $\epsilon_p$ small enough. There is $h_\epsilon \in \mathcal{O}(X)$ (which can be found by approximating a suitable $C^\infty(X)$ function) such that

$$\bigcup_{p \in D} U(p, \epsilon_p/2) \subset \{h_\epsilon > 0\} \text{ and } X \setminus \bigcup_{p \in D} U(p, \epsilon_p) \subset \{h_\epsilon < 0\}.$$

The function $h_\epsilon$ is positive at all points of $D \in \mathcal{U}_\beta$, so $h_\epsilon \in \beta$, cf. [DC-An], Lemma 1.1. But then, as $\mathcal{U}_\beta = \mathcal{U}_{\beta_i}$, we also have $h_\epsilon \in \beta_i$, for all $i$. Hence, $B_i' := \{h_\epsilon > 0\} \cap B_i \in \nu_{\beta_i}$, for any $i$ and any $\epsilon$. Thus, replacing $B_i$ by $B_i'$, we may consider the $B_i$'s in an arbitrarily small ball of the discrete set $D$.

As $\overline{B}_i \cap D \in \mathcal{U}_\beta$, replacing $D$ by $D' := (\bigcap_{i=1}^8 \overline{B}_i) \cap D$ we can suppose that $\overline{B}_i \cap D = D$, for all $i$.

Now, for any disjoint sets $B_1 \in \nu_{\beta_1}$ and $B_i \in \nu_{\beta_i}$, $\dim(\overline{B}_1 \cap \overline{B}_i)_p = 0$ or 1 for each $p \in D$. Thus, $D = \{p \in D \mid \dim(\overline{B}_1 \cap \overline{B}_i)_p = 0\} \cup \{p \in D \mid \dim(\overline{B}_1 \cap \overline{B}_i)_p = 1\} \in \mathcal{U}_\beta$. As $\mathcal{U}_\beta$ is an ultrafilter, only one of these two sets is in $\mathcal{U}_\beta$. Hence we have two cases:

*Case 1.* For any $i = 2, \ldots, 8$, there exist pairwise disjoint basic open sets $B_i \in \nu_{\beta_i}$ such that the discrete set $D_i' := \{p \in D \mid \dim(\overline{B}_1 \cap \overline{B}_i)_p = 0\}$ is in $\mathcal{U}_\beta$. In this case we replace $D$ by $\bigcap_2^8 D_i' \in \mathcal{U}_\beta$.

Let $p \in D$ and let $B_{1,p}, \ldots, B_{8,p}$ be the basic open semianalytic set germs of $B_1, \ldots, B_8$ at $p$. Since $t(X_p) = 2$, cf. [DC], we can write $B_{2,p} \cup \cdots \cup B_{8,p} = C_{1,p} \cup C_{2,p}$, for some basic open set germs $C_{1,p}$ and $C_{2,p}$. Now, by Theorem 1.5 of [An-DC] (see also Proposition 8.9 of [Br] for the semialgebraic analogue), we can take $h_{i,p} \in \mathcal{O}(X_p)$ separating $\overline{B}_{1,p}$ and $\overline{C}_{i,p}$, i.e., $\overline{B}_{1,p} \subset \{h_{i,p} > 0\} \cup \{p\}$ and $\overline{C}_{i,p} \subset \{h_{i,p} < 0\} \cup \{p\}$.

Hence we have $\overline{B}_{1,p} \subset \{h_{1,p} > 0,\, h_{2,p} > 0\} \cup \{p\}$ and $\overline{B}_{2,p} \cup \cdots \cup \overline{B}_{8,p} = \overline{C}_{1,p} \cup \overline{C}_{2,p} \subset \{h_{1,p} < 0\} \cup \{h_{2,p} < 0\} \cup \{p\}$. Now, by Lemma 1.3, there exists $l_p \in \mathbb{N}$ such that if $h'_{i,p} = h_{i,p} \bmod \mathfrak{m}_p^{l_p}$, then $\overline{B}_{1,p} \subset \{h'_{1,p} > 0,\, h'_{2,p} > 0\} \cup \{p\}$ and $\overline{B}_{2,p} \cup \cdots \cup \overline{B}_{8,p} \subset \{h'_{1,p} < 0\} \cup \{h'_{2,p} < 0\} \cup \{p\}$.

