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MONOMIAL IDEALS AND N-LISTS

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ABSTRACT. This paper generalizes a construction of Geramita, Harima, and Shin (Illinois J. Math. **45** (2001), 1–23). They give an inductive description of a certain set of elements called n-type vectors, and use these objects to prove various results about Hilbert functions of sets of points. We extend their notation by inductively describing the monomial ideals in R and identifying certain interesting subsets. We demonstrate that this new notation is useful by using it to calculate multiplicity and the degree of the Hilbert polynomial for quotients of Borel fixed ideals, and by giving another proof of the result of Geramita, Harima, and Shin: The set of n-type vectors is in bijective correspondence with all Hilbert functions of finite length cyclic R-modules over the polynomial ring $R = k[x_1, \ldots, x_n]$, where k is a field.

1. Introduction

In a recent paper [GeHaSh1], Geramita, Harima, and Shin explore the Hilbert function invariant in a novel way. First they inductively describe a new set of elements, called n-type vectors. These n-type vectors are then shown to be in one-to-one correspondence with the set of all Hilbert functions of finite length, cyclic $k[x_1, \ldots, x_n]$-modules, where k is a field.

This "alternative to the Hilbert function" has immediate applications. For instance, the set of n-type vectors can be used to classify the Hilbert functions of all depth 2 complete intersections. This new set is also sufficient to recover the numerical character of Gruson-Peskine and slightly generalize a result of M. Boij. In [GeHaSh2], Geramita, Harima, and Shin use n-type vector notation to calculate the graded Betti numbers of $I_{\mathbb{X}}$, where $\mathbb{X} \subset \mathbb{P}^n$ is any set of points with a given Hilbert function. This resolution is then shown to be extremal in the sense of Bigatti [Bi], Hulett [Hu], and Pardue [Pa]. Type vector notation is further used to calculate the graded Betti numbers of certain Gorenstein ideals. When the corresponding Hilbert functions have the weak Lefschetz property, n-type vectors expedite showing that these resolutions are extremal among those occurring for Gorensteins.

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The purpose of this paper is to generalize the set of n-type vectors and demonstrate that such a generalization is useful. In particular, we define a new set, the set of n-lists, obtained by relaxing the definition of the set of n-type vectors (in such a way that we can embed the set of n-type vectors in the set of n-lists). Then we develop an injective map from the set of n-lists to the set of monomial ideals in $k[x_1, \dots, x_n]$. This map and its inverse are both very important tools for manipulating n-lists, and they turn out to be the key ingredients used to prove the results we give as applications of our construction. These results are as follows. First we classify all Artinian monomial ideals using n-lists. Next, we use our notation to calculate the multiplicity and Hilbert polynomial degree for the quotient of any Borel fixed ideal. Finally, we use our results to give a new proof of Geramita, Harima, and Shin's correspondence between n-type vectors and the Hilbert functions of Artinian monomial ideals in $k[x_1, \dots, x_n]$. We conclude the paper by exploring a limitation of our new construction, and offer a slight generalization which addresses this constraint.

A more detailed outline is in order. In [GeHaSh1], the set of n-type vectors is defined roughly as follows (up to notation). A 1-type vector is an element of $\mathbb{N}^+$. A 2-type vector is a finite, ordered, strictly decreasing collection of 1-type vectors. A 3-type vector is a finite, ordered, "strictly decreasing" collection of 2-type vectors, where strictly decreasing is defined appropriately, and so on. Then there exists a bijective map, which we will call ψ_n, from the set of n-type vectors to the set of all Hilbert functions of Artinian ideals in $k[x_1, \dots, x_n]$. In [GeHaSh1] this map happens to be factored through the set of Hilbert functions of sets of points in $\mathbb{P}^n$.

It turns out that ψ_n can also be factored through the set of monomial ideals in $k[x_1, \dots, x_n]$. For example, let $\mathcal{T}$ be the 2-type vector $\mathcal{T} = (7, 4, 2)$. Then in this case $\psi_n(\mathcal{T})$ is the Hilbert function

$$\mathcal{H}(d) = H_7(d) + H_4(d-1) + H_2(d-2),$$

where

$$H_i(d) = \begin{cases} 1 & \text{for } 0 \le d < i, \\ 0 & \text{otherwise,} \end{cases}$$

that is, $\mathcal{H} = \{1, 2, 3, 3, 2, 1, 1\}$. We can also obtain a monomial ideal from $\mathcal{T}$. Let I be the ideal generated by monomials of the form $x_1^a x_2^i$, where a is the ith entry of $(7, 4, 2, 0)$, counting from zero. That is,

$$I = (x_1^7, x_1^4 x_2, x_1^2 x_2^2, x_2^3) \subset k[x_1, x_2].$$

Remarkably, the Hilbert function of R/I is exactly $H = \{1, 2, 3, 3, 2, 1, 1\}$.

The reason for generalizing the set of n-type vectors is that factoring ψ as demonstrated above does not require the definition of an n-type vector to be quite so rigid. Thus in Section 2 we define a new set, the set of n-lists, to capitalize on this latitude. Roughly, a 1-list is defined to be an element of $\mathbb{N}$.

A 2-list is an ordered, decreasing collection of 1-lists. A 3-list is an ordered, "decreasing" collection of 2-lists, where decreasing is defined appropriately, and so on. The apparent differences between n-lists and n-type vectors are that n-lists may be infinite, contain zero, and need only be decreasing. In Section 3, we show that n-type vectors can be thought of as n-lists.

Next, in Section 4, we give a map from the set of n-lists to the set of monomial ideals in $k[x_1, \ldots, x_n]$. This map, which we denote by Φ_n, is shown to be injective. Then in Section 5 we give an inverse to Φ_n. This pair of functions is crucial for successfully manipulating the set of n-lists.

In Section 6 we give the first application of our construction by characterizing all Artinian monomial ideals in terms of n-lists. Our second application is to calculate the multiplicities and Hilbert polynomial degrees for the quotients of Borel fixed ideals. This proceeds as follows. First, in Section 7 we give a bijection between the set of all Borel fixed ideals in $k[x_1, \ldots, x_n]$ and a certain subset of the set of n-lists. Then, in Section 8 we explore the characteristics of these "Borel" n-lists. Finally, in Section 9 we use our newfound knowledge about "Borel" n-lists to calculate the multiplicities and Hilbert polynomial degrees promised.

For our third application, in Section 10 we use the set of n-lists and the map Φ to give a new proof of Geramita, Harima, and Shin's result. In particular, we show that the set of n-type vectors, as a subset of the set of n-lists, is in bijective correspondence with the set of lex Artinian ideals in $k[x_1, \ldots, x_n]$. By Macaulay [Ma], this is enough to give a bijection between the set of n-type vectors and the set of all Hilbert functions of Artinian monomial ideals in $k[x_1, \ldots, x_n]$. We also demonstrate that Φ_n factors ψ_n, and thus that our construction actually yields the same map as in [GeHaSh1].

Finally, to be complete, in Section 11 we generalize the set of n-lists one step further to make Φ_n surjective. This allow us to classify all monomial ideals in $k[x_1, \ldots, x_n]$.

2. Definition of the set of n-lists

The set of n-lists is formed by relaxing the parameters defining the set of n-type vectors, and this requires changing the notation slightly. Therefore, the following definition of Geramita, Harima, and Shin has been reformatted to make the similarities between n-type vectors and n-lists more obvious.

Definition 2.1 (Geramita, Harima, Shin). A 1-type vector is a length 1 vector $\mathcal{T} \in \mathbb{N}$. For such a vector, define $\alpha_{\mathcal{T}} = \sigma(\mathcal{T}) = \mathcal{T}$.

Inductively, an n-type vector $\mathcal{T}$ is an ordered collection of $(n-1)$-type vectors $\mathcal{T}_0, \ldots, \mathcal{T}_{s-1}$,

$$\mathcal{T} = (\mathcal{T}_0, \ldots, \mathcal{T}_{s-1}),$$

where $\alpha_{\mathcal{T}_i} > \sigma(\mathcal{T}_{i+1})$, for $i = 0, \ldots, s-2$. Given such a $\mathcal{T}$ we define $\alpha_{\mathcal{T}} = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_0)$.

A more detailed explanation with several elucidating examples can be found in [GeHaSh1].

Next we give the definition of the set of n-lists. Note the following notational convention: If v is a vector in some set $\mathcal{D}^\infty$, say $v = (v_0, v_1, \dots)$, where $v_i \in \mathcal{D}$ for $i = 0, 1, \dots$, then we will write $v(i)$ to indicate the ith element, v_i.

Definition 2.2. A 1-list is a natural number $A \in \mathbb{N}$. For two 1-lists A and B we say that $A \geq B$ as 1-lists if $A \geq B$ as natural numbers. Inductively, an n-list for $n > 1$ is a vector $A \in \{(n-1)-\text{lists}\}^\infty$ such that $A(i) \geq A(i+1)$ as $(n-1)$-lists for all $i \in \mathbb{N}$. Two n-lists A and B have $A \geq B$ if $A(i) \geq B(i)$ as $(n-1)$-lists for all $i \in \mathbb{N}$.

The fact that n-lists are relaxed n-type vectors will require some proof, but is clear for $n = 1$ and $n = 2$. The set of 1-type vectors is $\mathbb{N}^+$, while the set of 1-lists is $\mathbb{N}$. A 2-type vector is a finite, ordered, strictly decreasing collection of non-zero natural numbers, while a 2-list is an ordered, decreasing collection of natural numbers.

In general, we want to show that the set of n-type vectors embeds in the set of n-lists. Of course, on the face of it this cannot be true, as n-type vectors are finite and n-lists are infinite, for $n \geq 2$. This turns out, however, to be a purely notational difficulty. Given an n-type vector $\mathcal{T}$, we simply suppose that $\mathcal{T}(i) = 0$ for all $i \geq \alpha_\mathcal{T}$, that is, wherever Geramita, Harima, and Shin leave $\mathcal{T}$ undefined. Then to show that $\mathcal{T}$ is an n-list, it is enough by induction to show that $\mathcal{T}(i) \geq \mathcal{T}(i+1)$ for all $i \in \mathbb{N}$. This is the goal of Section 3. First, however, we collect a few more notations.

