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A CLASS OF MAXIMAL OPERATORS WITH ROUGH KERNEL ON PRODUCT SPACES

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Dedicated to Professor Kôzô Yabuta on the occasion of his 60th birthday

ABSTRACT. In this note the authors prove the $L^p(\mathbb{R}^n\times\mathbb{R}^m)$-boundedness for a class of maximal singular integral operators with rough kernel on product spaces. This extends a result obtained by Chen and Wang in 1992.

1. Introduction

In 1992, L. K. Chen and X. Wang [2] considered the L^p-boundedness of the maximal operator $\sup_{K\in M}|T_K f|$, where the operator T_K is defined by

$$T_K f(x) = \int_0^\infty \int_{S^{n-1}} K(r\xi) f(x-r\xi) r^{n-1}\, d\xi\, dr,$$

and

$$M = \Big\{ K(r\xi) = r^{-n}\sum_j a_j(r)\Omega_j(\xi) \colon \int_0^\infty \sum_j |a_j(r)|^2 \frac{dr}{r} \le 1, \Omega_j \in L^2(S^{n-1}),$$
$$\int_{S^{n-1}} \Omega_j(\xi)\, d\xi = 0 \text{ for all } j, \text{ and } \sum_j \|\Omega_j\|^2_{L^2(S^{n-1})} < \infty \Big\}.$$

In [2] Chen and Wang proved the following result:

THEOREM A. *Let $2n/(2n-1) < p < \infty$, $n \ge 2$. Then the operator $\sup_{K\in M}|T_K f|$ is bounded on $L^p(\mathbb{R}^n)$; that is,*

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$$\Big\| \sup_{K\in M} |T_K f| \Big\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}$$

for all f in the Schwartz class.

In this note we extend this result to product spaces, with $\Omega_j \in L^q(S^{n-1}\times S^{m-1})$, $1 < q \le \infty$. Let $n, m \ge 2$ and let $\{\Omega_j\}$ be a countable subset of $L^q(S^{n-1}\times S^{m-1})$, $1 < q \le \infty$, satisfying the following conditions (for all j):

$$\Omega_j(t\xi, s\eta) = \Omega_j(\xi,\eta) \qquad \text{for any } t, s > 0, \tag{1.1}$$

$$\int_{S^{n-1}} \Omega_j(\xi,\eta)\, d\xi = 0 \qquad \text{for any } \eta \in S^{m-1}, \tag{1.2}$$

$$\int_{S^{m-1}} \Omega_j(\xi,\eta)\, d\eta = 0 \qquad \text{for any } \xi \in S^{n-1}. \tag{1.3}$$

Moreover, suppose $\sum_j \|\Omega_j\|^2_{L^q(S^{n-1}\times S^{m-1})} < \infty$. Let M denote the class of all kernels of the form $K(r\xi, s\eta) = r^{-n}s^{-m}\sum_j a_j(r,s)\Omega_j(\xi,\eta)$ (defined for $r > 0$, $s > 0$, and $(\xi,\eta) \in S^{n-1}\times S^{m-1}$), where

$$\int_0^\infty\int_0^\infty \sum_j |a_j(r,s)|^2 \frac{dr\,ds}{rs} \le 1. \tag{1.4}$$

Let us define the singular integral operator T_K by

$$T_K f(x,y) = \int_0^\infty\int_0^\infty \iint_{S^{n-1}\times S^{m-1}} K(r\xi, s\eta) f(x - r\xi, y - s\eta) r^{n-1}s^{m-1}\, d\xi\, d\eta\, dr\, ds.$$

We denote by q' the conjugate index of q; that is, $1/q + 1/q' = 1$. We shall prove the following theorem:

THEOREM 1. *Suppose that $\{\Omega_j\} \subseteq L^q(S^{n-1}\times S^{m-1})$ satisfies the above conditions and T_K is defined as above. Suppose that q and p satisfy one of the following conditions:*

(a) $1 < q < 2$ *and* $\max\{2nq'/(2n + nq' - 2), 2mq'/(2m + mq' - 2)\} < p < 2q'/(q' - 2)$,
(b) $2 \le q \le \max\{2(n-1)/(n-2), 2(m-1)/(m-2)\}$ *and* $\max\{2nq'/(2n + nq' - 2), 2mq'/(2m + mq' - 2)\} < p < \infty$,
(c) $q > \max\{2(n-1)/(n-2), 2(m-1)/(m-2)\}$ *and* $1 < p < \infty$.