By Cartan's Theorem B there are $H_1, H_2 \in \mathcal{O}(X)$ whose germs at each $p \in D$ coincide with $h_{1,p}$ and $h_{2,p}$, respectively, till order $l_p$. Hence it follows that $H_1$ and $H_2$ separate locally $B_{1,p}$ from $B_{2,p}, \ldots, B_{8,p}$ at each $p \in D$. Replacing, if necessary, each $B_i$ by $U \cap B_i$ for some small enough open neighbourhood $U$ of $D$, we can conclude that $H_1$ and $H_2$ separate globally $B_1$ from $B_2, \ldots, B_8$ and, therefore, $\beta_1$ from $\{\beta_2, \ldots, \beta_8\}$. This shows that the subset $\{\beta_1, \ldots, \beta_8\}$ is not a fan, cf. [An-Br-Rz], Proposition III.3.8.

*Case 2.* For some $i \in \{2, \ldots, 8\}$ is not possible to find disjoint sets $B_1 \in \nu_{\beta_1}$ and $B_i \in \nu_{\beta_i}$ such that $\{p \in D \mid \dim(\overline{B}_1 \cap \overline{B}_i)_p = 0\}$ is in $\mathcal{U}_\beta$. We suppose $i = 2$ and replace $D$ by $D' := \{p \in D \mid \dim(\overline{B}_1 \cap \overline{B}_2)_p = 1\}$.

If $U$ is an open global semianalytic neighbourhood of $(\overline{B}_1 \cap \overline{B}_2) \setminus D$, then $U \in \nu_{\beta_1}$. For otherwise $U' := \mathrm{Int}(X \setminus U) \in \nu_{\beta_1}$, since $\nu_{\beta_1}$ is an ultrafilter, and defining $B'_1 := B_1 \cap U' \in \nu_{\beta_1}$ we will have $\dim \overline{B'}_1 \cap \overline{B}_2 = 0$, contradicting the hypothesis. Analogously, $U \in \nu_{\beta_2}$.

Let us denote by $\gamma_p^1, \ldots, \gamma_p^{r_p}$ the half-branches of $(\overline{B}_1 \cap \overline{B}_2)_p$. Clearly $(\gamma_p^1 \cup \cdots \cup \gamma_p^{r_p}) \cap (\overline{B}_3 \cup \cdots \cup \overline{B}_8)_p = \{p\}$, so using again Cartan's Theorem B we can find a global analytic function $f$ whose germ at each $p \in D$ is positive on $(\gamma_p^1 \cup \cdots \cup \gamma_p^{r_p})$ and negative on $(\overline{B}_{3,p} \cup \cdots \cup \overline{B}_{8,p}) \setminus \{p\}$. Then $\{f > 0\} \in \nu_{\beta_1} \cap \nu_{\beta_2}$, but $\{f < 0\} \in \nu_{\beta_3} \cap \cdots \cap \nu_{\beta_8}$, which is impossible for 8-element fans, cf. [An-Br-Rz], Proposition III.3.8. $\qquad\square$

THEOREM 3.5. *Let $X$ be a coherent analytic surface with affine normalization. Then $s(\mathcal{M}(X)) = 2$ and $t(\mathcal{M}(X)) = 2$.*

*Proof.* Suppose first that $X$ is a normal surface. By the stability formula we have to show that $\#F \leq 4$ for any fan $F \subset \mathrm{Spec}_r \mathcal{M}(X)$. Since $F \subset F_{W_\beta} \cup F_{W_{\beta'}}$ for some $\beta, \beta' \in F$, we have to show that if $\#F_{W_\beta} = \#F_{W_{\beta'}} = 4$, then $F_{W_\beta} \cup F_{W_{\beta'}}$ is not a fan (unless $F_{W_\beta} = F_{W_{\beta'}}$).

We can suppose that $\dim \mathcal{U}_\beta = \dim \mathcal{U}_{\beta'} = 0$. If the ultrafilters $\mathcal{U}_\beta$ and $\mathcal{U}_{\beta'}$ coincide, then by the proof of the previous proposition $F_{W_\beta} \cup F_{W_{\beta'}}$ is not a fan. If, on the other hand, $\mathcal{U}_\beta \neq \mathcal{U}_{\beta'}$, then there are discrete disjoint sets $D \subset \mathcal{U}_\beta$ and $D' \subset \mathcal{U}_{\beta'}$. Now, let $F_{W_\beta} = \{\beta = \beta_1, \ldots, \beta_4\}$ and denote by $\nu_{\beta_i}$