Definition 2.3. Let A be an n-list. Then inductively we define

$$\sigma_n(A) = \begin{cases} A & \text{for } n = 1, \text{ and} \\ \sigma_{n-1}(A(0)) & \text{for } n > 1. \end{cases}$$

This extends Geramita, Harima, and Shin's σ notation. We will shortly define α in such a way that it also recovers Geramita, Harima, and Shin's use of the term.

Remark 2.4. Let A be an n-list. Then $A(i)$ is an $(n-1)$-list, $A(i)(j)$ is an $(n-2)$-list, and so on. More generally, $A(i_n)(i_{n-1})\dots(i_l)$ is an $(l-1)$-list for $2 \leq l \leq n$ and $A(i_n)(i_{n-1})\dots(i_2)$ is a 1-list.

If there is a smallest $\alpha \in \mathbb{N}$ with $A(\alpha) = A(\alpha + i) \neq 0$ for all $i \in \mathbb{N}$, then we will write

$$A = (A(0), A(1), \dots, A(\alpha) \rightarrow),$$

where the arrow indicates that the $(n-1)$-list $A(\alpha)$ is to be infinitely repeated. If instead there is a smallest α such that $A(\alpha) = A(\alpha + i) = 0$ for all $i \in \mathbb{N}$, then we will write

$$A = (A(0), A(1), \dots, A(\alpha - 1)).$$

Shortly, in Section 4, we will show that all n-lists stabilize, and thus that we can write down any n-list using this notation.

We also need a notion of the "zero" n-list.

DEFINITION 2.5. Let A be an n-list. We will write $A = 0$ and call A the zero n-list if, for $n = 1$, $A = 0$, and inductively for $n > 1$, $A(i) = 0$ as an $(n-1)$-list for all $i \in \mathbb{N}$.

EXAMPLE 2.6. The integers 5 and 0, for instance, are 1-lists, with $5 \geq 0$. Note that 5 is a 1-type vector, but of course 0 is not.

The vector $A = (5, 4, 4, 2)$ is a 2-list. Here A is not a 2-type vector because it is not strictly decreasing.

Since $(5, 4, 4, 2) \geq (4, 2)$, the vector $B = ((5, 4, 4, 2), (4, 2))$ is clearly a 3-list. It fails to be a 3-type vector, both because $(5, 4, 4, 2)$ is not a 2-type vector, as mentioned before, and because $\alpha_{(5,4,4,2)} = 4 \not> \sigma((4, 2)) = 4$.

Note that $C = (5, 4, 4, 2 \rightarrow)$ is a 2-list, where the arrow indicates that the 1-list 2 is to be infinitely repeated. Furthermore $D = (4, 2)$ and $E = (2, 1)$ are 2-lists and $C \geq D \geq E$, so

$$(C, D, E \rightarrow) = \Big((5, 4, 4, 2 \rightarrow), (4, 2), (2, 1) \rightarrow\Big)$$

is a 3-list, where the final arrow indicates that the 2-list $(2, 1)$ is to be infinitely repeated. Here $\sigma_3((C, D, E \rightarrow)) = \sigma_2(C) = \sigma_2((5, 4, 4, 2 \rightarrow)) = \sigma_1(5) = 5$.

The vector $F = \big((5, 4, 3, 1), (2, 1 \rightarrow)\big)$ fails to be a 3-list because $F(0) \not\geq F(1)$, that is, because $F(0)(4) = 0 \not\geq 1 = F(1)(4)$.

3. Embedding the set of n-type vectors in the set of n-lists

We now demonstrate that the set of n-type vectors embeds naturally into the set of n-lists. This is not difficult if one proves the right lemmas. Note that we overcome the simple obstruction to this embedding, that n-type vectors are not infinite, by supposing that they take the value zero everywhere Geramita, Harima, and Shin left them undefined. Thus for an n-type vector $\mathcal{T}$, we let $\mathcal{T}(i) = 0$ for all $i \geq \alpha_{\mathcal{T}}$.

We also extend the definitions of σ and α in the expected way by taking $\sigma(0) = \alpha_0 = 0$. It is helpful to note that if $\mathcal{T} \neq 0$, then $\alpha_{\mathcal{T}} \neq 0$ by definition. In particular, $\alpha_{\mathcal{T}(\alpha_{\mathcal{T}}-1)} \neq 0$.

EXAMPLE 3.1. Consider the 3-type vector

$$\mathcal{T} = \Big((7, 5, 3, 2, 1), (4, 3, 2), (2, 1)\Big).$$

Then $\alpha_{\mathcal{T}} = 3$, $\mathcal{T}(\alpha_{\mathcal{T}} - 1) = \mathcal{T}(2) = (2, 1)$, and $\alpha_{\mathcal{T}(\alpha_{\mathcal{T}}-1)} = \alpha_{(2,1)} = 2$.

LEMMA 3.2. *Let $\mathcal{T}$ be an n-type vector, $n \geq 2$. Then $\sigma(\mathcal{T}) \geq \alpha_{\mathcal{T}}$, and $\alpha_{\mathcal{T}(i)} \geq \alpha_{\mathcal{T}} - i$ for $i = 0, \ldots, (\alpha_{\mathcal{T}} - 1)$.*

Proof. Note that if $n = 1$, then $\sigma(\mathcal{T}) \geq \alpha_{\mathcal{T}}$ by definition. Thus when $n \geq 2$ we may assume that $\sigma(\mathcal{T}(i)) \geq \alpha_{\mathcal{T}(i)}$, for $i = 0, \ldots, (\alpha_{\mathcal{T}} - 1)$. Recall that by definition $\alpha_{\mathcal{T}(i)} > \sigma(\mathcal{T}(i+1))$. Also, as mentioned above, $\alpha_{\mathcal{T}(\alpha_{\mathcal{T}}-1)} > 0$, so writing $\lambda = \alpha_{\mathcal{T}} - i - 1$, we conclude that

$$\begin{aligned}\alpha_{\mathcal{T}(i)} &\geq \sigma(\mathcal{T}(i+1)) + 1 \geq \alpha_{\mathcal{T}(i+1)} + 1 \geq \sigma(\mathcal{T}(i+2)) + 2 \\ &\geq \alpha_{\mathcal{T}(i+2)} + 2 \geq \cdots \geq \alpha_{\mathcal{T}(i+\lambda)} + \lambda \\ &= \alpha_{\mathcal{T}(\alpha_{\mathcal{T}}-1)} + \lambda \geq 1 + \lambda = \alpha_{\mathcal{T}} - i.\end{aligned}$$

Moreover, when $i = 0$, we have that $\sigma(\mathcal{T}) = \sigma(\mathcal{T}(0)) \geq \alpha_{\mathcal{T}(0)} \geq \alpha_{\mathcal{T}}$, as required. □

LEMMA 3.3. *Let $\mathcal{T}$ be an n-type vector, $n \geq 2$. Then $\sigma(\mathcal{T}) \geq \sigma(\mathcal{T}(i)) + i$ for all $i = 0, \ldots, (\alpha_{\mathcal{T}} - 1)$.*

Proof. Because $\sigma(\mathcal{T}) = \sigma(\mathcal{T}(0))$, it is enough to show that $\sigma(\mathcal{T}(i)) \geq \sigma(\mathcal{T}(i+1)) + 1$ for $1 \leq i \leq \alpha_{\mathcal{T}} - 2$. This follows because $\sigma(\mathcal{T}(i)) \geq \alpha_{\mathcal{T}(i)}$ by Lemma 3.2, while $\alpha_{\mathcal{T}(i)} \geq \sigma(\mathcal{T}(i+1)) + 1$ because $\mathcal{T}$ is an n-type vector. □

LEMMA 3.4. *Let $\mathcal{T}$ be an n-type vector, for $n \geq 2$. Then*

$$\alpha_{\mathcal{T}(i_n)(i_{n-1})\ldots(i_l)} > \alpha_{\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)}$$

for all $2 \leq l \leq n$, where $0 \leq i_n < \alpha_{\mathcal{T}} - 1$, $0 \leq i_{n-1} < \alpha_{\mathcal{T}(i_n+1)}$, and $0 \leq i_t < \alpha_{\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_{t+1})}$, for $t = (n-2), \ldots, l$.

Proof. The proof is by descending induction on l. For $l = n$, $\alpha_{\mathcal{T}(i_n)} > \sigma(\mathcal{T}(i_n+1)) \geq \alpha_{\mathcal{T}(i_n+1)}$ by the definition of an n-type vector and Lemma 3.2.

For $l < n$ we can conclude that $\alpha_{\mathcal{T}(i_n)\ldots(i_p)} > \alpha_{\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_p)}$ for each $p = (n-1), \ldots, (l+1)$. This allows us to iterate Lemma 3.2 whence we conclude that $\alpha_{\mathcal{T}(i_n)\ldots(i_l)} \geq \alpha_{\mathcal{T}(i_n)} - \sum_{t=l}^{n-1} i_t$.