Then the maximal operator $\sup_{K\in M} |T_K f|$ can be extended to a bounded operator on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$. That is,

$$\Big\| \sup_{K\in M} |T_K f| \Big\|_p \le C\|f\|_p,$$

where, here and below, we denote $\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$ by $\|f\|_p$.

Remark. Clearly, if we take $q = 2$ and consider the case of one parameter, then the conclusion of Theorem 1 under condition (b) is identical to that of Theorem A. If either $n = 2$ or $m = 2$, condition (b) means $2 \le q \le \infty$, and condition (c) means $q = \infty$, which is a special case of (b).

2. Preliminaries

Let us begin by proving some lemmas.

Lemma 1. *Suppose that $\Omega \in L^q(S^{n-1} \times S^{m-1})$ for $q > 1$ satisfies (1.1)–(1.3). Then for any $0 < \sigma < 1/q'$, there exist $0 < \varepsilon, \theta < 1$, and a constant $C = C(\sigma, \varepsilon, \theta)$ such that*

$$\int_1^2 \int_1^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(\xi,\eta) e^{-i(rx\cdot\xi + sy\cdot\eta)}\, d\xi\, d\eta \Bigg|^2 \frac{dr\, ds}{rs}$$
$$\le C\|\Omega\|_q^2 \min\left\{ |x||y|, |x|^{-\sigma}|y|^{-\sigma}, |x|^{\varepsilon}|y|^{-\theta}, |x|^{-\theta}|y|^{\varepsilon} \right\},$$

where, here and below, we denote $\|\Omega\|_{L^q(S^{n-1}\times S^{m-1})}$ by $\|\Omega\|_q$.

Proof. Let

$$I(x,y) = \int_1^2 \int_1^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(\xi,\eta) e^{-i(rx\cdot\xi + sy\cdot\eta)}\, d\xi\, d\eta \Bigg|^2 \frac{dr\, ds}{rs}.$$

By the cancellation conditions (1.2) and (1.3), we have

$$\begin{aligned} I(x,y) &= \int_1^2 \int_1^2 \Bigg| \iint_{S^{n-1}\times S^{m-1}} \Omega(\xi,\eta) e^{-isy\cdot\eta} e^{-irx\cdot\xi_0} \left[e^{-irx\cdot(\xi-\xi_0)} - 1 \right] d\xi\, d\eta \Bigg|^2 \frac{dr\, ds}{rs} \\ &\le C \Bigg\{ \iint_{S^{n-1}\times S^{m-1}} |\Omega(\xi,\eta)| \bigg(\int_1^2 \int_1^2 |rx\cdot(\xi-\xi_0)|^2 \frac{dr\, ds}{rs} \bigg)^{1/2} d\xi\, d\eta \Bigg\}^2 \\ &\le C|x|^2 \bigg(\iint_{S^{n-1}\times S^{m-1}} |\Omega(\xi,\eta)||\xi-\xi_0|\, d\xi\, d\eta \bigg)^2 \le C|x|^2 \|\Omega\|_q^2. \end{aligned}$$

The same argument gives $I(x,y) \le C|y|^2\|\Omega\|_q^2$. Thus,

$$I(x,y) \le C\|\Omega\|_q^2 \min\{|x|^2, |y|^2\}. \tag{2.1}$$

On the other hand,

$$\begin{aligned}\Bigg|\iint\limits_{S^{n-1}\times S^{m-1}} \Omega(\xi,\eta)e^{-i(rx\cdot\xi+sy\cdot\eta)}\,d\xi\,d\eta\Bigg|^2 \\ = \iint\limits_{(S^{n-1}\times S^{m-1})^2} \Omega(\xi,\eta)\overline{\Omega(\xi',\eta')}e^{-i(rx\cdot\xi+sy\cdot\eta)}e^{i(rx\cdot\xi'+sy\cdot\eta')}\,d\xi\,d\eta\,d\xi'\,d\eta',\end{aligned}$$

where $\overline{\Omega}$ denotes the conjugate of Ω. Let

$$J = \int_1^2\int_1^2 e^{-i[rx\cdot(\xi-\xi')+sy\cdot(\eta-\eta')]}\,\frac{dr\,ds}{rs}.$$

Clearly, $|J| \le (\log 2)^2$. Moreover, by [3] there is a constant C such that

$$|J| \le C\frac{1}{|x\cdot(\xi-\xi')||y\cdot(\eta-\eta')|}.$$

Thus, for any $0<\sigma<1$, we have

$$|J| \le C_\sigma\frac{1}{|x\cdot(\xi-\xi')|^\sigma|y\cdot(\eta-\eta')|^\sigma}.$$