the corresponding ultrafilters of $\mathcal{S}$. Then, as in the previous proposition, there exist basic open sets $B_i \in \nu_{\beta_i}$ and function germs $f_p$ such that (possibly after taking a smaller $D \in \mathcal{U}_\beta$ and renumbering the elements of $F_{W_\beta}$) any function germ $g_p$ equal to $f_p$ up to a sufficiently high order separates $B_{1,p} \cup B_{2,p}$ from $B_{3,p} \cup B_{4,p}, \forall p \in D$. Defining $f_p = -1$ for $p \in D'$ and using Cartan's Theorem B we find a global analytic function which separates two orderings from the other six, showing that $F_{W_\beta} \cup F_{W_{\beta'}}$ is not a fan, cf. [An-Br-Rz], Proposition III.3.8.

If $X$ is a coherent surface, the result follows from the birationality of the normalization.

The result for $t$ follows immediately from [An-Br-Rz], Corollary IV.7.9.a), because $\mathrm{Spec}_r \, \mathcal{M}(X)$ is a space of orderings. □

After computing the generic invariants we can apply Proposition 3.2 and the pasting lemmas to compute the invariants $s(X)$, $\overline{s}(X)$, $t(X)$ and $\overline{t}(X)$.

THEOREM 3.6. *Let $X$ be a coherent analytic surface with affine normalization. Then $s(X) = 2$, that is, every basic open subset of $X$ can be written with only two inequalities.*

*Proof.* Let $B = \{f_1 > 0, \ldots, f_r > 0\}$ be any basic open set and consider the constructible set $\tilde{B} = \{\alpha \in \mathrm{Spec}_r \, \mathcal{M}(X) \,|\, f_1 >_\alpha 0, \ldots, f_r >_\alpha 0\}$. As $s(\mathcal{M}(X)) = 2$, we have that $\tilde{B} = \{\alpha \in \mathrm{Spec}_r \, \mathcal{M}(X) \,|\, f >_\alpha 0, g >_\alpha 0\}$ for some $f, g \in \mathcal{O}(X)$. Then $B' := \{x \in X \,|\, f(x) > 0, g(x) > 0\}$ is a basic open set that is generically equal to $B$. Hence, by Proposition 3.2, $B$ can be written with two inequalities. □

PROPOSITION 3.7. *Let $X \subset \mathbb{R}^n$ be a coherent analytic surface with affine normalization. Then:*

(a) $\overline{s}(X) = 3$.
(b) $\overline{t}(X) = 2$.
(c) $t(X) = 3$.

*Proof.* (a) Let $B \subset X$ be a basic closed set, say $B = \{f_1 \geq 0, \ldots, f_r \geq 0\}$, and define $B' := \{f_1 > 0, \ldots, f_r > 0\}$. By Theorem 3.6 there are functions $g_1, g_2 \in \mathcal{O}(X)$ such that $B' = \{g_1 > 0, g_2 > 0\}$. Hence $B$ and $B'' := \{g_1 \geq 0, g_2 \geq 0\}$ are generically equal, that is, there exists a one dimensional analytic subset $Y \subset X$ such that $B \backslash Y = B'' \backslash Y$. Now, $B \cap Y$ is a global closed semianalytic subset of $Y$, so there is $h \in \mathcal{O}(X)$ such that $B \cap Y = \{h \geq 0\} \cap Y$, cf. [DC-An], Theorem 4.4. Thus we can conclude that $B = \{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$, for suitable $c_1, c_2, c_3 \in \mathcal{O}(X)$, cf. Lemma 1.5. This shows that $\overline{s}(X) \leq 3$.

To check the opposite inequality we just point out that the argument of [An-Br-Rz], Example VI.7.2.b), and [Br], Theorem 7.6, shows that the basic

closed set $B := \{y \geq 0,\, y^2(x-1) \geq 0,\, x \geq 0\}$ (see Figure 3), where $x$ and $y$ are global analytic functions which are local coordinates at some regular point of $X$, cannot be written with only two inequalities.
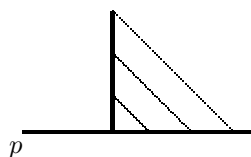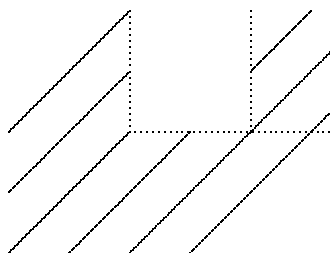


FIGURE 3                    FIGURE 4

(b) Let $S \subset X$ be a closed semianalytic set. By Theorem 3.5 there exist basic closed sets $B_1$, $B_2$ and an analytic set of dimension 1, $Y$, such that $S \setminus Y = (B_1 \cup B_2) \setminus Y$. Defining $B_i' := (B_i \cup Y) \cap S$, we have that $S = B_1' \cup B_2'$. Now, $B_i'$ is closed and generically basic, so from Lemma 1.5 it follows that $B_i'$ is basic closed.