Then because $\mathcal{T}$ is an n-type vector,

$$\alpha_{\mathcal{T}(i_n)} - \sum_{t=l}^{n-1} i_t > \sigma(\mathcal{T}(i_n+1)) - \sum_{t=l}^{n-1} i_t,$$

and by iterating Lemma 3.3, we conclude that

$$\begin{aligned}\sigma(\mathcal{T}(i_n+1)) - \sum_{t=l}^{n-1} i_t &\geq \sigma(\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)) + \sum_{t=l}^{n-1} i_t - \sum_{t=l}^{n-1} i_t \\ &= \sigma(\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)).\end{aligned}$$

Finally by Lemma 3.2, we know that

$$\sigma(\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)) \geq \alpha_{\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)}.$$

Thus $\alpha_{\mathcal{T}(i_n)(i_{n-1})\ldots(i_l)} > \alpha_{\mathcal{T}(i_n+1)(i_{n-1})\ldots(i_l)}$ as required. □

PROPOSITION 3.5. *Let $\mathcal{T}$ be an n-type vector. Then $\mathcal{T}$ is an n-list.*

Proof. If $n = 1$ this is clear. If $n > 1$, then it is enough to show that $\mathcal{T}(i) \geq \mathcal{T}(i+1)$ for all $i \in \mathbb{N}$. This is only interesting if $0 \leq i < (\alpha_{\mathcal{T}} - 1)$. Given such an i, we are required to show that $\mathcal{T}(i)(i_{n-1})\dots(i_2) \geq T(i+1)(i_{n-1})\dots(i_2)$ for all $\{i_{n-1},\dots,i_2\} \in \mathbb{N}^{n-2}$. This in turn is only interesting if $0 \leq i_{n-1} < \alpha_{\mathcal{T}(i+1)}$ and $0 \leq i_t < \alpha_{\mathcal{T}(i+1)(i_{n-1})\dots(i_{t+1})}$ for $t = (n-2),\dots,2$. But for such i_t, $\mathcal{T}(i)(i_{n-1})\dots(i_2)$ and $\mathcal{T}(i+1)(i_{n-1})\dots(i_2)$ are 1-type vectors, that is,

$$\mathcal{T}(i)(i_{n-1})\dots(i_2) = \alpha_{\mathcal{T}(i)(i_{n-1})\dots(i_2)}$$

and

$$T(i+1)(i_{n-1})\dots(i_2) = \alpha_{T(i+1)(i_{n-1})\dots(i_2)}.$$

Thus Lemma 3.4 completes the proof. $\square$

REMARK 3.6. Let $\mathcal{T}$ be an n-type vector. Then in Proposition 3.5 we not only show that $\mathcal{T}(i) \geq \mathcal{T}(i+1)$, but also that $\mathcal{T}(i) \neq \mathcal{T}(i+1)$ for all $i = 0,\dots,(\alpha_{\mathcal{T}} - 1)$.

4. The injective map

We now give a map from the set of n-lists to the set of monomial ideals in $k[x_1,\dots,x_n]$.

DEFINITION 4.1. Let A be an n-list. Then let

$$\Phi_n : \{n - \text{lists}\} \longrightarrow \{\text{monomial ideals in } k[x_1,\dots,x_n]\}$$

be the function defined inductively as

$$\Phi_n[A] = \begin{cases} (x_1^A) \subseteq k[x_1] & \text{for } n = 1, \\ \Phi_{n-1}[A(0)] + x_n\Phi_{n-1}[A(1)] + x_n^2\Phi_{n-1}[A(2)] + & \\ \quad \cdots + x_n^j\Phi_{n-1}[A(j)] + \cdots \subseteq k[x_1,\dots,x_n] & \text{for } n > 1. \end{cases}$$

When the meaning is clear, we will drop the subscript from Φ_n.

EXAMPLE 4.2. Let us evaluate Φ_n on two of the n-lists found in Example 2.6.

First

$$\begin{aligned}\Phi_2\big[(5,4,4,2\rightarrow)\big] &= \Phi_1[5] + x_2\Phi_1[4] + x_2^2\Phi_1[4] + x_2^3\Phi_1[2] + x_2^4\Phi_1[2] + \cdots \\ &= (x_1^5, x_1^4x_2, x_1^2x_2^3) \subset k[x_1,x_2].\end{aligned}$$

Likewise,

$$\begin{aligned}
&\Phi_3\big[((5,4,4,2\rightarrow),(4,2),(2,1)\rightarrow)\big]\\
&\quad = (x_1^5, x_1^4x_2, x_1^2x_2^3) + x_3(\Phi_2[(4,2)])\\
&\qquad\qquad + x_3^2(\Phi_2[(2,1)]) + x_3^3(\Phi_2[(2,1)]) + \cdots\\
&\quad = (x_1^5, x_1^4x_2, x_1^2x_2^3) + x_3(\Phi_1[4] + x_2\Phi_1[2] + x_2^2\Phi_1[0])\\
&\qquad\qquad + x_3^2(\Phi_1[2] + x_2\Phi_1[1] + x_2^2\Phi_1[0]) + \cdots\\
&\quad = (x_1^5, x_1^4x_2, x_1^2x_2^3) + x_3(x_1^4, x_1^2x_2, x_2^2) + x_3^2(x_1^2, x_1x_2, x_2^2) + \cdots\\
&\quad = (x_1^5, x_1^4x_2, x_1^2x_2^3, x_1^4x_3, x_1^2x_2x_3, x_2^2x_3, x_1^2x_3^2, x_1x_2x_3^2) \subset k[x_1,x_2,x_3].
\end{aligned}$$

Before we prove that Φ is injective, it is helpful to collect the following facts. We omit the proof of the lemma.

LEMMA 4.3. *Let A and B be n-lists. If $A \geq B$ and $A = 0$, then $B = 0$. If $A \geq B \geq A$, then $A = B$. Finally, $A = 0$ if and only if $\Phi_n[A] = k[x_1,\dots,x_n]$.*

PROPOSITION 4.4. *Let A and B be n-lists. Then $A \geq B$ if and only if $\Phi_n[A] \subseteq \Phi_n[B]$.*

Proof. The proof is by induction. If $n = 1$, then $A \geq B$ if and only if $\Phi_1[A] = (x_1^A) \subseteq (x_1^B) = \Phi_1[B]$.

So suppose that $n > 1$. If $A \geq B$, then $A(i) \geq B(i)$ for all $i \in \mathbb{N}$. Hence by induction $\Phi_{n-1}[A(i)] \subseteq \Phi_{n-1}[B(i)]$ for all $i \in \mathbb{N}$. By considering the definition of Φ, this implies that $\Phi_n[A] \subseteq \Phi_n[B]$.

Suppose, on the other hand, that $\Phi_n[A] \subseteq \Phi_n[B]$. Now because B is an n-list, we know that $B(i) \geq B(i+1)$ for all $i \in \mathbb{N}$. Thus $\Phi_{n-1}[B(i)] \subseteq \Phi_{n-1}[B(i+1)]$ by induction, and hence

$$\Phi_{n-1}[B(0)] \oplus x_n\Phi_{n-1}[B(1)] \oplus x_n^2\Phi_{n-1}[B(2)] \oplus \cdots \oplus x_n^j\Phi_{n-1}[B(j)] \oplus \cdots$$

is the unique x_n-graded decomposition of $\Phi_n[B]$. Then $x_n^i\Phi_{n-1}[A(i)] \subseteq \Phi_n[A] \subseteq \Phi_n[B] \Longrightarrow \Phi_{n-1}[A(i)] \subseteq \Phi_{n-1}[B(i)]$, whence by induction we conclude that $A(i) \geq B(i)$ for all $i \in \mathbb{N}$, or $A \geq B$. □

COROLLARY 4.5. *Φ is injective.*

Proof. Apply Proposition 4.4 and Lemma 4.3. □

REMARK 4.6. As we used in the proof of Proposition 4.4, if A is an n-list, then

$$\Phi_n[A] = \Phi_{n-1}[A(0)] \oplus x_n\Phi_{n-1}[A(1)] \oplus \cdots \oplus x_n^j\Phi_{n-1}[A(j)] \oplus \cdots$$

is the unique x_n-graded decomposition of $\Phi_n[A]$. Thus $x_1^{i_1}\dots x_n^{i_n} \in \Phi_n[A]$ if and only if $x_1^{i_1}\dots x_{n-1}^{i_{n-1}} \in \Phi_{n-1}[A(i_n)]$. Notice also that A stabilizes, that is, there is a smallest $\alpha \in \mathbb{N}$ such that $A(\alpha) = A(\alpha + i)$ for all $i \in \mathbb{N}$. This result, which follows from Proposition 4.4 and its corollary as well as the fact that $k[x_1, \ldots, x_n]$ is Noetherian, makes it apparent that we can write down any n-list using the notation in Remark 2.4.

Definition 4.7. Let A be an n-list. If $n > 1$, then we define α_A to be the smallest integer such that $A(\alpha_A) = A(\alpha_A + i)$ for all $i \in \mathbb{N}$. If $n = 1$ our convention will be to let $\alpha_A = A$.

Example 4.8. The 3-list

$$A = \Big((12, 5, 3, 2 \rightarrow), (6, 3, 2, 1), (2, 1) \rightarrow \Big)$$

and the 4-list

$$B = \Big(\Big((5, 2, 1 \rightarrow), (2, 1 \rightarrow) \rightarrow \Big), \Big((3, 1 \rightarrow), (2) \rightarrow \Big)\Big)$$

have $\alpha_A = \alpha_B = 2$.

Remark 4.9. Note that on n-type vectors, our definition of α coincides with that of Geramita, Harima, and Shin as given in Definition 2.1.

5. The inverse map

The next step is to construct an inverse to Φ.

Definition 5.1. Let $I \subseteq k[x_1, \ldots, x_n]$ be a non-zero monomial ideal with unique x_n-graded decomposition $I = I_0 \oplus x_n I_1 \oplus x_n^2 I_2 \cdots \oplus x_n^j I_j \oplus \cdots$. Here $I_j = \{x_1^{i_1} \ldots x_{n-1}^{i_{n-1}} \mid x_1^{i_1} \ldots x_{n-1}^{i_{n-1}} x_n^j \in I\} \subseteq k[x_1, \ldots, x_{n-1}]$. Note that if $n = 1$, then $I = (0) \oplus x_1(0) \oplus \cdots \oplus x_1^{\lambda(I)-1}(0) \oplus x_1^{\lambda(I)}(1) \oplus x_1^{\lambda(I)+1}(1) \oplus \cdots$ for some $\lambda(I) \in \mathbb{N}$, that is, $I = (x_1^{\lambda(I)})$. Then let

$$\rho_n(I)\colon \{\text{monomial ideals in } k[x_1, \ldots, x_n] \text{ containing a power of } x_1\} \rightarrow \{\text{n-lists}\}$$

be the function defined by

$$\rho_n(I) = \begin{cases} \lambda(I) & \text{for } n = 1, \\ \Big(\rho_{n-1}(I_0), \rho_{n-1}(I_1), \ldots\Big) & \text{for } n > 1. \end{cases}$$

This definition needs some justification, namely, we must show that $\rho_n(I)$ is an n-list. At the same time we can show that ρ_n provides an inverse to Φ_n.