Taking $0<\sigma<1/q'$, we get

$$\begin{aligned}(2.2)\quad I(x,y) &\le C_\sigma \iint\limits_{(S^{n-1}\times S^{m-1})^2} |\Omega(\xi,\eta)\overline{\Omega(\xi',\eta')}|\frac{d\xi\,d\eta\,d\xi'\,d\eta'}{|x\cdot(\xi-\xi')|^\sigma|y\cdot(\eta-\eta')|^\sigma} \\ &\le \frac{C_\sigma}{|x|^\sigma|y|^\sigma}\|\Omega\|_q^2\Bigg(\iint\limits_{S^{n-1}\times S^{n-1}}\frac{d\xi\,d\xi'}{|\xi_1-\xi_1'|^{\sigma q'}}\Bigg)^{1/q'} \\ &\qquad\times\Bigg(\iint\limits_{S^{m-1}\times S^{m-1}}\frac{d\eta\,d\eta'}{|\eta_1-\eta_1'|^{\sigma q'}}\Bigg)^{1/q'} \\ &\le C_\sigma\|\Omega\|_q^2|x|^{-\sigma}|y|^{-\sigma}.\end{aligned}$$

Combining (2.1) and (2.2), we see that for $0<\sigma<1/q'$

$$I(x,y) \le C_\sigma\|\Omega\|_q^2\cdot\min\{|x|^2,|y|^2,|x|^{-\sigma}|y|^{-\sigma}\}.$$

By interpolating we get

$$(2.3)\qquad I(x,y) \le C\|\Omega\|_q^2\cdot\min\{|x||y|,|x|^{-\sigma}|y|^{-\sigma},|x|^\varepsilon|y|^{-\theta},|x|^{-\theta}|y|^\varepsilon\},$$

where $0<\sigma<1/q'$ and $0<\varepsilon,\theta<1$. In fact, taking $\sigma/(2+\sigma)<\tau<(1+\sigma)/(2+\sigma)$, we have

$$\begin{aligned}I(x,y) &= I(x,y)^\tau I(x,y)^{1-\tau} \\ &\le C\|\Omega\|_q^2|x|^{2\tau}\cdot\{|x|^{-\sigma}|y|^{-\sigma}\}^{1-\tau} = C\|\Omega\|_q^2|x|^\varepsilon|y|^{-\theta},\end{aligned}$$

where $\varepsilon = 2\tau - \sigma(1-\tau)$ and $\theta = \sigma(1-\tau)$. Similarly, we have

$$I(x,y) \le C\|\Omega\|_q^2 |x|^{-\theta} |y|^{\varepsilon}.$$

This completes the proof of Lemma 1. □

The following lemma is related to Stein's spherical maximal function. Let us give some definitions. For a function $f(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^m$, we denote the spherical maximal function on the first variable x of f by

$$\mathcal{M}_1 f(x,y) = \sup_{r>0} \int\limits_{S^{n-1}} |f(x - r\theta, y)|\, d\theta,$$

the spherical maximal function on the second variable y of f by

$$\mathcal{M}_2 f(x,y) = \sup_{s>0} \int\limits_{S^{m-1}} |f(x, y - s\phi)|\, d\phi,$$

and the spherical maximal function on $S^{n-1} \times S^{m-1}$ of f by

$$\mathcal{M}^* f(x,y) = \sup_{r,s>0} \iint\limits_{S^{n-1}\times S^{m-1}} |f(x - r\theta, y - s\phi)|\, d\theta\, d\phi.$$

From the above definitions it is easy to see that

$$\mathcal{M}^* f(x,y) \le \mathcal{M}_2 \mathcal{M}_1 f(x,y). \tag{2.4}$$

LEMMA 2. *Suppose that $n, m \ge 2$ and $p > \max\{n/(n-1), m/(m-1)\}$. Then the spherical maximal function $\mathcal{M}^* f$ of f is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$; that is, $\|\mathcal{M}^* f\|_p \le C\|f\|_p$.*

