

AN ERROR ESTIMATE FOR THE ISOPERIMETRIC DEFICIT

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To Gerald Klain on his 70th birthday

ABSTRACT. A four part dissection and rearrangement provides a new proof of the isoperimetric inequality in the plane as well as a new approach to Bonnesen-type error estimates for the isoperimetric deficit of compact convex sets and of star bodies that are centrally symmetric with respect to the origin.

An *isoperimetric inequality* in \mathbb{R}^2 bounds the area (or related functional) of a compact set by some function of the perimeter (or related functional) of that same set.

The *classical isoperimetric inequality* asserts that, for any compact subset $S \subseteq \mathbb{R}^2$ having well-defined perimeter,

$$(1) \quad P(S)^2 \geq 4\pi A(S),$$

where $A(S)$ and $P(S)$ respectively denote the area and perimeter of the set S . Equality holds in (1) if and only if S is a disk. In other words, the largest planar region one can enclose with a string of length ℓ is the disk of radius $\ell/2\pi$. Proofs of (1) are as common as dandelions. See, for example, any of [2], [4], [5], [6], [10], [17], [19], [20]. A proof of (1) using cyclic rearrangement is described in Section 3 of this article.

In Section 1 we introduce *cyclic rearrangement*, a four part dissection and rearrangement that preserves much of the boundary structure of a planar star body (including its perimeter), while potentially increasing the area of the region enclosed by the boundary (itself a simple closed curve).

A *Bonnesen-type inequality* is an isoperimetric inequality together with an error estimate, usually involving radii or widths of the original set, or of sets that inscribe and circumscribe the original set. Bonnesen-type inequalities are surveyed in [15] and are also treated in [16], [17]. In Section 2 we give a new method of proof for some of these error estimates. Although these methods

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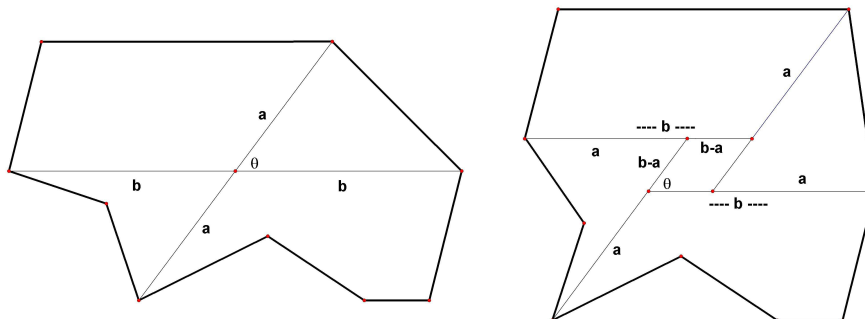


FIGURE 1. K and its cyclic rearrangement $\boxtimes(K)$.

do not give the strongest estimates possible (due to Fuglede [3]) the resulting error bounds do improve on the classical result of Bonnesen in a number of instances, while simplifying the proof substantially. Moreover, the method of rearrangement uses only a finite number of dissections, translations, and reflection, so that the methods of this article may be generalized to inequalities for other geometric functionals besides area and perimeter (certain invariant valuations [9], for example) provided those functionals are also invariant under such elementary geometric transformations.

The four part dissection and rearrangement technique for star-shaped sets developed in Section 1, and applied in the subsequent sections, was inspired in part by a much simpler problem (and solution) posed by Steinhaus [18, p. 87].

1. Cyclic rearrangement

A non-empty set K said to be *star-shaped* with respect to the origin o if, for all $x \in K$, the line segment \overline{ox} lies entirely inside K . Let \mathbb{S}^1 denote the unit circle in \mathbb{R}^2 centered at the origin. The *radial function* $\rho_K : \mathbb{S}^1 \rightarrow [0, \infty]$ of a star-shaped set K is a non-negative function on the unit circle, defined by

$$\rho_K(u) = \max\{a \in \mathbb{R} \mid au \in K\}.$$

In other words, $\rho_K(u)$ is the radius of the set K in the direction of the unit vector u .

A star-shaped set K is called a *star body* if ρ_K is a continuous function and K has non-empty interior; that is, $\rho_K > 0$. Note that a star body is always compact.

Let $\{u, v\}$ form a basis of unit vectors for \mathbb{R}^2 , and let θ denote the angle between u and v . Without loss of generality in what follows, we will assume $0 \leq \theta \leq \pi/2$.

Suppose that K is a star body such that K has equal radii in the directions $\pm u$ and also equal radii in the directions $\pm v$. Let $b = \rho_K(\pm u)$ and $a = \rho_K(\pm v)$ denote the radii of K in the direction of $\pm u$ and $\pm v$, as shown in Figure 1. We will define the *cyclic rearrangement* of K with respect to the basis $\{u, v\}$, denoted $\boxtimes(K)_{u,v}$, or simply $\boxtimes(K)$, to be the set formed by the four part dissection and rearrangement shown in Figure 1.