Proposition 5.2. *Let I be a monomial ideal in $k[x_1, \ldots, x_n]$ which contains some power of x_1. Then $\rho_n(I)$ is an n-list, and $\Phi_n[\rho_n(I)] = I$.*

Proof. We use induction. If $n = 1$, then $I = (x_1^{\lambda(I)})$ for some $\lambda(I) \in \mathbb{N}$. It is clear both that $\rho_1(I) = \lambda(I)$ is a 1-list and that $\Phi_1[\rho_1(x_1^{\lambda(I)})] = \Phi_1[\lambda(I)] = (x_1^{\lambda(I)})$.

If $n > 1$, then let $I = I_0 \oplus x_n I_1 \oplus x_n^2 I_2 \oplus \cdots \oplus x_n^j I_j \oplus \cdots$ be the unique x_n-graded decomposition of I. By induction, $\rho_{n-1}(I_j)$ is an $(n-1)$-list for all $j \in \mathbb{N}$. But because $I_j \subseteq I_{j+1}$ for all $j \in \mathbb{N}$, we also have by induction that $\Phi_{n-1}[\rho_{n-1}(I_j)] = I_j \subseteq I_{j+1} = \Phi_{n-1}[\rho_{n-1}(I_{j+1})]$. Thus by Proposition 4.4, $\rho_{n-1}(I_j) \geq \rho_{n-1}(I_{j+1})$. We conclude that $\rho_n(I)$ is an n-list. Also it is clear that

$$\begin{aligned}
\Phi_n[\rho_n(I)] &= \Phi_n\Big[(\rho_{n-1}(I_0), \rho_{n-1}(I_1), \rho_{n-1}(I_2), \ldots, \rho_{n-1}(I_j), \ldots)\Big] \\
&= \Phi_{n-1}[\rho_{n-1}(I_0)] \oplus x_n \Phi_{n-1}[\rho_{n-1}(I_1)] \oplus \cdots \\
&\qquad \cdots \oplus x_n^j \Phi_{n-1}[\rho_{n-1}(I_j)] \oplus \cdots \\
&= I_0 \oplus x_n I_1 \oplus \cdots \oplus x_n^j I_j \oplus \cdots = I,
\end{aligned}$$

so that $\Phi_n[\rho_n(I)] = I$ as required. □

6. The correspondence with Artinian monomial ideals

We now give a few results which demonstrate the efficacy of n-list notation. Our first application is to use n-lists to classify the Artinian monomial ideals in $k[x_1, \ldots, x_n]$. As well as being independently interesting, this will prove useful in Section 9 when we calculate Hilbert polynomial degrees and multiplicities of Borel fixed monomial ideals, and in Section 10 when we recover Geramita, Harima, and Shin's bijection.

Our procedure is to identify a subset of the set of n-lists in bijective correspondence, under Φ_n, with the set of Artinian monomial ideals.

Definition 6.1. Let A be an n-list. Then if $n = 1$, we call A Artinian. For $n > 1$, we call A Artinian if $A(\alpha_A) = 0$ and $A(i)$ is Artinian for all $0 \leq i \leq \alpha_A - 1$.

We must, of course, justify Definition 6.1, but first we give an example. The justification follows.

Example 6.2. The 2-list $A = (5, 4, 4, 2)$ is Artinian, because $A(\alpha_A) = 0$. On the other hand, $B = (5, 4, 4, 2 \rightarrow)$ clearly fails in this regard.

Theorem 6.3. *The map Φ_n is a bijection between the set of Artinian n-lists and the set of Artinian monomial ideals in $k[x_1, \ldots, x_n]$.*

Proof. It is easy to show that Φ_n sends an Artinian n-list to an Artinian ideal. Indeed, if $n = 1$ this is trivial. For $n > 1$ we have by induction that $x_j^{i_j} \in \Phi_{n-1}[A(0)] \subseteq \Phi_n[A]$ for some $i_j \gg 0$, $j = 1, \ldots, (n-1)$. But $A(\alpha_A) = 0$, so $1 \in \Phi_{n-1}[A(\alpha_A)]$ by Lemma 4.3, and thus $x_n^{\alpha_A} \in \Phi_n[A]$ as well.

Thus it remains to demonstrate that the image of an Artinian monomial ideal under ρ_n is an Artinian n-list. So let $I \subseteq k[x_1, \dots, x_n]$ be an Artinian monomial ideal with unique x_n-graded decomposition $I = I_0 \oplus x_n I_1 \oplus x_n^2 I_2 \oplus \cdots \oplus x_n^j I_j \oplus \cdots$. If $n = 1$, then the result is obvious. If $n > 1$, then write $\rho_n(I) = A$. By induction, $\rho_{n-1}(I_j) = A(j)$ is Artinian for all $j \in \mathbb{N}$, so we only need to show that $A(\alpha_A) = 0$. Because I is Artinian, $x_n^\alpha \in I$ for some $\alpha \gg 0$. In particular, this implies that $1 \in I_\alpha = \Phi_{n-1}[A(\alpha)]$ or that $\Phi_{n-1}[A(\alpha)] = k[x_1, \dots, x_{n-1}]$. Thus by Lemma 4.3, $A(\alpha) = 0$ and because $A(\alpha) \geq A(\alpha_A)$ we conclude that $A(\alpha_A) = 0$. $\square$

Remark 6.4. It is clear that an n-type vector is an Artinian n-list. Thus Φ_n sends any n-type vector to an Artinian ideal.

7. The correspondence with Borel fixed ideals

Our next application is somewhat more involved. The goal is to calculate the multiplicities and Hilbert polynomial degrees of quotients of Borel fixed ideals using n-list notation. This program will require three steps. First, in this section, we will identify the subset of n-lists in bijective correspondence with the set of Borel fixed ideals in $k[x_1, \dots, x_n]$. Next, in Section 8 we will explore the properties of these so-called "Borel" n-lists. Finally, in Section 9, we will use the results of Sections 7 and 8 to make the calculations promised.

In order to identify the n-lists corresponding to Borel fixed ideals, we must first specify an order on the variables. Note that given any n-list A, some power of x_1 must be in $\Phi[A]$. Borel fixed ideals also have this property when x_1 is the largest variable, so we take $x_1 > \cdots > x_n$. The inability of n-list notation to represent monomial ideals which do not contain a power of x_1 will be discussed in Section 11.

To describe the set of "Borel" n-lists, we need a new n-list relation.

Definition 7.1. Let A and B be n-lists. Then we say $A \leq_{\mathfrak{B}} B$ when, for $n = 1$, $A - B \leq 1$, and inductively for $n > 1$, $A(i+1) \leq B(i)$ for all $i \in \mathbb{N}$.

Example 7.2. We have

$$\Big((6,3,1 \to),(3,2,1 \to),(2,1) \to\Big) \leq_{\mathfrak{B}} \Big((4,3,1 \to),(2,2,1),(2,1) \to\Big),$$

but

$$\Big((6,3,1 \to),(3,2,1 \to) \to\Big) \not\leq_{\mathfrak{B}} \Big((3,1 \to)\Big) \text{ as } (3,2,1 \to) \not\leq (3,1 \to).$$

Definition 7.3. Let A be an n-list. Then if $n = 1$, we call A Borel. For $n > 1$ we call A Borel if $A(i) \leq_{\mathfrak{B}} A(i+1)$ and $A(i)$ is Borel as an $(n-1)$-list for all $i \in \mathbb{N}$.

Example 7.4. It is easy to check that the 3-list

$$\Big((6,5,5,4,3,2\rightarrow),(6,5,4,3,2\rightarrow),(5,4,4,3,2\rightarrow)\rightarrow\Big),$$

for example, is Borel. The 3-list

$$A=\Big((6,4,3,2\rightarrow),(3,3,2\rightarrow),(3,2,1\rightarrow)\rightarrow\Big)$$

is not Borel, for three separate reasons. First, $A(0)(0)\not\leq_{\mathfrak{B}} A(0)(1)$ or $6-4\not\leq 1$, so that $A(0)$ is not Borel. Second, $A(0)\not\leq_{\mathfrak{B}} A(1)$ as $A(0)(1)=4\not\leq 3=A(1)(0)$. Finally, $A(1)\not\leq_{\mathfrak{B}} A(2)$ as $A(1)(3)=2\not\leq 1=A(2)(2)$.

To prove that Definition 7.3 is justified we first show that Φ takes Borel n-lists to Borel fixed ideals. Then we demonstrate that the image of ρ on any Borel fixed ideal is Borel.

Proposition 7.5. *Let A be a Borel n-list. Then $\Phi_n[A]$ is Borel fixed.*

Proof. The proof is by induction. If $n=1$, then all 1-lists are Borel and all ideals $\Phi_1[A]=(x_1^A)\subseteq k[x_1]$ are Borel fixed.

If $n=2$, then we must show that $(x_1/x_2)(x_1^{i_1}x_2^{i_2})\in\Phi_2[A]$ whenever $x_1^{i_1}x_2^{i_2}\in\Phi_2[A]$ and $i_2\neq 0$. This is equivalent to showing that $x_1^{i_1+1}x_2^{i_2-1}\in\Phi_2[A]$, or $x_1^{i_1+1}\in\Phi_1[A(i_2-1)]$. By hypothesis $A(i_2-1)\leq_{\mathfrak{B}} A(i_2)\Rightarrow A(i_2-1)-A(i_2)\leq 1\Rightarrow A(i_2-1)\leq 1+A(i_2)$. Also $x_1^{i_1}x_2^{i_2}\in\Phi_2[A]\Rightarrow x_1^{i_1}\in\Phi_1[A(i_2)]=(x_1^{A(i_2)})\Rightarrow i_1-A(i_2)\geq 0$. Therefore $x_1^{i_1+1}=x_1^{i_1-A(i_2)}x_1^{A(i_2)+1}\in(x_1^{A(i_2-1)})=\Phi_1[A(i_2-1)]$ and thus $x_1^{i_1+1}x_2^{i_2-1}\in\Phi_2[A]$.