Proof. By (2.4) and the results of J. Bourgain [1] and E. M. Stein [5], we have

$$\begin{aligned} \|\mathcal{M}^* f\|_p &\le \Big(\iint\limits_{\mathbb{R}^n\times\mathbb{R}^m} |\mathcal{M}_2\mathcal{M}_1 f(x,y)|^p\, dx\, dy \Big)^{1/p} \\ &\le C \Big(\int\limits_{\mathbb{R}^n} \|\mathcal{M}_1 f(x,\cdot)\|^p_{L^p(\mathbb{R}^m)}\, dx \Big)^{1/p} \\ &= C \Big(\int\limits_{\mathbb{R}^m} \big(\int\limits_{\mathbb{R}^n} |\mathcal{M}_1 f(x,y)|^p\, dx \big) dy \Big)^{1/p} \le C\|f\|_p. \end{aligned}$$

This is the desired conclusion. □

3. Proof of Theorem 1

Let us write $T_K f(x,y)$ as

$$T_K f(x,y) = \int_0^\infty \int_0^\infty \sum_j a_j(r,s) \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) f(x-r\xi, y-s\eta)\, d\xi\, d\eta \frac{dr\, ds}{rs}.$$

Applying Schwarz's inequality, we get

$$|T_K f(x,y)| \le \Big(\int_0^\infty \int_0^\infty \sum_j \big|a_j(r,s)\big|^2 \frac{dr\, ds}{rs}\Big)^{1/2} \times \Big(\int_0^\infty \int_0^\infty \sum_j \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) f(x-r\xi, y-s\eta)\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs}\Big)^{1/2}.$$

Take two Schwartz functions $\varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^m)$ satisfying

(i) $0 \le \varphi_1, \varphi_2 \le 1$, $\operatorname{supp} \varphi_1(x) \subset \{x\colon 1/2 \le |x| \le 2\}$, $\operatorname{supp} \varphi_2(y) \subset \{y\colon 1/2 \le |y| \le 2\}$;

(ii) $\sum_l \varphi_1(2^l|x|) = 1, \sum_k \varphi_2(2^k|y|) = 1$ for all $x \in \mathbb{R}^n\backslash\{0\}, y \in \mathbb{R}^m\backslash\{0\}$.

Define the operators S_l^1 and S_k^2 by

$$\widehat{S_l^1 g}(x) = \varphi_1(2^l|x|)\hat{g}(x), \qquad \widehat{S_k^2 h}(y) = \varphi_2(2^k|y|)\hat{h}(y).$$

Since $f(x,y) = \sum_u \sum_v \big(S_{l+u}^1 \otimes S_{k+v}^2 f\big)(x,y)$ for any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and $l,k \in \mathbb{Z}$, we have, by (1.4) and Minkowski's inequality,

$$\begin{aligned}
|T_K f(x,y)| &\le \Big(\int_0^\infty \int_0^\infty \sum_j \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) \\
&\qquad\qquad \times f(x-r\xi, y-s\eta)\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs}\Big)^{1/2} \\
&= \Big(\sum_j \sum_l \sum_k \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) \\
&\qquad\qquad \times f(x-r\xi, y-s\eta)\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs}\Big)^{1/2} \\
&= \Big(\sum_j \sum_l \sum_k \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \Big| \sum_u \sum_v \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) \\
&\qquad\qquad \times \big(S_{l+u}^1 \otimes S_{k+v}^2 f\big)(x-r\xi, y-s\eta)\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs}\Big)^{1/2}
\end{aligned}$$

$$\leq \sum_u \sum_v \Big(\sum_j \sum_l \sum_k \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) \times \big(S^1_{l+u} \otimes S^2_{k+v} f\big)(x-r\xi, y-s\eta)\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs} \Big)^{1/2}$$

$$:= \sum_u \sum_v B_{u,v}(f)(x,y).$$

We first estimate the $L^2(\mathbb{R}^n\times\mathbb{R}^m)$-norm of $B_{u,v}(f)(x,y)$. Let

$$L(x,y) = \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)\big(S^1_{l+u}\otimes S^2_{k+v} f\big)(x-r\xi, y-s\eta)\, d\xi\, d\eta.$$

We then have

$$\widehat{L}(x,y) = \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) e^{-i(x\cdot r\xi + y\cdot s\eta)} \times \varphi_1(2^{l+u}|x|)\varphi_2(2^{k+v}|y|)\hat{f}(x,y)\, d\xi\, d\eta. \tag{3.1}$$

