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A TANGENCY PRINCIPLE AND APPLICATIONS

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ABSTRACT. In this paper we obtain a tangency principle for hypersurfaces, with not necessarily constant r-mean curvature function H_r, of an arbitrary Riemannian manifold. That is, we obtain sufficient geometric conditions for two submanifolds of a Riemannian manifold to coincide, as a set, in a neighborhood of a tangency point. As applications of our tangency principle, we obtain, under certain conditions on the function H_r, sharp estimates on the size of the greatest ball that fits inside a connected compact hypersurface embedded in a space form of constant sectional curvature $c \leq 0$ and on the size of the smallest ball that encloses the image of an immersion of a compact Riemannian manifold into a Riemannian manifold with sectional curvatures limited from above. This generalizes results of Koutroufiotis, Coghlan-Itokawa, Pui-Fai Leung, Vlachos and Markvorsen. We also generalize a result of Serrin. Our techniques permit us to extend results of Hounie-Leite.

1. Introduction

Let N^{n+1} be a complete Riemannian manifold with metric $\langle\ ,\ \rangle$, Levi-Civita connection ∇ and the usual exponential mapping $\exp\colon\ TN \to N$. Consider a hypersurface M^n of N^{n+1}. Given $p \in M^n$ and a fixed unitary vector η_0 that is normal to M^n at p, we can parametrize a neighborhood of M^n containing p and contained in a normal ball of N^{n+1} as

$$\varphi(x) = \exp_p(x + \mu(x)\eta_0), \tag{1.1}$$

where the vector x varies in a neighborhood W of zero in T_pM and $\mu\colon\ W \to \mathbb{R}$ satisfies $\mu(0) = 0$. Observe that μ is unique. Consider now a local orientation $\eta\colon\ W \to T^{\perp}_{\varphi(W)}M$ of M^n with $\eta(0) = \eta_0$. Denote by $A_{\eta(x)}$ the second fundamental form of M^n in the direction $\eta(x)$. Choosing the principal curvatures of M^n at each $x \in W$ so that $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$, the functions λ_i become continuous functions on W. Denote by $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$

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the principal curvature vector at $x \in W$. The r-mean curvatures H_r, $1 \le r \le n$, are given by

$$H_r(x) = \frac{1}{\binom{n}{r}} \sigma_r(\lambda(x)), \tag{1.2}$$

where $\sigma_r(\lambda(x))$ is the value at $\lambda(x)$ of the r-elementary symmetric function $\sigma_r \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\sigma_r(z_1, z_2, \dots, z_n) = \sum_{i_1 < i_2 < \cdots < i_r} z_{i_1} z_{i_2} \dots z_{i_r}. \tag{1.3}$$

Denote by Γ_r the connected component in $\mathbb{R}^n$ of the set $\{\, \sigma_r > 0 \,\}$ that contains the vector $a_0 = (1, 1, \dots, 1)$. Observe that Γ_n is precisely the positive cone $\mathcal{O}^n$, defined by

$$\mathcal{O}^n = \{\, (z_1, z_2, \dots, z_n) \in \mathbb{R}^n \;\mid\; z_i > 0 \quad \text{for } 1 \le i \le n \,\}, \tag{1.4}$$

and that $\mathcal{O}^n \subset \Gamma_r$ for $1 \le r \le n$. In fact, we will show in Section 2 that, more generally, $\Gamma_{r+1} \subset \Gamma_r$ for $1 \le r \le n - 1$.

Definition. Let M_1^n and M_2^n be hypersurfaces of N^{n+1} that are tangent at p, i.e., which satisfy $T_p M_1 = T_p M_2$. Fix a unitary vector η_0 that is normal to M_1^n at p. We say that M_1^n remains above M_2^n in a neighborhood of p with respect to η_0 if, when we parametrize M_1^n and M_2^n by φ^1 and φ^2 as in (1.1), the corresponding functions μ^1 and μ^2 satisfy $\mu^1(x) \ge \mu^2(x)$ in a neighborhood of zero.

We note in passing that this definition is equivalent to requiring that the geodesics of N^{n+1} that are normal to the hypersurface which is totally geodesic at p (namely, $\exp_p(W)$), in a neighborhood of p intercept M_2^n before M_1^n.

In this paper we obtain the following tangency principle:

Theorem 1.1. *Let M_1^n and M_2^n be hypersurfaces of N^{n+1} that are tangent at p and let η_0 be a unitary vector that is normal to M_1^n at p. Suppose that M_1^n remains above M_2^n in a neighborhood of p with respect to η_0. Denote by $H_r^1(x)$ and $H_r^2(x)$ the r-mean curvature at $x \in W$ of M_1^n and M_2^n, respectively. Assume that, for some r, $1 \le r \le n$, we have $H_r^2(x) \ge H_r^1(x)$ in a neighborhood of zero; if $r \ge 2$, assume also that $\lambda^2(0)$, the principal curvature vector of M_2 at zero, belongs to Γ_r. Then M_1^n and M_2^n coincide in a neighborhood of p.*

For hypersurfaces with boundaries, as a consequence of the proof of Theorem 1.1, we obtain the following tangency principle:

Theorem 1.2. *Let M_1^n and M_2^n be hypersurfaces of N^{n+1} with boundaries ∂M_1 and ∂M_2, respectively. Suppose that M_1^n and M_2^n, as well as ∂M_1 and ∂M_2, are tangent at $p \in \partial M_1 \cap \partial M_2$, and let η_0 be normal to M_1^n at p.*

Suppose that M_1^n remains above M_2^n in a neighborhood of p with respect to η_0. Denote by $H_r^1(x)$ and $H_r^2(x)$ the r-mean curvatures at $x \in W$ of M_1^n and M_2^n, respectively. Assume that, for some r, $1 \leq r \leq n$, we have $H_r^2(x) \geq H_r^1(x)$ in a neighborhood of zero. If $r \geq 2$, assume also that $\lambda^2(0)$, the principal curvature vector of M_2 at zero, belongs to Γ_r. Then M_1^n and M_2^n coincide in a neighborhood of p.

In connection with the above results see also Remark 4.4.

In order to state our applications, we need to introduce some notations. Denote by $\overline{B_\rho(p_0)}$ a geodesic closed ball centered at p_0 and of radius ρ in the ambient space, and let Q_c^{n+1} be the $(n+1)$-dimensional simply connected space form of constant curvature c. Consider the functions

$$\mu_c(t) = \begin{cases} t\sqrt{-c}\,\coth(t\sqrt{-c}), & c < 0, \\ 1, & c = 0, \\ t\sqrt{c}\,\cot(t\sqrt{c}), & c > 0. \end{cases} \tag{1.5}$$

As a first application of Theorem 1.1, we obtain the following result.