So suppose $n>2$ and $x_1^{i_1}\dots x_n^{i_n}\in\Phi_n[A]$. We are required to show that $(x_{j-1}/x_j)(x_1^{i_1}\dots x_n^{i_n})\in\Phi_n[A]$ whenever $i_j\neq 0$ for some $j\in\{2,\dots,n\}$. If in fact $j\in\{2,\dots,(n-1)\}$, then $(x_1^{i_1}\dots x_{n-1}^{i_{n-1}})\in\Phi_{n-1}[A(i_n)]$, and by induction, $(x_{j-1}/x_j)(x_1^{i_1}\dots x_{n-1}^{i_{n-1}})\in\Phi_{n-1}[A(i_n)]$ so that $(x_{j-1}/x_j)(x_1^{i_1}\dots x_n^{i_n})\in\Phi_n[A]$. If $j=n$, then in particular $i_n\neq 0$, and we must show that

$$(x_1^{i_1}\dots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}+1}x_n^{i_n-1})\in\Phi_n[A],$$

or that $(x_1^{i_1}\dots x_{n-2}^{i_{n-2}})\in\Phi_{n-2}[A(i_n-1)(i_{n-1}+1)]$. But $A(i_n-1)\leq_{\mathfrak{B}} A(i_n)\Rightarrow A(i_n-1)(i_{n-1}+1)\leq A(i_n)(i_{n-1})$. Hence by Proposition 4.4,

$$\Phi_{n-2}[A(i_n-1)(i_{n-1}+1)]\supseteq\Phi_{n-2}[A(i_n)(i_{n-1})].$$

Then $x_1^{i_1}\dots x_n^{i_n}\in\Phi_n[A]$ implies that

$$x_1^{i_1}\dots x_{n-2}^{i_{n-2}}\in\Phi_{n-2}[A(i_n)(i_{n-1})]\subseteq\Phi_{n-2}[A(i_n-1)(i_{n-1}+1)],$$

and thus $(x_1^{i_1}\dots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}+1}x_n^{i_n-1})\in\Phi_n[A]$ as required. $\square$

Theorem 7.6. *Φ_n is a bijection from the set of Borel n-lists to the set of non-zero Borel fixed ideals in $k[x_1,\dots,x_n]$.*

Proof. By Corollary 4.5 and Propositions 5.2 and 7.5 it is enough to show that ρ_n sends a non-zero Borel fixed ideal to a Borel n-list.

If $n = 1$, then this is obvious.

If $n > 1$, then suppose that $I \subseteq k[x_1, \dots, x_n]$ is Borel fixed, $I \neq 0$, and let $I = I_0 \oplus x_n I_1 \oplus x_n^2 I_2 \oplus \cdots \oplus x_n^j I_j \oplus \cdots$ be the unique x_n-graded decomposition of I. Write $\rho_n(I) = A$. Then each I_j is non-zero and Borel fixed, so $A(j)$ is Borel by induction. It remains to show that $A(j) \leq_{\mathfrak{B}} A(j+1)$ for all $j \in \mathbb{N}$. We treat separately the cases $n = 2$ and $n > 2$.

If $n = 2$, then for any $j \in \mathbb{N}$, $I_{j+1} = \Phi_1[A(j+1)] = (x_1^{A(j+1)})$ and $x_1^{A(j+1)} x_2^{j+1} \in I$. Thus $x_1^{A(j+1)+1} x_2^j \in I$, as I is Borel fixed, and $x_1^{A(j+1)+1} \in I_j = \Phi_1[A(j)] = (x_1^{A(j)})$ implies that $A(j) \leq A(j+1) + 1$, as required.

If $n > 2$, then we must show that $A(j)(t+1) \leq A(j+1)(t)$ for all $j, t \in \mathbb{N}$, which by Proposition 4.4 is equivalent to demonstrating that

$$\Phi_{n-2}[A(j)(t+1)] \supseteq \Phi_{n-2}[A(j+1)(t)].$$

Consider any $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} \in \Phi_{n-2}[A(j+1)(t)]$. Then $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} x_{n-1}^t x_n^{j+1} \in \Phi_n[A] = I$ and $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} x_{n-1}^{t+1} x_n^j \in I$ because I is Borel fixed. Therefore $x_1^{i_1} \dots x_{n-2}^{i_{n-2}} \in \Phi_{n-2}[A(j)(t+1)]$, and we conclude that the necessary inclusion is true. □

8. Some properties of Borel n-lists

In order to compute Hilbert polynomial degrees and multiplicities in Section 9 a short study of Borel n-lists is required.

LEMMA 8.1. *Let A and B be n-lists such that $A \geq B$ and $A \leq_{\mathfrak{B}} B$. If A is Borel, then $A(i_n) \dots (i_t) \leq_{\mathfrak{B}} B(i_n) \dots (i_t)$ for all $t = 2, \dots, n$.*

Proof. It is enough to show that $A(i_n) \leq_{\mathfrak{B}} B(i_n)$. For $n = 1$ this is vacuous. If $n = 2$, then $B(i_2) \geq A(i_2 + 1)$ and $A(i_2) - A(i_2 + 1) \leq 1$. This forces $A(i_2) - B(i_2) \leq 1$, or $A(i_2) \leq_{\mathfrak{B}} B(i_2)$ as required. If $n > 2$, then $A(i_n)(j+1) \leq A(i_n + 1)(j)$ for all $j \in \mathbb{N}$. But $A \leq_{\mathfrak{B}} B$ implies that $A(i_n + 1) \leq B(i_n)$ and thus $A(i_n)(j+1) \leq A(i_n+1)(j) \leq B(i_n)(j)$, that is, $A(i_n) \leq_{\mathfrak{B}} B(i_n)$ as required. □

DEFINITION 8.2. Let A be an n-list. If $A(\alpha_A) \neq 0$, then we say that A has an $(n-1)$-arrow. We say that A has a p-arrow if, for some $\{i_n, \dots, i_{p+2}\} \in \mathbb{N}^{n-p-1}$, the $(p+1)$-list $A(i_n) \dots (i_{p+2})$ has a p-arrow.

EXAMPLE 8.3. The 2-list $(5, 4, 4, 2 \rightarrow)$ has a 1-arrow (a 1-list is repeated) while the 2-list $(5, 4, 4, 2)$ does not have a 1-arrow. The 3-list $\big((4 \rightarrow), (2 \rightarrow)\big)$ has a 1-arrow, but does not have a 2-arrow. The terminology was chosen to be suggestive of the notation: A 1-arrow signifies the repeat of a 1-list, a 2-arrow repeats a 2-list, and so on.

LEMMA 8.4. *Let A and B be n-lists. If $A \geq B$ and B has an $(n–1)$-arrow, then A has an $(n-1)$-arrow. If $A \leq_{\mathfrak{B}} B$ and A has an $(n-1)$-arrow, then B has an $(n-1)$-arrow.*

Proof. For $n = 1$ this is vacuous, so we may take $n \geq 2$. If $A \geq B$ and B has an $(n-1)$-arrow, then $A(\alpha_A) \geq B(\alpha_A) \geq B(\alpha_B) \neq 0$. Thus A has an $(n-1)$-arrow.

If $A \leq_{\mathfrak{B}} B$ and A has an $(n–1)$-arrow, then $B(\alpha_B) \geq A(\alpha_B+1) \geq A(\alpha_A) \neq 0$. Thus B has an $(n-1)$-arrow. □

LEMMA 8.5. *If A is an n-list and the $(p+1)$-list $A(i_n)\dots(i_{p+2})$ has a p-arrow, then $A(i_n)\dots(i_{t+1})(i_t-1)(i_{t-1})\dots(i_{p+2})$ has a p-arrow for all $t \in \{(p+2),\dots,n\}$ such that $i_t \neq 0$.*

Proof. Because $A(i_n)\dots(i_{t+1})(i_t-1) \geq A(i_n)\dots(i_{t+1})(i_t)$, we know that $A(i_n)\dots(i_{t+1})(i_t-1)(i_{t-1})\dots(i_{p+2}) \geq A(i_n)\dots(i_{t+1})(i_t)(i_{t-1})\dots(i_{p+2})$, whence we apply Lemma 8.4. □

LEMMA 8.6. *If A is a Borel n-list and the $(p+1)$-list $A(i_n)\dots(i_{p+2})$ has a p-arrow, then $A(i_n)\dots(i_{t+1})(i_t+1)(i_{t-1})\dots(i_{p+2})$ has a p-arrow for all $t \in \{(p+2),\dots,n\}$.*

Proof. We know that $A(i_n)\dots(i_{t+1})(i_t) \leq_{\mathfrak{B}} A(i_n)\dots(i_{t+1})(i_t+1)$ because $A(i_n)\dots(i_{t+1})$ is Borel, and thus Lemmas 8.1 and 8.4 complete the proof. □

LEMMA 8.7. *If A is a Borel n-list and the $(p+1)$-list $A(i_n)\dots(i_{p+2})$ has a p-arrow, then $A(j_n)\dots(j_{p+2})$ has a p-arrow for all $\{j_n,\dots,j_{p+2}\} \in \mathbb{N}^{n-p-1}$.*

Proof. This is a clear result of Lemmas 8.5 and 8.6. □

Lemma 8.7 will be the main technical result we need to calculate multiplicities and Hilbert polynomial degrees in Section 10. To be complete, however, we also include the following corollary. We find this result useful when attempting to show by hand that an n-list is not Borel.

COROLLARY 8.8. *If A is a Borel n-list and $A(i_n)\dots(i_{p+2})$ fails to have a p-arrow for some $\{i_n,\dots,i_{p+2}\} \in \mathbb{N}^{n-p-1}$, then A fails to have an l-arrow for all $1 \leq l \leq p$.*

Proof. Whenever $A(i_n)\dots(i_{n-q+2})$ fails to have an $(n-q)$-arrow, we know that $A(i_n)\dots(i_{n-q+2})(\alpha_{A(i_n)\dots(i_{n-q+2})}) = 0$, and thus has no $(n-l)$-arrows for all $q+1 \leq l \leq (n-1)$. The contrapositive of Lemma 8.7 completes the proof. □

EXAMPLE 8.9. One can check that the 4-list

$$\Big(\Big((3,2,1),(2,1)\to\Big),\Big((2,1)\to\Big)\to\Big)$$

is Borel. However

$$A = \Big(\Big((3,2,1 \to), (2,1) \to\Big), \Big((2,1) \to\Big) \to\Big)$$

is clearly not Borel as $A(0)(0)$ has a 1-arrow while $A(0)(1)$ and $A(1)(0)$ do not. Likewise, it is easy to see that

$$A = \Big(\Big((3,2,1), (2,1) \to\Big), \Big((2,1) \to\Big)\Big)$$

is not Borel because it fails to have a 3-arrow, but has a 2-arrow, contradicting Corollary 8.8.