Applying Plancherel's theorem and (3.1), we get

$$\begin{aligned}
\big\|B_{u,v}(f)\big\|_2^2 &= \sum_j \sum_l \sum_k \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \big|L(x,y)\big|^2\, dx\, dy\, \frac{dr\, ds}{rs} \\
&= \sum_j \sum_l \sum_k \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \big|\widehat{L}(x,y)\big|^2\, dx\, dy\, \frac{dr\, ds}{rs} \\
&\leq \sum_j \sum_l \sum_k \iint_{\substack{1/2\leq|2^{l+u}x|\leq 2 \\ 1/2\leq|2^{k+v}y|\leq 2}} \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \Big| \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta) \\
&\qquad \times e^{-i(x\cdot r\xi + y\cdot s\eta)}\, d\xi\, d\eta \Big|^2 \frac{dr\, ds}{rs} |\hat{f}(x,y)|^2\, dx\, dy.
\end{aligned}$$

By Lemma 1,

$$\begin{aligned}
\big\|B_{u,v}(f)\big\|_2^2 &\leq C \sum_j \|\Omega_j\|_q^2 \sum_l \sum_k \iint_{\substack{1/2\leq|2^{l+u}x|\leq 2 \\ 1/2\leq|2^{k+v}y|\leq 2}} \Big(\min\big\{|2^l x||2^k y|, |2^l x|^{-\sigma}|2^k y|^{-\sigma}, \\
&\qquad |2^l x|^{\varepsilon}|2^k y|^{-\theta}, |2^l x|^{-\theta}|2^k y|^{\varepsilon}\big\}\Big) |\hat{f}(x,y)|^2\, dx\, dy \\
&\leq C \min\{2^{-u}2^{-v}, 2^{u\sigma}2^{v\sigma}, 2^{-u\varepsilon}2^{v\theta}, 2^{u\theta}2^{-v\varepsilon}\}\|f\|_2^2.
\end{aligned}$$

Thus, for $0<\sigma<1/q'$ and $0<\varepsilon,\theta<1$, we obtain

$$\|B_{u,v}(f)\|_2 \le C\min\{2^{-u/2}2^{-v/2}, 2^{u\sigma/2}2^{v\sigma/2}, 2^{-u\varepsilon/2}2^{v\theta/2}, 2^{u\theta/2}2^{-v\varepsilon/2}\}\|f\|_2. \tag{3.2}$$

We now estimate the $L^p(\mathbb{R}^n\times\mathbb{R}^m)$-norm of $B_{u,v}(f)(x,y)$. This will allow us to finish the proof of Theorem 1 under the conditions (a), (b), and (c), respectively.

Proof of Theorem 1 for condition (a). Let us first consider the case when $2\le p<2q'/(q'-2)$. Since

$$\big(B_{u,v}(f)(x,y)\big)^2 \le \sum_j \|\Omega_j\|_q^2 \sum_l\sum_k \int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} \cdot\Big(\iint_{S^{n-1}\times S^{m-1}} \big|S^1_{l+u}\otimes S^2_{k+v}f(x-r\xi,y-s\eta)\big|^{q'}d\xi\,d\eta\Big)^{2/q'}\frac{dr\,ds}{rs},$$

by duality there is a function $g(x,y)\in L^{(p/2)'}(\mathbb{R}^n\times\mathbb{R}^m)$ satisfying $\|g\|_{(p/2)'}\le 1$ such that

$$\begin{aligned}\|B_{u,v}(f)\|_p^2 &\le C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_l\sum_k\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\\ &\quad\cdot\Big(\iint_{S^{n-1}\times S^{m-1}}\big|S^1_{l+u}\otimes S^2_{k+v}f(x-r\xi,y-s\eta)\big|^{q'}\,d\xi\,d\eta\Big)^{2/q'}\frac{dr\,ds}{rs}\,|g(x,y)|\,dx\,dy.\end{aligned}$$

Changing variables and applying Hölder's inequality (note that $q'/2>1$), we get

$$\begin{aligned}\|B_{u,v}(f)\|_p^2 &\le C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_l\sum_k\big|S^1_{l+u}\otimes S^2_{k+v}f(x,y)\big|^2\int_1^2\int_1^2\\ &\quad\cdot\Big(\iint_{S^{n-1}\times S^{m-1}}\big|g(x+2^lr\xi,y+2^ks\eta)\big|^{q'/2}\,d\xi\,d\eta\Big)^{2/q'}\frac{dr\,ds}{rs}\,dx\,dy\\ &\le C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_l\sum_k\big|S^1_{l+u}\otimes S^2_{k+v}f(x,y)\big|^2\big(M^*(|g|^{q'/2})(x,y)\big)^{2/q'}\,dx\,dy \qquad (3.3)\\ &\le C\Big\|\sum_l\sum_k\big|S^1_{l+u}\otimes S^2_{k+v}f(\cdot,\cdot)\big|^2\Big\|_{p/2}\Big\|\big(M^*(|g|^{q'/2})(\cdot,\cdot)\big)^{2/q'}\Big\|_{(p/2)'},\end{aligned}$$

where M^* is the Hardy-Littlewood maximal operator on product spaces defined by

$$M^*g(x,y) = \sup_{r,s>0} \frac{1}{r^n s^m} \iint\limits_{\substack{|x-w|<r \\ |y-z|<s}} |g(w,z)|\, dw\, dz.$$