(c) Let $C \subset X$ be an open semianalytic set. By Theorem 3.5 it is generically equal to $B_1 \cup B_2$ for suitable basic open sets $B_1$ and $B_2$, that is, there is an analytic subset $Y \subset X$ of dimension 1 such that $C \setminus Y = (B_1 \cup B_2) \setminus Y$. Also $C \cap Y = B_3 \cap Y$ with $B_3$ basic open, since $t(Y) = 1$. Finally, by Lemma 1.6 we get $C = B_1' \cup B_2' \cup B_3'$, for some basic open sets $B_1'$, $B_2'$ and $B_3'$, which shows $t(X) \leq 3$.

Next, as in [An-Br-Rz], Example VI.7.2.e), and [Br], Proposition 9.6, it can be shown that the open semianalytic set $C := \{y < 0\} \cup \{x < -1\} \cup \{x > 1,\, y > 0\}$ (see Figure 4), where $x$ and $y$ are global analytic functions which are local coordinates at some regular point of $X$, cannot be written as a union of two basic open sets. This shows that $t(X) \geq 3$, so the proof is complete. $\quad\square$

## 4. Real Nullstellensatz

In this section we will suppose that $X$ is a normal surface in order to ensure that the multiplicity along any irreducible curve $Y$ is defined.

LEMMA 4.1.   *Let $X \subset \mathbb{R}^n$ be an irreducible normal analytic surface and let $\mathfrak{p} \subset \mathcal{O}(X)$ be a non-trivial real prime ideal such that $\mathcal{Z}(\mathfrak{p}) \neq \emptyset$. Then $\mathcal{Z}(\mathfrak{p})$ is either a point or an irreducible analytic curve.*

*Proof.* First of all, $\mathcal{Z}(\mathfrak{p})$ is a global analytic set of $X$, cf. [Br-Wh], and $\dim \mathcal{Z}(\mathfrak{p}) \leq 1$.

*Case 1:* $\dim \mathcal{Z}(\mathfrak{p}) = 0$. In this case $\mathcal{Z}(\mathfrak{p})$ is a non-empty discrete set $D$. We pick one point $p \in D$. Then there is some $f \in \mathfrak{p}$ such that the germ of $\mathcal{Z}(f)$ at $p$ is $\{p\}$ since otherwise $\dim_p \mathcal{Z}(f) = 1$ for all $f \in \mathfrak{p}$ and $\dim \mathcal{Z}(\mathfrak{p}) = 1$. Thus the germ of $f^2$ at $p$ is elliptic, i.e., there is a neighbourhood $U$ of $p$ such that $f^2 > 0$ on $U \setminus \{p\}$.

By [DC-An], Proposition 3.1, there is $h \in \mathcal{O}(X)$ such that $h_p = f_p^2 u_p$ for some unit $u_p \in \mathcal{O}_{X,p}$ and $\mathcal{Z}(h) = \{p\}$. Then we can factor $f^2$ as $f^2 = hg$ for some $g \in \mathcal{O}(X)$ with $g(p) \neq 0$, which implies $g \notin \mathfrak{p}$. As $f^2 = hg \in \mathfrak{p}$, we must have $h \in \mathfrak{p}$ and therefore $\mathcal{Z}(\mathfrak{p}) = \{p\}$.

*Case 2:* $\dim \mathcal{Z}(\mathfrak{p}) = 1$. In this case $\mathcal{Z}(\mathfrak{p}) = (\bigcup Y_j) \cup D$, where the $Y_j$'s are irreducible curves and $D$ is a discrete set.