THEOREM 1.3. *Let M^n be a compact connected embedded hypersurface of Q_c^{n+1}, $c \leq 0$. Suppose that $|H_r| \geq [\mu_c(\rho)/\rho]^r$ on M^n for some $\rho > 0$. Then the largest sphere which fits inside M^n has radius less than ρ, unless M^n is a sphere.*

Theorem 1.3 generalizes Theorem 1 in [11] and a result due to Blaschke ([3]; see also Theorem 3 in [11]). As a second application of Theorem 1.1, we generalize a result of Serrin, stated as Theorem 1 in [14], in the following theorem.

THEOREM 1.4. *Let M^n be a compact connected hypersurface in Q_c^{n+1} with boundary ∂M contained in the closed ball $\overline{B_\tau(p_0)}$. Suppose that, for some $\rho > 0$, we have $|H_r| \leq [\mu_c(\rho)/\rho]^r$ and that M^n is contained in the closed ball $\overline{B_\rho(p_0)}$; if $c > 0$, suppose further that $\rho < \pi/2\sqrt{c}$. Then M^n is contained in $\overline{B_\tau(p_0)}$.*

From Theorem 1.1 we also obtain the following result.

THEOREM 1.5. *Let $F\colon\ M^n \to N^{n+1}$ be a smooth isometric immersion of a compact connected Riemannian manifold into a Riemannian manifold N^{n+1}. Suppose that $F(M)$ is contained in a closed normal ball $\overline{B_\rho(p_0)}$ centered at p_0 and of radius ρ. Let c be the supremum of the sectional curvatures of N^{n+1} on $\overline{B_\rho(p_0)}$; if $c > 0$, assume also that $\rho < \pi/2\sqrt{c}$. If $|H_r| \leq [\mu_c(\rho)/\rho]^r$, then $F(M)$ is the boundary of $\overline{B_\rho(p_0)}$ and $B_\rho(p_0)$ is isometric to an open ball of radius ρ in Q_c^{n+1}.*

Corollary 1.6. *Let $F\colon\ M^n \to N^{n+1}$ be a smooth isometric immersion of a compact connected Riemannian manifold into a Riemannian manifold N^{n+1} with sectional curvature function satisfying $K_N \leq c$ for some real constant c. Suppose that $F(M)$ is contained in a closed normal ball $\overline{B_\rho(p_0)}$. If $c > 0$, assume furthermore that $\rho < \pi/2\sqrt{c}$. If $|H_r| \leq [\mu_c(\rho)/\rho]^r$ then $F(M)$ is the boundary of $\overline{B_\rho(p_0)}$ and $B_\rho(p_0)$ is isometric to an open ball of radius ρ in Q_c^{n+1}.*

For the case of mean curvature, i.e., the case $r = 1$, Theorem 1.5 was obtained by Markvorsen in [13]. We point out that Coghlan, Itokawa, and Kosecki [6], assuming $\sup_M |H| = \mu_c(\rho)/\rho$ for the length of the mean curvature vector H of an immersion $G\colon\ M^n \to N^m$ such that $G(M) \subset \overline{B_\rho(p_0)}$, concluded that F must be a minimal immersion on the boundary of $\overline{B_\rho(p_0)}$. Here M^n is a complete connected Riemannian manifold with scalar curvature bounded away from $-\infty$, c is the supremum of the sectional curvature over $\overline{B_\rho(p_0)}$, and $\rho < \pi/2\sqrt{c}$ if $c > 0$.

When N^{n+1} is the space form Q_c^{n+1}, rigidity theorems similar to Theorem 1.5 were obtained by Koutroufiotis [11] and Coghlan and Itokawa [5] for sectional curvature, by Pui-Fai Leung [12] for Ricci curvature, and by Vlachos [15] for all r-mean curvatures.

2. Elliptic operators and hyperbolic polynomials

For $d = (n(n+1)/2) + 2n + 1$, write an arbitrary point p at $\mathbb{R}^d$ as

$$p = \big(r_{11}, \ldots, r_{1n}, r_{22}, \ldots r_{2n}, \ldots, r_{(n-1)n}, r_{nn}, r_1, \ldots, r_n, z, x_1, \ldots, x_n\big)$$

or, in short, as $p = (r_{ij}, r_i, z, x)$ with $1 \leq i \leq j \leq n$ and $x = (x_1, \ldots, x_n)$. A C^1-function $\Phi\colon\ \Gamma \to \mathbb{R}$ defined in an open set Γ of $\mathbb{R}^d$ is said to be elliptic in $p \in \Gamma$ if

$$\sum_{i \leq j=1}^{n} \frac{\partial \Phi}{\partial r_{ij}}(p)\xi_i\xi_j > 0 \quad \text{for all nonzero} \quad (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n. \tag{2.1}$$

We say that Φ is elliptic in Γ if Φ is elliptic in p for all $p \in \Gamma$. Given a function $f\colon\ U \to \mathbb{R}$ of class C^2 defined in an open set $U \subset \mathbb{R}^n$ and $x \in U$, we associate a point $\Lambda(f)(x)$ in $\mathbb{R}^d$ by setting

$$\Lambda(f)(x) = (f_{ij}(x), f_i(x), f(x), x), \tag{2.2}$$

where $f_{ij}(x)$ and $f_i(x)$ stand for $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ and $\frac{\partial f}{\partial x_i}(x)$, respectively. Saying that the function Φ is elliptic with respect to f means that $\Lambda(f)(x)$ belongs to Γ and Φ is elliptic in $\Lambda(f)(x)$ for all $x \in U$. For elliptic functions we have the following maximum principle (see [1]).

Maximum Principle. *Let f, $g\colon\ U \to \mathbb{R}$ be C^2-functions defined in an open set U of $\mathbb{R}^n$ and let $\Phi\colon\ \Gamma \subset \mathbb{R}^d \to \mathbb{R}$ be a function of class C^1. Suppose* *that Φ is elliptic with respect to the functions $(1-t)f+tg$, $t\in[0,1]$. Assume also that*

$$\Phi(\Lambda(f)(x)) \geq \Phi(\Lambda(g)(x)) \text{ for all } x\in U, \tag{2.3}$$

and that $f\leq g$ on U. Then $f<g$ on U unless f and g coincide in a neighborhood of any point $x_0\in U$ such that $f(x_0)=g(x_0)$.