9. Multiplicity and the Hilbert polynomial

Given our increased understanding of Borel n-lists, we can now calculate multiplicities and Hilbert polynomial degrees. Our method computes these invariants using n-list notation and as such could be easily implemented in a computer algebra system. For other methods of computing multiplicities and dimensions we direct the reader to Bayer and Stillman's paper on computing Hilbert functions [BaSt], or Herzog and Srinivasan's paper on multiplicities [HeSr].

Definition 9.1. Let A be an n-list. Then define

$$\xi_n[A] = \begin{cases} A & \text{for } n = 1, \text{ and} \\ \xi_{n-1}[A(0)] + \cdots + \xi_{n-1}[A(\alpha_A)] & \text{for } n > 1. \end{cases}$$

Example 9.2. We can calculate that

$$\begin{aligned} &\xi_3\left[((5,4,4,2 \to), (4,2), (2,1) \to)\right] \\ &\quad = \xi_2\left[(5,4,4,2 \to)\right] + \xi_2\left[(4,2)\right] + \xi_2\left[(2,1)\right] \\ &\quad = \xi_1[5] + \xi_1[4] + \xi_1[4] + \xi_1[2] + \xi_1[4] + \xi_1[2] + \xi_1[0] + \xi_1[2] + \xi_1[1] + \xi_1[0] \\ &\quad = 5 + 4 + 4 + 2 + 4 + 2 + 0 + 2 + 1 + 0 = 24. \end{aligned}$$

Simply put, ξ counts up the numbers which appear when we write down an n-list.

Definition 9.3. Let A be an n-list. Then for $0 \le t \le n-1$, let

$$\mathcal{L}_t^n : \{n\text{-lists}\} \mapsto \{(n-t)\text{-lists}\}$$

be the function defined by

$$\mathcal{L}_t^n[A] = \begin{cases} A, \text{ an } n\text{-list} & \text{for } t = 0, \\ A(\alpha_A), \text{ an } (n-1)\text{-list} & \text{for } t = 1, \text{ and} \\ \mathcal{L}_{t-1}^n[A](\alpha_{\mathcal{L}_{t-1}^n[A]}), \text{ an } (n-t)\text{-list} & \text{for } t > 1. \end{cases}$$

EXAMPLE 9.4. For $A = (5, 4)$ we have $\mathcal{L}_1^2[A] = 0$. For $B = (5, 4 \rightarrow)$ we have $\mathcal{L}_1^2[B] = 4$, a 1-list. For

$$C = \Big(\Big((3,2,1),(2,1) \rightarrow\Big), \Big((2,1) \rightarrow\Big) \rightarrow\Big)$$

we have $\mathcal{L}_1^4[C] = ((2,1) \rightarrow)$, which is a 3-list, $\mathcal{L}_2^4[C] = (2,1)$, which is a 2-list, and $\mathcal{L}_3^4[C] = 0$. Notice that for any n-list A, $\mathcal{L}_t^n[A]$ is either zero, or the last $(n-t)$-list which appears when we write A down.

DEFINITION 9.5. Let A be a Borel n-list. If A is not Artinian, then define q_A to be the largest number such that A has an $(n–q_A)$-arrow, and let $m(A) = \xi_{n-q_A}[\mathcal{L}_{q_A}^n[A]]$. If A is Artinian, we take $q_A = 0$, and $m(A) = \xi_n[\mathcal{L}_0^n[A]] = \xi_n[A]$.

EXAMPLE 9.6. Suppose that A, B, C are as in Example 9.4. Then $q_A = 0$ and $m(A) = \xi_2[A] = 9$; $q_B = 1$ and $m(B) = \xi_1[\mathcal{L}_1^2[B]] = \xi_1[4] = 4$; $q_C = 2$, and $m(C) = \xi_2[\mathcal{L}_2^4[C]] = \xi_2[(2,1)] = 3$.

REMARK 9.7. The goal is to show that the Hilbert polynomial of $R/\Phi[A]$ has degree $q_A - 1$ and that the multiplicity of $R/\Phi[A]$ is $m(A)$. This will require several lemmas.

First we need to understand the Hilbert function of $R/\Phi[A]$.

DEFINITION 9.8. Let A be an n-list. Then if $n = 1$ define

$$\mathcal{H}_A(d) = \begin{cases} 1 & \text{for } 0 \le d < A, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If $n > 1$ inductively define

$$\mathcal{H}_A(d) = \sum_{i=0}^{d} \mathcal{H}_{A(i)}(d-i).$$

LEMMA 9.9. *Let A be an n-list. Then $H(R/\Phi[A], d)$, the Hilbert function of $R/\Phi[A]$, is equal to $\mathcal{H}_A(d)$.*

Proof. The proof is by induction. If $n = 1$ then $\Phi[A] = (x_1^A)$ and clearly $H(R/\Phi[A], d) = \mathcal{H}_A(d)$ as required.

If $n > 1$ then by induction

$$H\Big(k[x_1, \dots, x_{n-1}]/\Phi_{n-1}[A(i)], d\Big) = \mathcal{H}_{A(i)}(d)$$

for all $i \geq 0$. Write $S = k[x_1, \dots, x_{n-1}]$ and $R = S[x_n]$. Now

$$\begin{aligned}\Phi[A] = \Phi[A(0)] \oplus x_n\Phi[A(1)] \oplus \cdots \oplus x_n^{\alpha_A - 1}\Phi[A(\alpha_A - 1)] \\ \oplus x_n^{\alpha_A}(\Phi[A(\alpha_A)]) \oplus x_n^{\alpha_A+1}(\Phi[A(\alpha_A)]) \oplus \cdots,\end{aligned}$$

is the unique x_n graded decomposition of $\Phi[A]$, so

$$\frac{R}{\Phi[A]} = \frac{S}{\Phi[A(0)]} \oplus x_n \frac{S}{\Phi[A(1)]} \oplus \cdots \oplus x_n^{\alpha_A - 1} \frac{S}{\Phi[A(\alpha_A - 1)]} \\ \oplus x_n^{\alpha_A} \frac{S}{\Phi[A(\alpha_A)]} \oplus x_n^{\alpha_A + 1} \frac{S}{\Phi[A(\alpha_A)]} \oplus \cdots$$

is the unique $\overline{x}_n$-graded decomposition of $R/\Phi[A]$. It is clear then that

$$H\big(R/\Phi[A], d\big) = \sum_{i=0}^{\infty} H\Big(S/\Phi[A(i)], d - i\Big) = \sum_{i=0}^{\infty} \mathcal{H}_{A(i)}(d - i)$$

as required. □

The next few lemmas deal with the Artinian case.

Lemma 9.10. *Let A be an n-list, and suppose that $r \in \mathbb{N}$ is such that $\mathcal{H}_A(r) = 0$. Then $r \geq \alpha_A$ and $\mathcal{H}_{A(i)}(r - i) = 0$ for all $i \in \mathbb{N}$.*

Proof. The fact that $0 = \mathcal{H}_A(r) = \sum_{j=0}^{r} \mathcal{H}_{A(j)}(r-j)$ forces $\mathcal{H}_{A(i)}(r-i) = 0$ when $i \leq r$. By definition, $\mathcal{H}_{A(i)}(r-i) = 0$ for $i > r$. Furthermore, taking $i = r$ and using Lemma 9.9, we find that $H(R/\Phi_{n-1}[A(r)], 0) = \mathcal{H}_{A(r)}(0) = 0$. So $\Phi_{n-1}[A(r)] = k[x_1, \ldots, x_{n-1}]$, and by Lemma 4.3 this implies that $A(r) = 0$. We conclude that $r \geq \alpha_A$ as required. □

Lemma 9.11. *Let A be an n-list, and suppose that $r \in \mathbb{N}$ is such that $H(R/\Phi[A], r) = 0$. Then*

$$\sum_{d=0}^{r} H(R/\Phi[A], d) = \xi_n[A].$$

Proof. The proof is by induction. If $n = 1$, then by Lemma 9.9,

$$\sum_{d=0}^{r} H(R/\Phi[A], d) = \sum_{d=0}^{r} \mathcal{H}_A(d) = \sum_{d=0}^{A-1} 1 = A = \xi_1[A],$$

as required.

If $n > 1$, then again using Lemma 9.9, we find that

$$\begin{aligned} \sum_{d=0}^{r} H(R/\Phi[A], d) &= \sum_{d=0}^{r} \mathcal{H}_A(d) = \sum_{d=0}^{r} \sum_{i=0}^{d} \mathcal{H}_{A(i)}(d - i) \\ &= \sum_{d=0}^{r} \sum_{i=0}^{r} \mathcal{H}_{A(i)}(d - i) = \sum_{i=0}^{r} \sum_{d=0}^{r} \mathcal{H}_{A(i)}(d - i) \\ &= \sum_{i=0}^{r} \sum_{d=0}^{r-i} \mathcal{H}(R/\Phi[A(i)], d). \end{aligned}$$

By Lemma 9.10 and induction we furthermore conclude that

$$\sum_{i=0}^{r}\sum_{d=0}^{r-i}\mathcal{H}(R/\Phi[A(i)],d)=\sum_{i=0}^{r}\xi_{n-1}[A(i)].$$

Now because $H(R/\Phi[A],r)=0$, we know that $\Phi[A]$ is Artinian. By Theorem 6.3, A is Artinian, so that $\xi_{n-1}[A(i)]=0$ for $i\geq\alpha_A$. Note that $r\geq\alpha_A$ by Lemmas 9.9 and 9.10. Thus

$$\sum_{i=0}^{r}\xi_{n-1}[A(i)]=\sum_{i=0}^{\alpha_A-1}\xi_{n-1}[A(i)]=\xi_n[A],$$

as required. □

Remark 9.12. Recall that for any Artinian ideal I, the Hilbert polynomial of R/I is zero, hence has degree -1, and the multiplicity of R/I is $\sum_{d=0}^{\infty}\mathcal{H}(R/I,d)$. Lemma 9.11 therefore achieves the Artinian case of the goal set in Remark 9.7. That is, we have shown that for any Artinian Borel fixed ideal $\Phi[A]$, the multiplicity of $R/\Phi[A]$ is $m(A)=\xi_n[A]$. (Because $q_A=0$ for any Artinian n-list, it was already clear that $q_A-1=-1$ as desired.)