Take $Y_1 \in \mathcal{Z}(\mathfrak{p})$, $f \in \mathfrak{p}$ and $g \in \mathcal{O}(X)$ such that $m_{Y_1}(g) = 2m_{Y_1}(f)$ and $\mathcal{Z}(g) = Y_1$, cf. [An-DC-Rz], Proposition 2.1. Then $f^2/g$ is an analytic function up to a discrete set $D'$, so by Lemma 1.2 there is some $h \in \mathcal{O}(X)$ such that $G := h(f^2/g) \in \mathcal{O}(X)$. By construction $m_{Y_1}(G) = 0$, so $Y_1 \not\subset \mathcal{Z}(G)$ and $G \notin \mathfrak{p}$. As $Gg = f^2 h \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, we have $g \in \mathfrak{p}$, so $\mathcal{Z}(\mathfrak{p}) = \mathcal{Z}(g) = Y_1$. $\qquad\square$

LEMMA 4.2.  *Let $X \subset \mathbb{R}^n$ be a global irreducible analytic set and let $\mathfrak{p} \subset \mathcal{O}(X)$ be such that there is $g \in \mathfrak{p}$ with compact zero set. Suppose $f_1, \dots, f_m \in \mathcal{O}(X)$ are such that their classes modulo $\mathfrak{p}$ are positive in some total ordering of $\mathcal{O}(X)/\mathfrak{p}$. Then*

$$\{f_1 > 0, \dots, f_m > 0\} \cap \mathcal{Z}(\mathfrak{p}) \neq \emptyset.$$

*Proof.* This was obtained by J. Ruiz, cf. [Rz], Corollary 2.4, in the case of a real analytic manifold $X$ and the same proof works for any global analytic set $X$. $\qquad\square$

THEOREM 4.3 (Nullstellensatz for prime ideals).  *Let $X \subset \mathbb{R}^n$ be an irreducible normal analytic surface and let $\mathfrak{p} \subset \mathcal{O}(X)$ be a non-trivial real prime ideal such that $\mathcal{Z}(\mathfrak{p}) \neq \emptyset$. Then $\mathcal{I}\mathcal{Z}(\mathfrak{p}) = \mathfrak{p}$.*

*Proof.* According to Lemma 4.1 we distinguish two cases.

*Case 1:* $\mathcal{Z}(\mathfrak{p})$ *is a point $p$.* As in the proof of Lemma 4.1 (Case 1) we can take $f^2 \in \mathfrak{p}$ with elliptic germ at $p$ and factor $f^2$ as $f^2 = hg$ such that $\mathcal{Z}(h) = \{p\}$ and $g(p) \neq 0$, so that $h \in \mathfrak{p}$. Hence Lemma 4.2 can be applied in this case.

Suppose now that $\overline{f} \in \mathcal{I}\mathcal{Z}(\mathfrak{p}) \setminus \mathfrak{p}$. As $\overline{f} \notin \mathfrak{p}$, there is some total ordering $\alpha$ of $\mathcal{O}(X)/\mathfrak{p}$ such that $\overline{f}^2$ is strictly positive in $\alpha$ and, by Lemma 4.2, $\overline{f}(p) \neq 0$, that is, $\overline{f} \notin \mathcal{I}\mathcal{Z}(\mathfrak{p})$, a contradiction. Hence $\mathcal{I}\mathcal{Z}(\mathfrak{p}) \subset \mathfrak{p}$. On the other hand, it is trivial that $\mathfrak{p} \subset \mathcal{I}\mathcal{Z}(\mathfrak{p})$.

*Case 2:* $\mathcal{Z}(\mathfrak{p})$ *is an irreducible curve $Y$.* Let $f \in \mathcal{I}\mathcal{Z}(\mathfrak{p})$, so that $m_Y(f) > 0$, and take $g \in \mathfrak{p}$. Then for some positive integers $n, m$ we will have $m_Y(f^{2n}) =$

$m_Y(g^{2m})$. Now, we take $f_1$ such that $\mathcal{Z}(f_1) = \bigcup Y_j$ and $m_{Y_j}(f_1) = m_{Y_j}(f^{2n})$ (we suppose that $\mathcal{Z}(f) = Y \cup (\bigcup Y_j) \cup D$). Then $f^{2n}/f_1$ is an analytic function up to a discrete set, so there is $h_1$ such that $\mathcal{Z}(h_1)$ is a discrete set and $f^{2n}h_1/f_1$ is an analytic function, cf. Lemma 1.2. As $m_{Y_j}(f^{2n}h_1/f_1) = 0$, we have $\mathcal{Z}(f^{2n}h_1/f_1) = Y \cup D_1$, for some discrete set $D_1$.