To obtain this above maximum principle, which in the case $n=2$ is stated in [11], one linearizes in a well-known fashion,

$$\Phi(\Lambda(f)(x)) - \Phi(\Lambda(g)(x)) = L(f-g)(x) \geq 0,$$

and then applies Hopf's maximum principle for linear operators to conclude that if $f(x_0)=g(x_0)$ for some $x_0\in U$ then f and g coincide in a neighborhood of x_0 in U.

For our proofs we will also need the following result from [7]. Let $P\colon\ \mathbb{R}^n\to\mathbb{R}$ be a homogeneous polynomial of degree m and let $a\in\mathbb{R}^n$ be a fixed vector. We say that P is a-hyperbolic or hyperbolic with respect to the vector a if the s-polynomial $P(sa+x)$ has m real roots for all $x\in\mathbb{R}^n$. In [7], Gårding proved that the set

$$C(P,a) = \{\, x\in\mathbb{R}^n \mid P(sa+x)\neq 0, \text{ for all } s\geq 0\,\} \tag{2.4}$$

is an open convex cone that coincides with the connected component of $\{P\neq 0\}$ containing a and that if P is a-hyperbolic, then the homogeneous polynomial of degree $m-1$ given by

$$Q(x) = \frac{d}{d\,s}P(sa+x)|_{s=0} = \sum_{j=1}^{n} a_j\frac{\partial P}{\partial x_j}(x)$$

is also a-hyperbolic and $C(P,a)\subset C(Q,a)$.

Applying this result to the n-elementary symmetric function σ_n, which is a_0-hyperbolic with respect to $a_0=(1,1,\dots,1)$, and observing that

$$\sigma_r(x) = \frac{1}{(n-r)!}\frac{d^{n-r}}{d\,s^{n-r}}\sigma_n(sa+x)|_{s=0},$$

it is not difficult to see that the homogeneous polynomials σ_r of degree r, $1\leq r\leq n$, are a_0-hyperbolic and that the sets $\Gamma_r = C(\sigma_r, a_0)$, $1\leq r\leq n$, satisfy

$$\Gamma_n\subset\Gamma_{n-1}\subset\cdots\subset\Gamma_1. \tag{2.5}$$

As we have already noted in the Introduction, Γ_n is precisely the positive cone $\mathcal{O}^n$. Gårding also established an inequality for hyperbolic polynomials from which it is possible to prove (see [4], Proposition 1.1) that

$$D_i\sigma_r = \frac{\partial\sigma_r}{\partial z_i} > 0 \quad\text{on}\quad \Gamma_r,\ \ 1\leq i\leq n,\ 1\leq r\leq n. \tag{2.6}$$

3. r-mean curvatures and ellipticity

Given a hypersurface M^n of a complete Riemannian manifold N^{n+1} and $p \in M^n$, parametrize M^n in a neighborhood of p as in (1.1). Our goal now is to find a function Φ_r defined in some open set of $\mathbb{R}^d$, $d = \frac{n(n+1)}{2} + 2n + 1$, that contains the origin so that

$$H_r(x) = \Phi_r(\mu_{ij}(x), \mu_i(x), \mu(x), x) = \Phi_r(\Lambda(\mu)(x)), \quad x \in W.$$

To this end we fix an orthonormal basis $e_1, e_2, \dots, e_n$ in T_pM and introduce coordinates in T_pM by setting $x = \sum_{i=1}^n x_i e_i$ for all x in T_pM. Note that the function μ satisfies $\mu_i(0) = \frac{\partial \mu}{\partial x_i}(0) = 0$, $1 \le i \le n$. Recall that $\eta\colon W \to T^{\perp}_{\varphi(W)}M$ is a local orientation of M^n with $\eta(0) = \eta_0$ and $A_{\eta(x)}$ is the second fundamental form of M^n in the direction $\eta(x)$. Denote by $\varphi_i(x)$ the vector $\frac{\partial \varphi}{\partial x_i}(x)$. If $A(x) = (a_{ij}(x))$ is the matrix of $A_{\eta(x)}$ in the basis $\varphi_i(x)$, $1 \le i \le n$, then $A(x)$ satisfies $A_{\eta(x)}\varphi_i(x) = \sum_{j=1}^n a_{ji}(x)\varphi_j(x)$. It is not difficult to verify that

$$A(x) = I(x)^{-1} II(x), \tag{3.1}$$

where $I(x)$ and $II(x)$ are the matrices given by

$$I(x)_{ij} = \langle \varphi_i(x), \varphi_j(x) \rangle$$

and

$$II(x)_{ij} = \langle A_{\eta(x)}\varphi_i(x), \varphi_j(x) \rangle = \langle (\nabla_{\varphi_i}\varphi_j)_x, \eta(x) \rangle.$$

LEMMA 3.1. *There exists an $n \times n$-matrix valued function $\tilde{A}$ defined in an open set $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$ of $\mathbb{R}^d$ such that*

$$\tilde{A}(\mu_{ij}(x), \mu_i(x), \mu(x), x) = A(x), \quad x \in W. \tag{3.2}$$

Proof. We consider the entries in the matrices $I(x)$ and $II(x)$ given by (3.1). For simplicity of notation, we set $v(x) = \sum_{m=1}^n x_m e_m + \mu(x)\eta_0$. Since

$$\varphi_i(x) = d(\exp_p)_{v(x)}(e_i + \mu_i(x)\eta_0),$$

the $n \times n$-symmetric matrix $I(x)$ can be written as a function of x, $\mu(x)$ and $\mu_i(x)$, $1 \le i \le n$. Note that the point p, the orthonormal basis $e_i, 1 \le i \le n$, in T_pM, and η_0 are fixed. In the matrix $I(x)$ we replace, for all i, $\mu_i(x)$ by r_i, $\mu(x)$ by z, and x_i by y_i. We obtain an $n \times n$-symmetric matrix $\overline{F}(r_i, z, y_i)$ which has an inverse at points such that $d(\exp_p)_{(\sum_{i=1}^n y_i e_i + z\,\eta_0)}$ is a linear isomorphism. Take the maximal connected open set $\mathcal{N}$ in $\mathbb{R}^{n+1}$ that contains the origin and so that if $(z, y_1, \dots, y_n) \in \mathcal{N}$ then $d(\exp_p)_{(\sum_{i=1}^n y_i e_i + z\,\eta_0)}$ is a linear isomorphism. The existence of such a set $\mathcal{N}$ follows from the fact that $d(\exp_p)_0$ is the identity. Thus, restricting $\overline{F}$ to $\mathbb{R}^n \times \mathcal{N}$ and setting $F(r_i, z, y_i) = \overline{F}(r_i, z, y_i)^{-1}$, we have

$$I(x)^{-1} = F(\mu_i(x), \mu(x), x_i), \quad x \in W.$$

We now consider the entries in the $n \times n$-symmetric matrix $II(x)$. Observe first that

$$\begin{aligned}(3.3)\quad \langle \nabla_{\varphi_i}\varphi_j\, ,\, \eta\rangle_x = &\left\langle \nabla_{\varphi_i} d(\exp_p)_v e_j\, ,\, \eta\right\rangle_x \\ &+ \mu_{ij}(x)\langle d(\exp_p)_{v(x)}\eta_0\, ,\, \eta(x)\rangle + \mu_j(x)\left\langle \nabla_{\varphi_i} d(\exp_p)_v\eta_0, \eta\right\rangle_x .\end{aligned}$$