Note: We do not perform cyclic rearrangement of K with respect to $\{u, v\}$ if $\rho_K(u) \neq \rho_K(-u)$ or if $\rho_K(v) \neq \rho_K(-v)$.

A more careful definition of $\boxtimes(K)$ runs as follows:

Let Q_1, \dots, Q_4 denote the four quadrants of \mathbb{R}^2 induced by the basis $\{u, v\}$, listed counterclockwise starting with the positive quadrant. Let ψ denote reflection across the line passing through the origin and $u + v$. For each i , let $K_i = K \cap Q_i$, and let

$$\begin{aligned} \hat{K}_1 &= \psi K_1 + \frac{b-a}{2}(u-v) \\ \hat{K}_2 &= K_2 + \frac{b-a}{2}(u+v) \\ \hat{K}_3 &= \psi K_3 + \frac{b-a}{2}(-u+v) \\ \hat{K}_4 &= K_4 + \frac{b-a}{2}(-u-v) \end{aligned}$$

Let M be the rhombus, centered at the origin, with sides of length $b - a$ and parallel to u and v . Finally, define $\boxtimes(K)$ to be the union

$$\boxtimes(K) = \hat{K}_1 \cup \hat{K}_2 \cup \hat{K}_3 \cup \hat{K}_4 \cup M.$$

Note that $\boxtimes(K)$ is not necessarily a star-shaped set. However, $\boxtimes(K)$ is a compact simply-connected set, such that

$$P(\boxtimes(K)) = P(K) \quad \text{and} \quad A(\boxtimes(K)) = A(K) + (b-a)^2 |\sin \theta|,$$

where the term $(b-a)^2 |\sin \theta|$ is the area of the central rhombus M inside $\boxtimes(K)$. In other words, $\boxtimes(K)$ has the same perimeter as does K , while having equal or larger area. If $A(\boxtimes(K)) = A(K)$, then $a = b$; that is, K has the same radii in each of the directions u and v .

Evidently if D is a disk centered at the origin then $\boxtimes(D) = D$. While the converse of this assertion is false (see Figure 3), the following propositions describe all possible pre-images $\boxtimes^{-1}(D)$ of a disk D under certain conditions. These characterizations of \boxtimes^{-1} will be helpful in determining the equality cases for the geometric inequalities that follow.

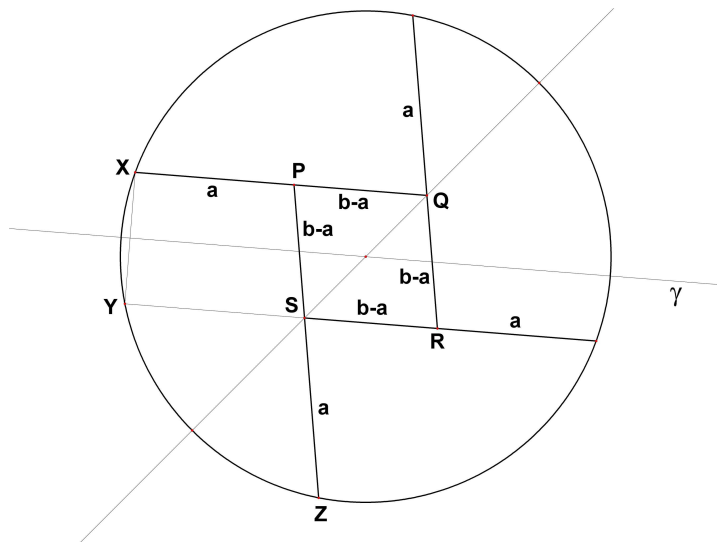


FIGURE 2. Concentric disk and rhombus (*impossible!*)

PROPOSITION 1. *Let K be a star body containing the origin in its interior such that $\rho_K(\pm u) = b$ and $\rho_K(\pm v) = a$. Suppose that $\boxtimes(K) = D$, where D is a disk. Then D is centered at the origin, and the rhombus M associated with this 4-part decomposition is a square concentric with D .*

Proof. Since $\boxtimes(K) = D$, the disk D is decomposed into 4 parts \hat{K}_i of the original set K , along with a rhombus M that is concentric with D . The definition of the rearrangement \boxtimes implies that each side of the rhombus M can be continued the same distance a to the boundary of the disk D . This implies, in turn, that D and M are concentric at the origin.

In Figure 2, the rhombus M is represented by $\square PQRS$. This rhombus purposely appears non-square in the figure so as not to assume the proposition beforehand—it will be seen that Figure 2 is in fact impossible as labelled (when we prove that M must be a square). The fact that M is square will follow from the assumptions about the lengths a and the concentricity of M and D . (An accurately illustrated cyclic decomposition of the disk appears in Figure 3.)