In the same way, there are functions $g_1, h_1' \in \mathcal{O}(X)$ such that $g^{2m}h_1'/g_1$ is an analytic function with zero set $Y \cup D_1'$ and $m_Y(f^{2n}h_1/f_1) = m_Y(g^{2m}h_1'/g_1)$. Then $f^{2n}h_1g_1/g^{2m}h_1'f_1$ is analytic up to a discrete set, so for some $h \in \mathcal{O}(X)$ with discrete zero set, the function $hf^{2n}h_1g_1/g^{2m}h_1'f_1$ is analytic; denote this function by $H$. Then $f^{2n}h_1g_1h = Hg^{2m}h_1'f_1 \in \mathfrak{p}$. Since $h_1g_1h \notin \mathfrak{p}$, we have $f^{2n} \in \mathfrak{p}$ and since $\mathfrak{p}$ is real, $f \in \mathfrak{p}$. This means that $\mathcal{I}\mathcal{Z}(\mathfrak{p}) \subset \mathfrak{p}$ and we are done. $\qquad\square$

THEOREM 4.4.   *Let $X \subset \mathbb{R}^n$ be an irreducible normal analytic surface and let $I \subset \mathcal{O}(X)$ be a non-trivial real ideal. Suppose also that $I = \bigcap \mathfrak{p}$, where the intersection is taken over all the real primes ideals $\mathfrak{p}$ such that $\mathfrak{p} \supset I$ and $\mathcal{Z}(\mathfrak{p}) \neq \emptyset$. Then $\mathcal{I}\mathcal{Z}(I) = I$, that is, the real Nullstellensatz holds.*

*Proof.* Let $\mathcal{Z}(I) = \bigcup C_i$ be the decomposition of $\mathcal{Z}(I)$ into irreducible components and let $\mathfrak{p}_i$ denote the real prime ideals $\mathcal{I}(C_i)$. It is clear that $\mathcal{Z}(I) = \mathcal{Z}(\bigcap \mathfrak{p}_i)$.

Let us take $\mathfrak{p} \supset I$ such that $\mathcal{Z}(\mathfrak{p}) \neq \emptyset$. By Theorem 4.3, $\mathfrak{p} = \mathcal{I}\mathcal{Z}(\mathfrak{p})$. Also, we have that $\mathcal{Z}(\mathfrak{p}) \subset \mathcal{Z}(I) = \bigcup C_i = \bigcup \mathcal{Z}(\mathfrak{p}_i)$, so $\mathfrak{p} = \mathcal{I}\mathcal{Z}(\mathfrak{p}) \supset \mathcal{I}(\bigcup \mathcal{Z}(\mathfrak{p}_i)) = \bigcap \mathcal{I}\mathcal{Z}(\mathfrak{p}_i) = \bigcap \mathfrak{p}_i$. Therefore

$$I = \bigcap_{\mathfrak{p} \supset I,\, \mathcal{Z}(\mathfrak{p}) \neq \emptyset} \mathfrak{p} \supset \bigcap \mathfrak{p}_i = \bigcap \mathcal{I}\mathcal{Z}(\mathfrak{p}_i) = \mathcal{I}\mathcal{Z}(\bigcap \mathfrak{p}_i) = \mathcal{I}\mathcal{Z}(I).$$

The other inclusion, $I \subset \mathcal{I}\mathcal{Z}(I)$, is trivial. $\qquad\square$

From this theorem we can obtain a real Nullstellensatz for finitely generated real ideals.

COROLLARY 4.5.   *Let $X \subset \mathbb{R}^n$ be an irreducible normal analytic surface and let $I \subset \mathcal{O}(X)$ be a non-trivial finitely generated real ideal. Then $\mathcal{I}\mathcal{Z}(I) = I$.*

*Proof.* As $I$ is real, we have that $I = \sqrt[r]{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$, where the $\mathfrak{p}$'s are real prime ideals, cf. [La], Theorem 6.5.

We define

$$A := \bigcap_{\mathfrak{p} \supset I,\, \mathcal{Z}(\mathfrak{p}) \neq \emptyset} \mathfrak{p} \quad \text{and} \quad B := \bigcap_{\mathfrak{p} \supset I,\, \mathcal{Z}(\mathfrak{p}) = \emptyset} \mathfrak{p},$$

so that $I = A \cap B$. We also define two sheafs of ideals, $\mathcal{I}$ and $\mathcal{A}$, whose stalks are $\mathcal{I}_x = I\mathcal{O}_{X,x}$ and $\mathcal{A}_x = A\mathcal{O}_{X,x}$, respectively.