The vector-valued function $\eta(x)$ depends on x, $\mu(x)$ and the first order derivatives of $\mu(x)$, since $\eta(x)$ is determined by the basis $\varphi_i, 1 \le i \le n$, and the metric of N^{n+1} at $\varphi(x)$. Let $G(r_{ij}, r_i, z, y_i)$ be the $n \times n$-symmetric matrix defined as follows: if $k \le l$ then $G(r_{ij}, r_i, z, y_i)_{kl}$ is obtained from $II(x)_{kl}$ by replacing, on the right hand side of (3.3), $\mu_{kl}(x)$ by r_{kl}, $\mu_m(x)$ by r_m, $\mu(x)$ by z, and finally x_m by $y_m, 1 \le m \le n$; that is, if $k \le l$, then

$$\begin{aligned}(3.4)\quad G(r_{ij}, r_i, z, y_i)_{kl} = &\left\langle \nabla_{\psi_k} d(\exp_p)_v e_l\, ,\, \eta\right\rangle_{(r_i,z,y_i)} \\ &+ r_{kl}\langle d(\exp_p)_v\eta_0\, ,\, \eta\rangle_{(r_i,z,y_i)} + r_l\left\langle \nabla_{\psi_k} d(\exp_p)_v\eta_0, \eta\right\rangle_{(r_i,z,y_i)} ,\end{aligned}$$

where

$$v(z, y_i) = \sum_{m=1}^{n} y_m e_m + z\,\eta_0, \quad \psi_k(r_i, z, y_i) = d(\exp_p)_{v(z,y_i)}(e_k + r_k\,\eta_0)$$

and $\eta(r_i, z, y_i)$ is a unitary vector that is normal to the hyperplane spanned by $\psi_m(r_i, z, y_i)$, $1 \le m \le n$. Hence the $n\times n$-symmetric matrix $G(r_{ij}, r_i, z, y_i)$ defined in $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$ satisfies

$$II(x) = G(\mu_{ij}(x), \mu_i(x), \mu(x), x_i).$$

Taking

$$(3.5)\qquad \tilde{A}(r_{ij}, r_i, z, y_i) = F(r_i, z, y_i)G(r_{ij}, r_i, z, y_i),$$

we obtain an $n\times n$-matrix valued function $\tilde{A}$ in the open subset $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$ of $\mathbb{R}^d$ such that $\tilde{A}(\mu_{ij}(x), \mu_i(x), \mu(x), x) = A(x)$, $x \in W$. □

We point out that, since $F(r_i, z, y_i)$ is a definite positive symmetric matrix and $G(r_{ij}, r_i, z, y_i)$ is symmetric, the matrix $\tilde{A}(r_{ij}, r_i, z, y_i)$ given by (3.5) is diagonalizable (see e.g. [8], p. 120); that is, there exists an $n \times n$-invertible real matrix P, depending on (r_{ij}, r_i, z, y_i), such that $P^{-1}\tilde{A}(r_{ij}, r_i, z, y_i)P$ is diagonal.

Proposition 3.2. *There exists a function* $\Phi_r\colon\ \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N} \to \mathbb{R}$ *satisfying*

$$(3.6)\qquad \Phi_r(\Lambda(\mu)(x)) = \Phi_r(\mu_{ij}(x), \mu_i(x), \mu(x), x) = H_r(x).$$

Proof. Consider the function Φ_r defined by

$$(3.7)\qquad \Phi_r = \frac{1}{\binom{n}{r}}\sigma_r \circ \lambda \circ \tilde{A}.$$

Here $\lambda(\tilde{A}) = (\lambda_1(\tilde{A}), \lambda_2(\tilde{A}), \dots, \lambda_n(\tilde{A}))$, where $\lambda_1(\tilde{A}) \leq \lambda_2(\tilde{A}) \leq \cdots \leq \lambda_n(\tilde{A})$ are the eigenvalues of $\tilde{A}$. Now, (3.6) is an immediate consequence of (3.7), (1.2) and (3.2). □

If A is an arbitrary $n \times n$-real matrix, the eigenvalues $\lambda_i(A)$, $1 \leq i \leq n$, of A are not necessarily real, but we can consider

$$(\sigma_r \circ \lambda)(A) = \sigma_r(\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)),$$

where σ_r is given by (1.3). The value $(\sigma_r \circ \lambda)(A)$ does not depend on the order of the eigenvalues of A we choose. The function $\sigma_r \circ \lambda \colon\ \mathcal{M}^n(\mathbb{R}) \to \mathbb{R}$ defined in the set of all $n \times n$-real matrices is differentiable since $(\sigma_r \circ \lambda)(A)$ is a homogeneous polynomial of degree r in the entries of A.

In order to establish some ellipticity properties of Φ_r, we will need the following lemma.

Lemma 3.3. *If $A_0 \in \mathcal{M}^n(\mathbb{R})$ is symmetric and $\lambda(A_0) \in \Gamma_r$ then*

$$\sum_{i,j=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{ij}}(A_0)\xi_i\xi_j > 0 \quad \text{for all nonzero} \quad (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n. \tag{3.8}$$

Proof. We divide the proof into three steps.