It is well known that M^* is a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p > 1$ (see [4]). Since $p < 2q'/(q'-2)$ implies $2(p/2)'/q' > 1$, by the choice of g, we get

$$\left\| \big(M^*(|g|^{q'/2})(\cdot,\cdot)\big)^{2/q'} \right\|_{(p/2)'} = \left\| M^*(|g|^{q'/2})(\cdot,\cdot) \right\|_{2(p/2)'/q'}^{2/q'} \le C\|g\|_{(p/2)'} \le C.$$

It follows from the Littlewood-Paley theorem and (3.3) that

$$\|B_{u,v}(f)\|_p \le C\|f\|_p. \tag{3.4}$$

Interpolating between (3.2) and (3.4) and applying Minkowski's inequality, we obtain

$$\left\| \sup_{K\in M} |T_K(f)| \right\|_p \le C\|f\|_p \qquad \text{for} \quad 2 \le p < 2q'/(q'-2). \tag{3.5}$$

We next consider the case of $\max\{2nq'/(2n+nq'-2), 2mq'/(2m+mq'-2)\} < p < 2$. Set

$$E_{r,s}f(x,y) = \iint\limits_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)\big(S^1_{l+u} \otimes S^2_{k+v} f\big)(x - 2^l r\xi, y - 2^k s\eta)\, d\xi\, d\eta.$$

Then,

$$B_{u,v}(f)(x,y) = \left(\sum_j \sum_l \sum_k \int_1^2 \int_1^2 \big|E_{r,s}f(x,y)\big|^2 \frac{dr\, ds}{rs} \right)^{1/2}.$$

Hence, to prove $B_{u,v}(f) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$, it suffices to show

$$E_{r,s}f(x,y) \in L^p \left\{ l^2 \left(l^2 \left\{ l^2 \left[L^2 \left([1,2]\times[1,2], \frac{dr\, ds}{rs} \right), k \right], l \right\}, j \right), dx\, dy \right\}.$$

By duality again, there is a function g depending on the indices j, l, and k and satisfying

$$g(x,y,r,s,j,l,k) \in L^{p'} \left\{ l^2 \left(l^2 \left\{ l^2 \left[L^2 \left([1,2]\times[1,2], \frac{dr\, ds}{rs} \right), k \right], l \right\}, j \right), dx\, dy \right\}$$

with $\|g\| \le 1$ such that

$$\|B_{u,v}(f)\|_p$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_j \sum_l \sum_k \int_1^2 \int_1^2 \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)$$

$$\times \big(S^1_{l+u}\otimes S^2_{k+v} f\big)(x-2^l r\xi, y-2^k s\eta)\, d\xi\, d\eta\, g(x,y,r,s,j,l,k) \frac{dr\, ds}{rs}\, dx\, dy$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \sum_j \int_1^2 \int_1^2 \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)$$

$$\times g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)\big(S^1_{l+u}\otimes S^2_{k+v} f\big)(x,y)\, d\xi\, d\eta\, \frac{dr\, ds}{rs}\, dx\, dy$$

$$\le \Bigg\| \Bigg\{ \sum_l \sum_k \Bigg(\sum_j \int_1^2 \int_1^2 \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)$$

$$\times g(\cdot+2^l r\xi, \cdot+2^k s\eta, r,s,j,l,k)\, d\xi\, d\eta \frac{dr\, ds}{rs}\Bigg)^2 \Bigg\}^{1/2} \Bigg\|_{p'}$$

$$\times \Bigg\| \Bigg(\sum_l \sum_k \big|\big(S^1_{l+u}\otimes S^2_{k+v} f\big)(\cdot,\cdot)\big|^2 \Bigg)^{1/2} \Bigg\|_p.$$

Let

$$Ug(x,y) = \sum_l \sum_k \Bigg(\sum_j \int_1^2 \int_1^2 \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)$$

$$\times g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)\, d\xi\, d\eta\, \frac{dr\, ds}{rs}\Bigg)^2.$$