To prove the main theorem, we still require two lemmas.

Lemma 9.13. *Let A be an n-list with $n\geq 2$. Then $\mathcal{L}_t^n[A]=\mathcal{L}_{t-1}^{n-1}[A(\alpha_A)]$ for $1\leq t\leq n-1$.*

Proof. The proof is by induction on t. If $t=1$, then

$$\mathcal{L}_1^n[A]=A(\alpha_A)=\mathcal{L}_0^{n-1}[A(\alpha_A)].$$

If $t>1$, then

$$\begin{aligned}\mathcal{L}_t^n[A]&=\mathcal{L}_{t-1}^n[A](\alpha_{\mathcal{L}_{t-1}^n[A]})=\mathcal{L}_{t-2}^{n-1}\big[A(\alpha_A)\big](\alpha_{\mathcal{L}_{t-2}^{n-1}[A(\alpha_A)]})\\&=\mathcal{L}_{t-1}^{n-1}\big[A(\alpha_A)\big].\end{aligned}$$

□

Lemma 9.14. *Let A be a Borel n-list for $n\geq 3$ such that $q_A>1$. Then $m(A)=m(A(\alpha_A))$.*

Proof. Notice that $q_{A(\alpha_A)}=q_A-1$. Then using Lemma 9.13, we have

$$\begin{aligned}m(A)&=\xi_{n-q_A}[\mathcal{L}_{q_A}^n[A]]=\xi_{n-1-(q_A-1)}\Big[\mathcal{L}_{q_A-1}^{n-1}\big[A(\alpha_A)\big]\Big]\\&=\xi_{n-1-q_{A(\alpha_A)}}\Big[\mathcal{L}_{q_{A(\alpha_A)}}^{n-1}\big[A(\alpha_A)\big]\Big]=m(A(\alpha_A)).\end{aligned}$$

□

Theorem 9.15. *Let A be a Borel n-list. Then the Hilbert polynomial of $R/\Phi[A]$ has degree (q_A-1) and the multiplicity of $R/\Phi[A]$ is $m(A)$.*

Proof. If A is Artinian, then it is clear that the Hilbert polynomial of $R/\Phi[A]$ has degree $(q_A - 1) = (0 - 1) = -1$, and the multiplicity is

$$\sum_{d=0}^{\infty} H(R/\Phi[A], d) = \xi_n[A] = m(A)$$

by Lemma 9.11. We may thus assume that A is not Artinian and $n \geq 2$. To proceed we use induction.

If $n = 2$, then $A = (A(0), A(1), \ldots, A(\alpha_A) \rightarrow)$, so it is enough to show that the Hilbert polynomial of $R/\Phi[A]$ is the constant $m(A)/(q_A - 1)! = \xi_1[\mathcal{L}_1^2(A)] = \xi_1[A(\alpha_A)] = A(\alpha_A)$, which has degree $q_A - 1 = 0$. Consider any $t > A(\alpha_A) + \alpha_A$. Then

$$x_1^{A(\alpha_A)+i} x_2^{t-A(\alpha_A)-i} \in \Phi[A]$$

for all $i = 0, \ldots, (t - A(\alpha_A))$ because $x_1^{A(\alpha_A)} x_2^{\alpha_A} \in \Phi[A]$ and A is Borel fixed. We also have that

$$x_1^{A(\alpha_A)-i} x_2^{t-A(\alpha_A)+i} \notin \Phi[A]$$

for $i = 1, \ldots, A(\alpha_A)$, because otherwise $A(t - A(\alpha_A) + i) \neq A(\alpha_A)$, a contradiction. Thus the Hilbert polynomial of $R/\Phi[A]$ is the constant $A(\alpha_A)$ as required.

If $n > 2$, we first suppose that $q_A \neq 1$, and write $S = k[x_1, \ldots, x_{n-1}]$ and $R = S[x_n]$. By Lemma 8.7, each of $A(0), \ldots, A(\alpha_A)$ is an $(n-1)$-list with an $(n-1-(q_A-1))$-arrow, and no $(n-1-t)$-arrows for $t > q_A - 1$. In particular, this means that $A(\alpha_A) \neq 0$. Let $f_i = f_{\Phi[A(i)]}$ be the Hilbert polynomial of $R/\Phi[A(i)]$. Then by induction, f_i has degree $(q_{A(i)} - 1) = ((q_A - 1) - 1) = (q_A - 2)$ and leading coefficient $m(A(i))/(q_A - 2)!$. For $i = 0, \ldots, \alpha_A$, let t_i be the smallest integer such that $H(S/\Phi[A(i)], j) = f_i(j)$ for all $j \geq t_i$. We have that

$$\begin{aligned}\Phi[A] = \Phi[A(0)] \oplus x_n \Phi[A(1)] \oplus \cdots \oplus x_n^{\alpha_A} \Phi[A(\alpha_A)] \\ \oplus\, x_n^{\alpha_A+1} \Phi[A(\alpha_A)] \oplus \cdots,\end{aligned}$$

so

$$\begin{aligned}\frac{R}{\Phi[A]} = \frac{S}{\Phi[A(0)]} \oplus x_n \frac{S}{\Phi[A(1)]} \oplus \cdots \oplus x_n^{\alpha_A} \frac{S}{\Phi[A(\alpha_A)]} \\ \oplus\, x_n^{\alpha_A+1} \frac{S}{\Phi[A(\alpha_A)]} \oplus x_n^{\alpha_A+2} \frac{S}{\Phi[A(\alpha_A)]} \oplus \cdots\end{aligned}$$

is the unique $\overline{x}_n$-graded decomposition of $R/\Phi[A]$. Thus we may write the Hilbert polynomial of $R/\Phi[A]$ for $t > \max\{t_i + i \mid 0 \leq i \leq \alpha_A\}$ as

$$\begin{aligned} f_{\Phi[A]}(t) &= \sum_{j=0}^{\alpha_A-1} f_j(t-j) + \sum_{j=\alpha_A}^{t-t_{\alpha_A}} f_{\alpha_A}(t-j) + \sum_{j=0}^{t_{\alpha_A}-1} H(S/\Phi[A(\alpha_A)], j) \\ &= \sum_{j=0}^{t-\alpha_A} f_{\alpha_A}(j) - \sum_{j=0}^{t_{\alpha_A}-1} f_{\alpha_A}(j) + \sum_{j=0}^{\alpha_A-1} f_j(t-j) \\ &\qquad + \sum_{j=0}^{t_{\alpha_A}-1} H(S/\Phi[A(\alpha_A)], j). \end{aligned}$$

Here $\sum_{j=0}^{\alpha_A-1} f_j(t-j)$ is a polynomial of degree (q_A-2), while $\sum_{j=0}^{t_{\alpha_A}-1} f_{\alpha_A}(j)$ and $\sum_{j=0}^{t_{\alpha_A}-1} H(S/\Phi[A(\alpha_A)], j)$ are constants. The remaining expression, the sum $\sum_{j=0}^{t-\alpha_A} f_{\alpha_A}(j)$, is a polynomial of degree $(q_A-2+1) = (q_A-1)$, and using Lemma 9.14, has leading coefficient $(m(A(\alpha_A))/(q_A-2)!)/(q_A-1) = m(A)/(q_A-1)!$. Thus the Hilbert polynomial of $R/\Phi[A]$ has degree q_A-1 and the multiplicity of $R/\Phi[A]$ is $m(A)$, as required.

Finally, suppose that $n > 2$ and $q_A = 1$, so that each $A(i)$ is Artinian. Then mimicking the construction above, we have the constant Hilbert polynomial $f_{\Phi[A]}(t) = \sum_{j=0}^{t_{\alpha_A}-1} H(S/\Phi[A(\alpha_A)], j)$, so by Lemma 9.11 and the fact that $A(\alpha_A)$ is Artinian,

$$\sum_{i=0}^{t_{\alpha_A}-1} H(S/\Phi[A(\alpha_A)], i) = \sum_{i=0}^{\infty} H(S/\Phi[A(\alpha_A)], i) = \xi_{n-1}[A(\alpha_A)].$$

Thus the multiplicity is $\xi_{n-1}[A(\alpha_A)](q_A-1)! = \xi_{n-1}[\mathcal{L}_1^n(A)] = m(A)$ and the degree of the Hilbert polynomial is $q_A - 1 = 0$ as required. $\square$

No simple procedure for determining the actual Hilbert polynomial, either in the Borel fixed or the general case, seems to be forthcoming.

10. Recovering Geramita, Harima, and Shin's correspondence

In [GeHaSh1], Geramita, Harima, and Shin show that the set of n-type vectors is in one-to-one correspondence with the set of all Hilbert functions of finite length $k[x_1, \ldots, x_n]$-modules. We will now use n-list notation and the tools developed thus far to give a new proof of this correspondence.

Recall that the set of n-type vectors can be considered a subset of the set of n-lists in a natural way, by Proposition 3.5. In this section we will show that Φ maps the set of n-type vectors bijectively onto the set of lex Artinian monomial ideals in $k[x_1, \ldots, x_n]$. This is enough to imply the advertised result, because by Macaulay [Ma] we know that the set of lex Artinian ideals and the set of Artinian Hilbert functions correspond bijectively.

To show that we have recovered Geramita, Harima, and Shin's particular bijection, we will then demonstrate that Φ factors their map ψ_n. That is, we show that given an n-type vector $\mathcal{T}$, we have $H(R/\Phi_n[\mathcal{T}]) = \psi_n(\mathcal{T})$.