Step 1. Suppose that A_0 is a diagonal matrix with distinct eigenvalues. In this case, it is well known that the functions λ_i, $1 \leq i \leq n$, are differentiable in a neighborhood of A_0, in $\mathcal{M}^n(\mathbb{R})$. Therefore,

$$\frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(A_0) = \sum_{i=1}^{n} \frac{\partial \sigma_r}{\partial z_i}(\lambda(A_0))\, \frac{\partial \lambda_i}{\partial A_{kl}}(A_0). \tag{3.9}$$

Let E^{kl} be the matrix defined by $\left(E^{kl}\right)_{ij} = \delta_{ki}\,\delta_{lj}$. Using the multilinearity of the determinant, we see that the matrices A_0 and $A_0 + tE^{kl}$ have the same characteristic polynomial for all t and $k \neq l$. This implies that

$$\frac{\partial \lambda_i}{\partial A_{kl}}(A_0) = 0 \quad \text{for} \quad k \neq l \quad \text{and} \quad 1 \leq i \leq n. \tag{3.10}$$

We now compute the above derivatives for $k = l$. Consider first the unique permutation θ of $\{1, 2, \dots, n\}$ such that $\lambda_{\theta(j)} = (A_0)_{jj}$. Since the functions λ_i, $1 \leq i \leq n$, are differentiable in a neighborhood of A_0, we have that in a neighborhood of zero the functions $\lambda_i(A_0 + t\,E^{kk})$, $1 \leq i \leq n$, are differentiable functions of t. Moreover, for t sufficiently small, the eigenvalues $\lambda_i(A_0 + t\,E^{kk})$, $1 \leq i \leq n$, are distinct since the values $\lambda_i(A_0)$ are distinct by assumption. Consequently, for small t, we have

$$\lambda_{\theta(j)}(A_0 + t\,E^{kk}) = (A_0 + tE^{kk})_{jj}.$$

Therefore,

$$\frac{d}{dt}\lambda_i(A_0 + t\,E^{kk})|_{t=0} = \begin{cases} 0, & k \neq \theta^{-1}(i), \\ 1, & k = \theta^{-1}(i), \end{cases}$$

and so

$$\frac{\partial \lambda_i}{\partial A_{kk}}(A_0) = \begin{cases} 0, & k \neq \theta^{-1}(i), \\ 1, & k = \theta^{-1}(i). \end{cases} \tag{3.11}$$

From (3.9), (3.10) and (3.11), it follows that

$$\frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(A_0) = \begin{cases} 0, & k \neq l, \\ D_{\theta(k)}\sigma_r(\lambda(A_0)), & k = l. \end{cases}$$

The last equality and (2.6) show that (3.8) holds.

Step 2. Suppose A_0 is diagonal. In this case, define $\mathrm{A}(t)$ by

$$\mathrm{A}(t)_{kl} = \begin{cases} 0, & k \neq l, \\ (A_0)_{kk} + \frac{t}{k}, & k = l. \end{cases}$$

For small nonzero t we have:

(i) $\mathrm{A}(t)$ is diagonal with distinct eigenvalues;
(ii) $\lambda(\mathrm{A}(t)) \in \Gamma_r$;
(iii) There exists an unique permutation θ of $\{1, 2, \ldots, n\}$ such that $\lambda_{\theta(j)}(\mathrm{A}(t)) = \mathrm{A}(t)_{jj}$ for $1 \leq j \leq n$.

By Step 1 we have

$$\frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(\mathrm{A}(t)) = \begin{cases} 0, & k \neq l, \\ D_{\theta(k)}\sigma_r(\lambda(\mathrm{A}(t))), & k = l. \end{cases}$$

Since σ_r is of class C^1 and $\lim_{t\to 0} \lambda(\mathrm{A}(t)) = \lambda(A_0)$, we conclude that

$$\frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(A_0) = \begin{cases} 0, & k \neq l, \\ D_{\theta(k)}\sigma_r(\lambda(A_0)) > 0, & k = l, \end{cases}$$

and that (3.8) holds.

Step 3. Suppose that A_0 is symmetric. In this case, there exists an orthogonal matrix P so that $P^t A_0 P$ is diagonal. Observe that $\lambda(P^t A_0 P) = \lambda(A_0) \in \Gamma_r$ and that $(\sigma_r \circ \lambda)(P^t A P) = (\sigma_r \circ \lambda)(A)$ for all matrices A. Setting $C = P^t A P$, we have

$$\begin{aligned} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(A_0) &= \sum_{i,j=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial C_{ij}}(P^t A_0 P)\,\frac{\partial C_{ij}}{\partial A_{kl}}(A_0) \\ &= \sum_{i,j=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial C_{ij}}(P^t A_0 P)\,P^t_{ik} P^t_{jl}. \end{aligned}$$

Thus,

$$\sum_{k,l=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{kl}}(A_0)\xi_k\xi_l = \sum_{i,j,k,l=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial C_{ij}}(P^t A_0 P)P^t_{ik}P^t_{jl}\xi_k\xi_l$$
$$= \sum_{i,j=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial C_{ij}}(P^t A_0 P)w_i w_j,$$

where $w = P^t\xi \neq 0$ for $\xi \neq 0$. Since the right hand side of the above expression is positive by Step 2, we have proved Lemma 3.3. □

We observe that, in Lemma 3.3, we can replace the assumption $\lambda(A_0) \in \Gamma_r$ by the less restrictive assumption that $D_k\sigma_r(\lambda(A_0)) > 0$, $1 \leq k \leq n$. This is an immediate consequence of the proof of Lemma 3.3. We note also that Lemma 3.3 is a reformulation of a result in [2]. We have included a proof here only for the convenience of the reader.

Proposition 3.4. *The functions $\Phi_r\colon \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N} \to \mathbb{R}$, $2 \leq r \leq n$, are elliptic at any point $p^0 = (r^0_{ij}, r^0_i, z^0, x^0_i)$ in the open set $\Omega_r = (\lambda \circ \tilde{A})^{-1}(\Gamma_r)$, such that $F(r^0_i, z^0, x^0_i)$ is the identity. The function Φ_1 is elliptic over $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$.*

Proof. The set Ω_r is open because $\lambda \circ \tilde{A}$ is continuous and Γ_r is open. Assume first that $r \geq 2$. For $k \leq l$, we have

$$\frac{\partial(\sigma_r \circ \lambda \circ \tilde{A})}{\partial r_{kl}}(p^0) = \sum_{m,t=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{mt}}(\tilde{A}(p^0))\frac{\partial \tilde{A}_{mt}}{\partial r_{kl}}(p^0). \tag{3.12}$$

We now compute the numbers $\frac{\partial \tilde{A}_{mt}}{\partial r_{kl}}(p^0)$. By the definition of $\tilde{A}$, we have

$$\tilde{A}_{mt}(r_{ij}, r_i, z, y_i) = \sum_{\ell} F(r_i, z, y_i)_{m\ell} G(r_{ij}, r_i, z, y_i)_{\ell t}.$$