Then, by the Littlewood-Paley theory we have

$$\|B_{u,v}(f)\|_p \le \|(Ug)^{1/2}\|_{p'} \|f\|_p. \tag{3.6}$$

Note that $\|(Ug)^{1/2}\|_{p'} = \|Ug\|^{1/2}_{p'/2}$ for $p'>2$. Therefore, there is a function $h \in L^{(p'/2)'}(\mathbb{R}^n\times\mathbb{R}^m)$ with $\|h\|_{(p'/2)'} \le 1$ such that

$$\|Ug\|_{p'/2} = \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \Bigg(\sum_j \int_1^2 \int_1^2 \iint_{S^{n-1}\times S^{m-1}} \Omega_j(\xi,\eta)$$

$$\times g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)\, d\xi\, d\eta \frac{dr\, ds}{rs}\Bigg)^2 h(x,y)\, dx\, dy.$$

By Hölder's inequality and Schwarz's inequality, we have

$$
\begin{aligned}
&\|Ug\|_{p'/2} \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \Big\{ \sum_j \int_1^2 \int_1^2 \Big(\iint_{S^{n-1}\times S^{m-1}} |\Omega_j(\xi,\eta)|^q \, d\xi\, d\eta \Big)^{1/q} \\
&\quad \cdot \Big(\iint_{S^{n-1}\times S^{m-1}} \big|g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)\big|^{q'} d\xi\, d\eta \Big)^{1/q'} \frac{dr\, ds}{rs} \Big\}^2 h(x,y) dx dy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \Big\{ \sum_j \|\Omega_j\|_q \int_1^2 \int_1^2 \\
&\quad \cdot \Big(\iint_{S^{n-1}\times S^{m-1}} |g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)|^{q'} d\xi\, d\eta \Big)^{1/q'} \frac{dr\, ds}{rs} \Big\}^2 h(x,y) dx dy \\
&\le \Big(\sum_j \|\Omega_j\|_q^2 \Big) \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \sum_j \Big\{ \int_1^2 \int_1^2 \\
&\quad \cdot \Big(\iint_{S^{n-1}\times S^{m-1}} |g(x+2^l r\xi, y+2^k s\eta, r,s,j,l,k)|^{q'} d\xi\, d\eta \Big)^{1/q'} \frac{dr\, ds}{rs} \Big\}^2 h(x,y) dx dy.
\end{aligned}
$$

Using the hypotheses $\sum_j \|\Omega_j\|_q^2 < \infty$ and changing variables, we obtain

$$
\begin{aligned}
&\|Ug\|_{p'/2} \\
&\le C \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \sum_j \Big\{ \int_1^2 \int_1^2 \Big(\iint_{S^{n-1}\times S^{m-1}} |g(x,y,r,s,j,l,k)|^{q'} \\
&\qquad\qquad \times \big| h(x-2^l r\xi, y-2^k s\eta) \big|^{q'/2} \, d\xi\, d\eta \Big)^{1/q'} \frac{dr\, ds}{rs} \Big\}^2 dx\, dy \\
&\le C \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \sum_j \Big\{ \int_1^2 \int_1^2 \Big(\mathcal{M}^*\big(h^{q'/2}\big)(x,y) \Big)^{1/q'} \\
&\qquad\qquad \times |g(x,y,r,s,j,l,k)| \frac{dr\, ds}{rs} \Big\}^2 dx\, dy \\
&\le C \iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_l \sum_k \sum_j \int_1^2 \int_1^2 |g(x,y,r,s,j,l,k)|^2 \frac{dr\, ds}{rs} \Big(\mathcal{M}^*\big(h^{q'/2}\big)(x,y) \Big)^{2/q'} dx\, dy
\end{aligned}
$$

$$\leq C\Big\| \sum_l \sum_k \sum_j \int_1^2\int_1^2 |g(\cdot,\cdot,r,s,j,l,k)|^2 \frac{dr\,ds}{rs}\Big\|_{p'/2} \Big\| \Big(\mathcal{M}^*\big(h^{q'/2}\big)(\cdot,\cdot)\Big)^{2/q'}\Big\|_{(p'/2)'}.$$

Since $\max\{2nq'/(2n+nq'-2), 2mq'/(2m+mq'-2)\} < p$, we have $(2/q')\cdot(p'/2)' > \max\{n/(n-1), m/(m-1)\}$. Using Lemma 2, we get

$$\Big\|\big(\mathcal{M}^*(h^{q'/2})(\cdot,\cdot)\big)^{2/q'}\Big\|_{(p'/2)'} \leq C\|h(\cdot,\cdot)\|_{(p'/2)'}.$$