Note that in this section we will use the lex order for which $x_n > \cdots > x_1$. This choice, although necessary to factor ψ_n, was disappointing for two reasons. First, by taking $x_n > \cdots > x_1$ while studying lex ideals, we restrict ourselves to the Artinian case. This is because every ideal in the image of Φ contains a power of x_1, and in a lex ideal this forces powers of all larger variables to be present. Second, the set of lex ideals for which $x_1 > \cdots > x_n$ is a natural subset of the set of Borel fixed ideals characterized in Section 7, an inclusion we did not wish to forgo.

The first step in our argument is an easy lemma concerning Φ and σ.

LEMMA 10.1. *Let A be an n-type vector. Then $x_1^{\sigma_n(A)}$ is a minimal generator of $\Phi_n[A]$.*

Proof. The proof is by induction. If $n = 1$, the result is obvious. If $n > 1$, then $\sigma_n(A) = \sigma_{n-1}(A(0))$, and by induction $x_1^{\sigma_{n-1}(A(0))}$ is a minimal generator of $\Phi_{n-1}[A(0)]$, so that $x_1^{\sigma_n(A)} \in \Phi_n[A]$. Furthermore, $x_1^j \in \Phi_n[A]$ implies that $x_1^j \in \Phi_{n-1}[A(0)]$, and by induction, $\sigma_n(A) = \sigma_{n-1}(A(0)) \leq j$. We conclude that $x_1^{\sigma_n(A)}$ is a minimal generator of $\Phi_n[A]$ as required. □

The proof that Φ is a bijection between the set of n-type vectors and the set of lex Artinian ideals is similar in form to the argument in Section 7. We begin by showing that the image of Φ is contained in the set of lex Artinian ideals.

PROPOSITION 10.2. *Let A be an n-type vector. Then $\Phi_n[A]$ is a lex Artinian ideal.*

Proof. As we noted in Remark 6.4, Φ_n send n-type vectors to Artinian ideals, so it only remains to show that $\Phi_n[A]$ is lex. We use induction. If $n = 1$, the result is obvious. If $n > 1$ we must show that, given $x_n^{i_n} \ldots x_1^{i_1} \in \Phi_n[A]$, where $i_j \neq 0$ for some $j \in \{1, \ldots, n\}$, then the next largest monomial in lex order is also contained in $\Phi_n[A]$. If $x_n^{i_n} \ldots x_1^{i_1}$ is actually of the form $x_n^{i_n}$, then the theorem is obvious, so we may assume that $i_j \neq 0$ for some $j \in \{1, \ldots, (n-1)\}$. Note that $x_n^{i_n} \ldots x_1^{i_1} \in x_n^{i_n} \Phi_{n-1}[A(i_n)]$. If the next largest element in lex order is of the form $x_n^{i_n} x_{n-1}^{i'_{n-1}} \ldots x_1^{i'_1}$, then we are done because $\Phi_{n-1}[A(i_n)]$ is lex by induction. This is always the case unless $x_n^{i_n} \ldots x_1^{i_1}$ is of the form $x_n^{i_n} x_{n-1}^{i_{n-1}}$ for $i_{n-1} \geq 1$, whence we must show that $x_n^{i_n+1} x_1^{i_{n-1}-1} \in \Phi_n[A]$. Since $x_n^{i_n} x_{n-1}^{i_{n-1}} \in \Phi_n[A]$, we have that $x_{n-1}^{i_{n-1}} \in \Phi_{n-1}[A(i_n)]$, and $\alpha_{A(i_n)} \leq i_{n-1}$. By hypothesis, $\sigma_{n-1}(A(i_n + 1)) < \alpha_{A(i_n)} \leq i_{n-1}$, and by Lemma 10.1, $x_1^{\sigma_{n-1}(A(i_n+1))} \in \Phi_{n-1}[A(i_n + 1)]$, so that $x_n^{i_n+1} x_1^{i_{n-1}-1} \in \Phi_n[A]$. □

In order to establish the bijection, we next show the other inclusion, that the image under ρ of any lex Artinian ideal is an n-type vector.

THEOREM 10.3. *The map Φ_n is a bijection from the set of n-type vectors to the set of lex Artinian ideals in $k[x_1, \dots, x_n]$.*

Proof. Let $I \subseteq k[x_1, \dots, x_n]$ be a lex Artinian ideal with unique x_n-graded decomposition $I = I_0 \oplus x_n I_1 \oplus x_n^2 I_2 \oplus \cdots \oplus x_n^j I_j \oplus \cdots$. By Corollary 4.5 and Propositions 5.2 and 10.2, it is enough to show that $\rho_n(I)$ is an n-type vector. We proceed by induction.

If $n = 1$, then the result is obvious. If $n > 1$, then write $\rho_n(I) = A$. For all $0 \leq j \leq \alpha_A - 1$ we have by induction that $A(j)$ is an $(n-1)$-type vector, and by Theorem 6.3 it is clear that $(A(\alpha_A)) = 0$, so it remains to show that $\alpha_{A(j)} > \sigma_{n-1}(A(j+1))$ for $0 \leq j \leq \alpha_A - 2$.

Note that $\alpha_{A(j)} \neq 0$. Then because $A(j)$ is Artinian we have $x_{n-1}^{\alpha_{A(j)}} \in \Phi_{n-1}[A(j)] = I_j$ so that $x_n^j x_{n-1}^{\alpha_{A(j)}} \in I$. Because I is lex we have $x_n^{j+1} x_1^{\alpha_{A(j)}-1} \in I$, or $x_1^{\alpha_{A(j)}-1} \in I_{j+1} = \Phi_{n-1}[A(j+1)]$. Thus Lemma 10.1 implies that $\alpha_{A(j)} - 1 \geq \sigma_{n-1}(A(j+1))$, or $\alpha_{A(j)} > \sigma_{n-1}(A(j+1))$. We conclude that $\rho_n(I)$ is an n-type vector as required. □

As mentioned above, Macaulay's bijection between lex Artinian ideals is enough to show that the set of n-type vectors and the set of Artinian Hilbert functions are in one-to-one correspondence.

We have not yet shown, however, that our correspondence recovers the particular bijection given in [GeHaSh1]. For that, we must show that Φ factors ψ, that is, that given an n-type vector $\mathcal{T}$, the Hilbert function $H(R/\Phi[\mathcal{T}])$ is exactly $\psi(\mathcal{T})$.

This is, however, obvious given the definition of ψ_n. From [GeHaSh1], if $\mathcal{T} = (t_{\mathcal{T}})$ is a 1-type vector, then $\psi_1(\mathcal{T})$ is the Hilbert function

$$\mathcal{H}_{\mathcal{T}}(d) = \begin{cases} 1 & \text{for } 0 \leq d < t_{\mathcal{T}}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{T} = (\mathcal{T}_0, \dots, \mathcal{T}_{s-1})$ is an n-type vector, then $\psi(\mathcal{T})$ is the Hilbert function $H_{\mathcal{T}}(d) = \sum_{i=0}^{s-1} H_{\mathcal{T}_i}(d-i)$.

In Lemma 9.9, we showed that this is exactly $H(R/\Phi[\mathcal{T}], d)$.

11. Recovering all ideals

It should be noted that in all the preceding material, the map from n-lists to monomial ideals is clearly not surjective. All ideals in the image of Φ contain some power of x_1. This did not prove detrimental when considering the subsets corresponding to Artinian, Borel fixed, or lex Artinian ideals, because each of these in fact contains a power of x_1 as required. In this section we generalize our discussion again so that we might characterize all monomial ideals in $k[x_1,\dots,x_n]$. We are especially interested in the computational uses of such an approach, because various calculations, such as intersecting ideals, become quite trivial when using n-list notation.

We generalize the set of n-lists as follows. Let f be the homomorphism taking $k[x_1,\dots,x_n]$ to $k[x_1,\dots,x_{n+1}]$ by the rule $f(x_i)=x_{i+1}$ for $i=1,\dots,n$. Then f induces an isomorphism $k[x_1,\dots,x_n]\cong k[x_1,\dots,x_{n+1}]/(x_1)$ and homogeneous ideals $I\subseteq k[x_1,\dots,x_n]$ are in one-to-one correspondence with ideals of the form $\hat{I}=(x_1,f(I))\subset k[x_1,\dots,x_{n+1}]$. Because $\hat{I}$ contains x_1 we can write it as an $(n+1)$-list which has all entries either 1 or 0. To avoid confusion, call the $(n+1)$-list we obtain an $\overline{(n+1)}$-list to indicate that we mean to associate it with a monomial ideal in $k[x_1,\dots,x_n]$. This new rule gives us a one-to-one correspondence between the non-zero monomial ideals in $k[x_1,\dots,x_n]$ and the set of $\overline{(n+1)}$-lists.

Although this new construction maintains all the inductive structure which proved so useful in the proofs of this paper, it is somewhat unwieldy for hand calculations.

Example 11.1. The ideal $I=(x_1x_2)\subset k[x_1,x_2]$ corresponds under this new construction to the $\overline{3}$-list $((1\ \rightarrow),(1)\ \rightarrow)$, that is, $(x_1,x_2x_3)\subset k[x_1,x_2,x_3]$. The simple ideal $(x_1^4,x_2^4,x_3^4)\subset k[x_1,x_2,x_3]$, or

$$\Big((4,4,4,4),(4,4,4,4),(4,4,4,4),(4,4,4,4)\Big)$$

in our standard n-list notation, would appear as

$$\begin{aligned}\Bigg(&\Big((1,1,1,1),(1,1,1,1),(1,1,1,1),(1,1,1,1)\Big),\\ &\Big((1,1,1,1),(1,1,1,1),(1,1,1,1),(1,1,1,1)\Big),\\ &\Big((1,1,1,1),(1,1,1,1),(1,1,1,1),(1,1,1,1)\Big),\\ &\Big((1,1,1,1),(1,1,1,1),(1,1,1,1),(1,1,1,1)\Big)\Bigg)\end{aligned}$$

as a $\overline{4}$-list.

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