Since $F(r^0_i, z^0, x^0_i)$ is the identity, we obtain that

$$\frac{\partial \tilde{A}_{mt}}{\partial r_{kl}}(p^0) = \sum_{\ell} F(r^0_i, z^0, x^0_i)_{m\ell}\frac{\partial G_{\ell t}}{\partial r_{kl}}(p^0) = \frac{\partial G_{mt}}{\partial r_{kl}}(p^0).$$

It is not hard to verify that

$$\frac{\partial G_{mt}}{\partial r_{kl}}(p^0) = \begin{cases} \omega(r^0_i, z^0, x^0_i), & \text{if}(\delta_{mk}\delta_{tl} + \delta_{ml}\delta_{tk}) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.13}$$

where $\omega(r^0_i, z^0, x^0_i)$ is given by

$$\omega(r^0_i, z^0, x^0_i) = \left\langle d\left(\exp_p\right)_{v(z^0, x^0_i)} \eta_0 \,,\, \eta(r^0_i, z^0, x^0_i)\right\rangle$$

with $v(z^0, x_i^0) = \sum_{i=1}^n x_i^0 e_i + z^0 \eta_0$. Since at any point $(r_i^0, z^0, x_i^0) \in \mathbb{R}^n \times \mathcal{N}$, $\eta(r_i^0, z^0, x_i^0)$ is a unitary vector that is orthogonal to the space spanned by the vectors

$$\psi_\ell(r_i^0, z^0, x_i^0) = d\left(\exp_p\right)_{v(z^0, x_i^0)} (e_\ell + r_\ell^0 \eta_0), \quad 1 \le \ell \le n,$$

and $d\left(\exp_p\right)_{v(z^0, x_i^0)}$ is an linear isomorphism, the function $\omega(r_i^0, z^0, x_i^0)$ does not change sign on $\mathbb{R}^n \times \mathcal{N}$. Since $\omega(0,0,0) = 1$, we conclude that $\omega(r_i^0, z^0, x_i^0)$ is positive in $\mathbb{R}^n \times \mathcal{N}$. Now (3.12) becomes

$$\frac{\partial(\sigma_r \circ \lambda \circ \tilde{A})}{\partial r_{kl}}(p^0) = \begin{cases} \omega(r_i^0, z^0, x_i^0) \left(\frac{\partial(\sigma_r \circ \lambda)}{\partial r_{kl}} + \frac{\partial(\sigma_r \circ \lambda)}{\partial r_{lk}}\right)(\tilde{A}(p^0)), & \text{if } k < l \\ \\ \omega(r_i^0, z^0, x_i^0) \frac{\partial(\sigma_r \circ \lambda)}{\partial r_{kk}}(\tilde{A}(p^0)), & \text{if } k = l. \end{cases}$$

Since $F(r_i^0, z^0, x_i^0)$ is the identity matrix, the matrix $\tilde{A}(p^0) = G(p^0)$ is symmetric. Consequently,

$$\sum_{k \le l = 1}^n \frac{\partial \Phi_r}{\partial r_{kl}}(p^0) \xi_k \xi_l = \frac{\omega(r_i^0, z^0, x_i^0)}{\binom{n}{r}} \sum_{k,l=1}^n \frac{\partial(\sigma_r \circ \lambda)}{\partial r_{kl}}(\tilde{A}(p^0)) \xi_k \xi_l$$

is positive for all nonzero vector $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ by Lemma 3.3. This proves Proposition 3.4 for $r \ge 2$.

If $r = 1$, we have, by (3.5) and (3.7),

$$\Phi_1 = \frac{1}{n} \sum_i \tilde{A}_{ii} = \frac{1}{n} \sum_{i,m} F_{im} G_{mi}$$

at any point in $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$. Using (3.13) and the fact that F does not depend on r_{kl}, it is not difficult to verify that

$$\sum_{k \le l} \frac{\partial \Phi_1}{\partial r_{kl}} \xi_k \xi_l = \frac{\omega}{n} \sum_{k,l} F_{kl} \xi_k \xi_l \quad \text{for all} \quad (\xi_1, \xi_2, \dots, \xi_n).$$

Since F is a definite positive symmetric matrix at any point of $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$, we obtain the ellipticity of Φ_1 over $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$. This completes the proof of Proposition 3.4. □

4. Proofs of the main results

For the proof of Theorem 1.1, we will need the following lemma.

LEMMA 4.1. *If $p \in \Gamma_r$ and $v \in \overline{\mathcal{O}^n}$ then $p + t\,v \in \Gamma_r$ for all $t \ge 0$.*

Proof. If the conclusion does not hold, then there exists $t_0 > 0$ such that $\sigma_r(p + t\,v) > 0$ in $[0, t_0)$ and $\sigma_r(p + t_0\,v) = 0$. This implies that $\frac{d}{dt}\sigma_r(p +$

$t\,v)|_{t=t'} < 0$ for some $t' \in (0, t_0)$. But

$$\frac{d}{d\,t}\sigma_r(p + t\,v)|_{t=t'} = \sum_{i=1}^{n} D_i\sigma_r(p + t'\,v)\,v_i \geq 0$$

by (2.6). Thus we have obtained a contradiction. □

Proof of Theorem 1.1. Restricting W if necessary, our assumptions and (3.6) imply

$$\Phi_r(\Lambda(\mu^2)(x)) = H_r^2(x) \geq H_r^1(x) = \Phi_r(\Lambda(\mu^1)(x)), \quad x \in W.$$

In order to apply the Maximum Principle of Section 2 and conclude that μ^1 coincides with μ^2 in a neighborhood of zero, we will prove that, by restricting W if necessary, the function Φ_r is elliptic with respect to the functions $(1-t)\mu^2 + t\mu^1$, $t \in [0, 1]$. To this end, observe first that if $\mu\colon\ W \to \mathbb{R}$ is a function satisfying $\mu(0) = 0$ and $\mu_i(0) = 0$ for $1 \leq i \leq n$, then $F(\mu_i(0), \mu(0), 0) = F(0, 0, 0)$ is the identity matrix and, consequently,