By (3.6) and the choice of $g(x,y,r,s,j,l,k)$ and h, we obtain

$$\|B_{u,v}(f)\|_p \leq C_p \|Ug\|_{p'/2}^{1/2} \|f\|_p \leq C_p\|g\|\cdot\|h\|_{(p'/2)'}^{1/2}\|f\|_p \leq C_p\|f\|_p$$

for $\max\{2nq'/(2n+nq'-2), 2mq'/(2m+mq'-2)\} < p < 2$. Again, interpolating between (3.2) and the above inequality and using Minkowski's inequality, we get

$$\Big\|\sup_{K\in M}|T_K(f)|\Big\|_p \leq C\|f\|_p$$

for $\max\{2nq'/(2n+nq'-2), 2mq'/(2m+mq'-2)\} < p < 2$, which together with (3.5) proves Theorem 1 under condition (a). □

Proof of Theorem 1 for condition (b). As in the preceding proof, we first consider the case $2 \leq p < \infty$. Since $q' \leq 2$, using Hölder's inequality twice, we obtain

$$\begin{aligned}
&\big(B_{u,v}(f)(x,y)\big)^2\\
&\quad\leq \Big(\sum_j \|\Omega_j\|_q^2\Big)\sum_l\sum_k \int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\\
&\qquad\cdot\Big(\iint_{S^{n-1}\times S^{m-1}} \big|S^1_{l+u}\otimes S^2_{k+v} f(x-r\xi, y-s\eta)\big|^{q'}\,d\xi\,d\eta\Big)^{2/q'}\frac{dr\,ds}{rs}\\
&\quad\leq C\sum_l\sum_k \int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\\
&\qquad\cdot\iint_{S^{n-1}\times S^{m-1}} \big|S^1_{l+u}\otimes S^2_{k+v} f(x-r\xi, y-s\eta)\big|^2\,d\xi\,d\eta\frac{dr\,ds}{rs}.
\end{aligned}$$

By the same argument as in the proof of (3.3), we obtain

$$\|B_{u,v}(f)\|_p^2 \leq C\Big\|\sum_l\sum_k \big|S^1_{l+u}\otimes S^2_{k+v} f(\cdot,\cdot)\big|^2\Big\|_{p/2}\|M^*(g)(\cdot,\cdot)\|_{(p/2)'}, \tag{3.7}$$

where $g(x,y)\in L^{(p/2)'}(\mathbb{R}^n\times\mathbb{R}^m)$ satisfies $\|g\|_{(p/2)'}\leq 1$. Hence, by (3.7), the $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ $(p>1)$ boundedness of M^*, and the choice of g, we obtain

(3.4) for $2 \le p < \infty$. Interpolating between (3.2) and (3.4) and applying Minkowski's inequality, we get

$$\Big\| \sup_{K\in M} |T_K(f)| \Big\|_p \le C\|f\|_p \qquad \text{for} \quad 2 \le p < \infty.$$

The proof of this inequality for $\max\{2nq'/(2n+nq'-2), 2mq'/(2m+mq'-2)\} < p < 2$ is exactly the same as in the case (a). Thus we obtain the conclusion of Theorem 1 under condition (b). □

Proof of Theorem 1 for condition (c). In this case, we have $1 \le q' < \min\{2(n-1)/n, 2(m-1)/m\} < 2$. The proof for the case $2 \le p < \infty$ is the same as the proof in case (b), so we only consider the case $1 < p < 2$. Using the same idea and notations as in case (a), we have

$$\|Ug\|_{p'/2} \le C \Big\| \sum_l \sum_k \sum_j \int_1^2 \int_1^2 |g(\cdot,\cdot,r,s,j,l,k)|^2 \frac{dr\,ds}{rs} \Big\|_{p'/2} \times \Big\| \big(\mathcal{M}^*(h^{q'/2})(\cdot,\cdot)\big)^{2/q'} \Big\|_{(p'/2)'}.$$

Since $(p'/2) > 1$, we have $(2/q')\cdot(p'/2)' > (2/q') > \max\{n/(n-1), m/(m-1)\}$. Using Lemma 2 together with the choice of $g(x,y,r,s,j,l,k)$ and h, we obtain $\|B_{u,v}(f)\|_p \le C_p\|f\|_p$ for $1 < p < 2$. It is now easy to see that the conclusion of Theorem 1 holds under condition (c). □

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