We will show that $\mathcal{I}_x = \mathcal{A}_x$. Obviously, we have $\mathcal{I}_x \subset \mathcal{A}_x$. For the other inclusion, suppose $f \in \mathcal{A}_x$. Then $f = \sum f_i g_i$, where $f_i \in A$ and $g_i \in \mathcal{O}_{X,x}$. Now, take any $h \in B$ such that $h(x) \neq 0$ and write $f = \sum (f_i h)(g_i/h)$. As $f_i h \in A$ and $g_i/h \in \mathcal{O}_{X,x}$, we conclude that $f \in \mathcal{I}_x$.

As $\mathcal{A}_x \subset \mathcal{I}_x$ for all $x \in X$ (in fact, we have equality here) and $I$ is finitely generated, using Cartan's Theorem B we can conclude that $A \subset I$. Hence we have $A = I$ and we finish by applying the previous theorem.         $\square$

REMARKS AND EXAMPLES 4.6.     (a) If $X$ is non-compact, then there are real primes ideals $\mathfrak{p}$ such that $\mathcal{Z}(\mathfrak{p}) = \emptyset$. For example, take an infinite discrete set $D = \bigcup_{i \in \mathbb{N}} x_i$ and let $N_\alpha$ be any free ultrafilter of $\mathbb{N}$, that is, $\bigcap_{J \in N_\alpha} J = \emptyset$. If we set $N_f := \{i \in \mathbb{N} \mid x_i \in \mathcal{Z}(f)\}$, then $\mathfrak{p} := \{f \in \mathcal{O}(X) \mid N_f \in N_\alpha\}$ is a real prime ideal and $\mathcal{Z}(\mathfrak{p}) = \emptyset$.

(b) Suppose that the regular locus of $X$ is not compact and let $Y = \bigcup_{i \in \mathbb{N}} Y_i$ be a global analytic subset, where the $Y_i$ are irreducible curves. Let $N_\alpha$ be any free ultrafilter of $\mathbb{N}$ and define $N_{a,b}(f) := \{i \in \mathbb{N} \mid m_{Y_i}(f) \geq ai+b\}$. Then the ideal

$$I := \{f \in \mathcal{O}(X) \mid Y \subset \mathcal{Z}(f) \text{ and } \forall a,b \in \mathbb{R}, \ N_{a,b}(f) \in N_\alpha\}$$

is real, $\mathcal{Z}(I) = Y$ and $I \neq \mathcal{IZ}(I)$. (For example, if $f \in \mathcal{O}(X)$ is such that $m_{Y_i}(f) = 1$ for all $i$, then $f \in \mathcal{IZ}(I)$, but clearly $f \notin I$.)

(c) It is known that if $I$ is a finitely generated ideal and $\mathcal{Z}(I)$ is compact, then $\sqrt[r]{I} = \mathcal{IZ}(I)$, cf. [An-Br-Rz], Theorem VIII.5.7. The next example shows that the assumption of compactness cannot be dropped, even for finitely generated ideals.

Let us take $Y = \bigcup_1^\infty Y_i$ and let $f \in \mathcal{O}(X)$ be such that $m_{Y_i}(f) = 2i$, for all $i$, and $\mathcal{Z}(f) = Y$. The ideal $I = (f)$ is finitely generated, $\mathcal{Z}(I) = Y$ is not compact and $\sqrt[r]{I} \neq \mathcal{IZ}(I)$. Take $g \in \mathcal{O}(X)$ such that $m_{Y_i}(g) = 1$ for all $i$, so that $g \in \mathcal{IZ}(I)$. If $g \in \sqrt[r]{I}$, then $h := g^{2k} + \sum a_i^2 \in I$ for some $k \in \mathbb{N}$ and some $a_i \in \mathcal{O}(X)$. But then $m_{Y_j}(h) \leq 2k$, $\forall j$, that is, $h$ cannot be a multiple of $f$ and so $h \notin I$.

(d) The above results (Theorem 4.5 and Corollary 4.5) remain true for any irreducible analytic surface $X$ if $\mathcal{Z}(I) \cap \text{Sing} \, X$ is a discrete set. In fact, we only need that for any irreducible component $Y$ of $\mathcal{Z}(I)$ the multiplicity along $Y$ can be defined.

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