$$\begin{aligned}
\tilde{A}(\Lambda(\mu)(0))_{kl} &= \tilde{A}(\mu_{ij}(0), 0, 0, 0)_{kl} = G(\mu_{ij}(0), 0, 0, 0)_{kl}\\
&= \left\langle \nabla_{e_k} d(\exp_p)_v e_l|_{x=0}\,,\, \eta_0 \right\rangle + \mu_{kl}(0)\\
&= \left\langle \frac{D}{d\,t} d(\exp_p)_{v(te_k)} e_l|_{t=0}\,,\, \eta_0 \right\rangle + \mu_{kl}(0)\\
&= \left\langle \frac{D}{d\,t}\frac{D}{d\,s} \exp_p(v(te_k) + s\,e_l)|_{t=0,\, s=0}\,,\, \eta_0 \right\rangle + \mu_{kl}(0)\\
&= \left\langle \frac{D}{d\,s} d(\exp_p)_{s\,e_l} e_k|_{s=0}\,,\, \eta_0 \right\rangle + \mu_{kl}(0)
\end{aligned}$$

by (3.3). Therefore,

$$\begin{aligned}
\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0))_{kl} &= \tilde{A}((1-t)\,\mu_{ij}^2(0) + t\,\mu_{ij}^1(0), 0, 0, 0)_{kl}\\
&= \left\langle \frac{D}{d\,s} d(\exp_p)_{s\,e_l} e_k|_{s=0}\,,\, \eta_0 \right\rangle + (1-t)\,\mu_{kl}^2(0) + t\,\mu_{kl}^1(0)\\
&= \tilde{A}(\Lambda(\mu^2)(0))_{kl} + t\,(\mu_{kl}^1(0) - \mu_{kl}^2(0));
\end{aligned}$$

that is,

$$\begin{aligned}
&\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0)) - \tilde{A}(\Lambda(\mu^2)(0))\\
&\qquad = t\,[(\operatorname{Hess}\mu^1)(0) - (\operatorname{Hess}\mu^2)(0)].
\end{aligned}$$

Since $\mu^1 \geq \mu^2$ in a neighborhood of zero, $\mu^1(0) = 0 = \mu^2(0)$ and $\mu_i^j(0) = 0$, for $1 \leq i \leq n$, $j = 1, 2$, we have $(\operatorname{Hess}\mu^1)(0) - (\operatorname{Hess}\mu^2)(0) \geq 0$ in the sense that

$$\sum_{k,l=1}^{n} (\operatorname{Hess}\mu^1 - \operatorname{Hess}\mu^2)_{kl}(0)\xi_k\xi_l \geq 0 \ \text{ for all } \ (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n.$$

We deduce

$$\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0)) - \tilde{A}(\Lambda(\mu^2)(0)) \geq 0, \ \ t \in [0,1].$$

Hence (see [8], p. 130), for $1 \leq i \leq n$ we have

$$\lambda_i(\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0))) - \lambda_i(\tilde{A}(\Lambda(\mu^2)(0))) \geq 0, \ \ t \in [0,1],$$

and thus

$$\lambda(\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0))) - \lambda(\tilde{A}(\Lambda(\mu^2)(0))) \in \overline{\mathcal{O}^n}, \ \ 0 \leq t \leq 1,$$

where $\overline{\mathcal{O}^n}$ is the closure of $\mathcal{O}^n$. Thus, by Lemma 4.1, $\lambda(\tilde{A}((1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0)))$ belongs to Γ_r, $0 \leq t \leq 1$. Proposition 3.4 then shows that Φ_r is elliptic at the points given by $(1-t)\,\Lambda(\mu^2)(0) + t\,\Lambda(\mu^1)(0)$, $t \in [0,1]$. Since ellipticity is an open condition and Ω_r is open, restricting W if necessary, we conclude by continuity and by the compactness of $[0,1]$ that Φ_r is elliptic at the points $(1-t)\,\Lambda(\mu^2)(x) + t\,\Lambda(\mu^1)(x)$, $x \in W$, $t \in [0,1]$. This means that Φ_r is elliptic with respect to the functions $(1-t)\,\mu^2 + t\,\mu^1$, $t \in [0,1]$. The Maximum Principle now enables us to conclude that μ_1 and μ_2 coincide in a neighborhood of zero. This proves Theorem 1.1. □

For the remaining proofs we will make use of the fact that the functions $\mu_c(t)/t$ are monotone decreasing on $t > 0$.

Proof of Theorem 1.3. Let $\partial\overline{B_{\rho'}(p_0)}$ be the largest sphere that fits inside M^n. Suppose that $\rho' > \rho$. Then, $\mu_c(\rho')/\rho' < \mu_c(\rho)/\rho$ and thus

$$|H_r| \geq \left[\frac{\mu_c(\rho)}{\rho}\right]^r > \left[\frac{\mu_c(\rho')}{\rho'}\right]^r \ \ \text{on } M^n. \tag{4.1}$$

Since M^n is compact and embedded, we can orient M^n by the normals pointing inward and find a point $q \in M^n$ where all principal curvatures are positive; that is, the principal curvature vector of M^n at q belongs to the positive cone $\mathcal{O}^n \subset \Gamma_r$. Let $\lambda\colon\ M^n \to \mathbb{R}^n$ be the continuous function that associates to each point in M^n its principal curvature vector with the choices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Since, by assumption, H_r does not change sign on M^n, and $H_r(q) > 0$, we deduce that $H_r > 0$ on M^n. Hence, $\lambda(M^n)$ is a connected compact set in $\mathbb{R}^n$, contained in the connected component of $\{\,\sigma_r > 0\,\}$ that contains $\mathcal{O}^n$, and therefore $\lambda(M^n) \subset \Gamma_r$. Observe now that M^n and $\partial\overline{B_{\rho'}(p_0)}$ are tangent at p. We can apply Theorem 1.1 and conclude that M^n and $\partial\overline{B_{\rho'}(p_0)}$ coincide in a neighborhood of p, since $[\mu_c(\rho')/\rho']^r$ is precisely the constant value of the r-mean curvature of $\partial\overline{B_{\rho'}(p_0)}$, oriented by the normals pointing inward, at any point. But this contradicts (4.1). Therefore, $\rho' \leq \rho$. If equality holds here, then Theorem 1.1 applies again and shows that M^n and $\partial\overline{B_\rho(p_0)}$ coincide in a neighborhood of points of tangency, and a standard argument using the connectedness ensures that these hypersurfaces are identical. □

Proof of Theorem 1.4. If $\rho \leq \tau$, then there is nothing to prove. Suppose that $\rho > \tau$ and $M^n \not\subset \overline{B_\tau(p_0)}$. In this case, if p is a point in M^n farthest from p_0, then $p \in M - \partial M$, $\rho' = d(p_0, p) > \tau$ and $\overline{B_{\rho'}(p_0)}$ is the smallest ball centered at p_0 enclosing M^n. Here $d(p_0, .)$ stands for the distance function from p_0 in the space form Q_c^{n+1}. The farthest point p from p_0, since it is an interior point of M^n, is a point where M^n and $\partial\overline{B_{\rho'}(p_0)}$ are tangent. Orient M^n at p with the unitary normal vector η_0 pointing inward to $\partial\overline{B_{\rho'}(p_0)}$. Since $\rho' \leq \rho$ and $\mu_c(t)/t$ is positive and monotone decreasing on t, we have over M^n

$$\left[\frac{\mu_c(\rho')}{\rho'}\right]^r \geq \left[\frac{\mu_c(\rho)}{\rho}\right]^r \geq |H_r| \geq H_r.$$

Since $[\mu_c(\rho')/\rho']^r$ is the constant value of the r-mean curvature of $\partial\overline{B_{\rho'}(p_0)}$, oriented by the normals pointing inward, we can apply Theorem 1.1 and conclude that M^n coincides with $\partial\overline{B_{\rho'}(p_0)}$ in a neighborhood of p. Arguing via connectedness, we obtain that $M - \partial M$ is contained in $\partial\overline{B_{\rho'}(p_0)}$. But ∂M is also contained in $\partial\overline{B_{\rho'}(p_0)}$, contradicting the relation $\partial M \subset \overline{B_\tau(p_0)}$ and $\tau < \rho'$. □

Proof of Theorem 1.5. Consider the function $g = \frac{1}{2} d_{p_0}(\,.\,)^2$, where $d_{p_0}(\,.\,)$ stands for the distance function from p_0 on N^{n+1}. Note that the function g is differentiable in a neighborhood of $\overline{B_\rho(p_0)}$. Let $\varphi\colon\ M \to \mathbb{R}$ be given by $\varphi = g \circ F$. The function φ is differentiable since $F(M)$ is contained in the closed normal ball $\overline{B_\rho(p_0)}$. We now show that $\overline{B_\rho(p_0)}$ is the smallest ball centered at p_0 that contains $F(M)$. If this is not the case, there exists a closed ball $\overline{B_{\rho'}(p_0)}$ with $\rho' < \rho$ that contains $F(M)$. Let $p \in M^n$ be a point such that $d_{p_0}(F(p)) = \rho'$. It is well known that if η is the unitary vector that is normal to M^n at p, pointing inward to $\partial\overline{B_{\rho'}(p_0)}$, then

$$\eta = -\frac{\operatorname{grad} g_{F(p)}}{\left|\operatorname{grad} g_{F(p)}\right|}$$

with $\left|\operatorname{grad} g_{F(p)}\right| = d_{p_0}(F(p)) = \rho'$. Here $\operatorname{grad} g_{F(p)}$ is the value at $F(p)$ of the gradient of g in N^{n+1}. It follows from Lemma 2.5 in [10] and the fact that, for fixed t, $\mu_c(t)$ is monotone decreasing in c, that the Hessian of φ in p satisfies

$$\operatorname{Hess}\varphi_p(X, X) \geq \mu_c(d_{p_0}(F(p)))\langle X, X\rangle + \langle \operatorname{grad} g_{F(p)}, \alpha(X, X)\rangle$$

for all $X \in T_pM$, where α is the second fundamental form of F at p. Consider now an arbitrary principal curvature λ_i of A_η with unitary principal direction e_i. Since φ attains a maximum at p, we deduce that

$$0 \geq \operatorname{Hess}\varphi_p(e_i, e_i) \geq \mu_c(\rho') - \rho'\lambda_i,$$

that is, $\lambda_i \geq \mu_c(\rho')/\rho'$. Consequently, we have

$$H_r(p) \geq \left[\frac{\mu_c(\rho')}{\rho'}\right]^r > \left[\frac{\mu_c(\rho)}{\rho}\right]^r,$$

which contradicts the hypothesis. Therefore, $\rho' = \rho$ and $\overline{B_\rho(p_0)}$ is the smallest ball centered at p_0 that contains $F(M)$. Observe that if we consider the constant function defined as the restriction of g to $\partial\overline{B_\rho(p_0)}$, then proceeding as above we deduce that for $\partial\overline{B_\rho(p_0)}$, oriented by the normals pointing inward, at any point all principal curvatures are greater than or equal to $\mu_c(\rho)/\rho$. This implies that at any point the principal curvature vector of $\partial\overline{B_\rho(p_0)}$ belongs to $\mathcal{O}^n$ and that the r-mean curvature H_r' of $\partial\overline{B_\rho(p_0)}$ satisfies

$$H_r' \geq \left[\frac{\mu_c(\rho)}{\rho}\right]^r \geq H_r.$$

By Theorem 1.1, this implies that $F(M)$ and $\partial\overline{B_\rho(p_0)}$ coincide in a neighborhood of $F(p)$. Arguing via connectedness, we conclude that $F(M)$ is the boundary of $\overline{B_\rho(p_0)}$. Since now M^n has all principal curvatures greater than or equal to $\mu_c(\rho)/\rho$ and, by assumption, $|H_r| \leq [\mu_c(\rho)/\rho]^r$, it follows that all principal curvatures are equal to $\mu_c(\rho)/\rho$. In particular, if H is the mean curvature vector function on M^n then $|H| = \mu_c(\rho)/\rho$. Theorem 1.5 now follows from Proposition 3.4 in [13]. □

Remark 4.2. It is clear from the proofs of our results that when r is even we can assume the less restrictive hypothesis $H_r \leq [\mu_c(\rho)/\rho]^r$ in Theorems 1.4 and 1.5.

Remark 4.3. It follows from Theorem 1.1 that, in any ambient space, if a hypersurface remains on one side of another hypersurface in a neighborhood of a tangency point and both hypersurfaces have the same constant mean curvature, then they coincide in a neighborhood of such a point.

Remark 4.4. In [9], J. Hounie and M.L. Leite have obtained tangency principles for hypersurfaces in Euclidean space satisfying $H_r = 0$. The proofs of their tangency principles are based on the fact that such hypersurfaces satisfy a nonlinear equation $G_r(\operatorname{Hess}\mu, \operatorname{grad}\mu) = 0$ and on algebraic results. In any Riemannian manifold, if we have a hypersurface with $H_r = 0$ then, as we have seen above, the nonlinear equation $\Phi_r(\Lambda(\mu)(x)) = 0$ is also satisfied. This fact permits us to extend their tangency principles, stated as Theorem 0.1 and Theorem 0.2, to hypersurfaces in any Riemannian manifold. The proofs are identical.

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