Since $M = \square PQRS$ arises from a 4-part decomposition, we know that $|PX| = |SZ| = a$. Since M and D are concentric and both symmetric under reflection across the line through the segment \overline{SQ} , it follows that $|SY| = |SZ| = a$ as well. Since the segments \overline{SY} and \overline{PX} are parallel (being extensions of the edges of the rhombus M), it now follows that $\square XPSY$ is a parallelogram (contrary to appearances in Figure 2).

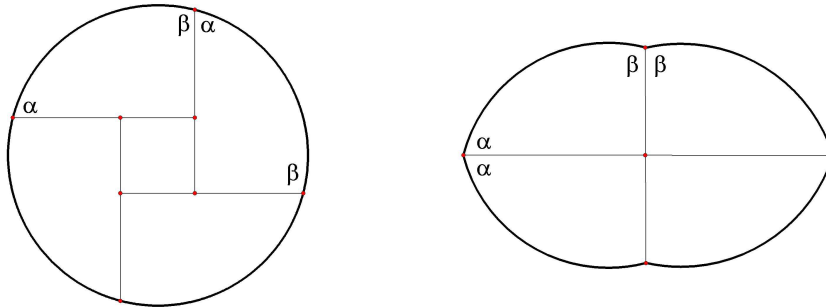


FIGURE 3. 4-part decomposition of a disk D and its pre-image $\boxtimes^{-1}(D)$

The line γ that bisects both \overline{PS} and \overline{QR} is a diameter of D , and is parallel to \overline{PX} and \overline{SY} . Therefore, the reflection ψ_γ across γ exchanges Y and X as well as the lines through \overline{SY} and \overline{PX} . Since $|PX| = |SY| = a$, it follows that ψ_γ exchanges the points P and S as well, so that the parallelogram $\square XPSY$ is symmetric under the reflection ψ_γ . It follows that $\square XPSY$ is a *rectangle*, so that the rhombus $M = \square PQRS$ must be a *square*. \square

Proposition 1 has the following immediate consequence.

PROPOSITION 2. *Let K be a star body containing the origin in its interior such that $\rho_K(u) = \rho_K(-u)$ and $\rho_K(v) = \rho_K(-v)$. Suppose also that $\boxtimes(K) = D$, where D is a disk centered at the origin.*

If K is not a disk then $u \perp v$.

When the original body K is *convex*, the situation is even more straightforward.

PROPOSITION 3. *Suppose that K is a convex body containing the origin in its interior such that $\rho_K(\pm u) = b$ and $\rho_K(\pm v) = a$. If $\boxtimes(K)$ is a disk then K must also be a disk centered at the origin.*

Proof. Since $\boxtimes(K)$ is a disk D , Proposition 1 implies that the associated rhombus M is a square concentric with D , so that the remaining parts \hat{K}_i are all congruent under 90° rotations. Subdivide $D = \boxtimes(K)$ into the square M and the congruent parts \hat{K}_i as defined above. Recall that the sets \hat{K}_i can be reflected and translated to reconstitute K , as in Figure 3.

Let α, β denote the two interior angles of each sector \hat{K}_i where its straight boundary edges meet the circular portion of the boundary of \hat{K}_i . See Figure 3. Since $\boxtimes(K)$ is a disk, we have $\alpha + \beta = \pi$. In order for K to be convex, we must also have $\alpha + \alpha \leq \pi$ and $\beta + \beta \leq \pi$. These identities and inequalities

imply that $\alpha = \beta = \pi/2$. But this implies that the square M is a singleton, $M = \{o\}$, so that $a = b$ and $\boxtimes(K) = K$. In other words, K is also a disk. \square

If $\boxtimes(K)$ is a disk when the original star body K is *not* convex, then K is still formed by gluing four congruent convex portions \hat{K}_i of the disk along their boundaries, so that the boundary ∂K of K is a locally convex curve at every point except possibly where the radial lines from the origin through $\pm u$ and $\pm v$ meet ∂K . In other words, there are at most these *four* points on the boundary at which ∂K fails to be locally convex. For example, in Figure 3 the star body $K = \boxtimes^{-1}(D)$ has exactly two points on its boundary where locally convexity fails to hold.

PROPOSITION 4. *Suppose that K is a star body containing the origin in its interior. Suppose that $\{u, v\}$ and $\{u', v'\}$ are two unit vector bases for the plane such that $\{\pm u, \pm v\} \cap \{\pm u', \pm v'\} = \emptyset$.*

If $\boxtimes_{u,v}(K)$ is a disk and $\boxtimes_{u',v'}(K)$ is a disk, then K must be a disk centered at the origin.

Proof. As noted above, the boundary ∂K of K is a locally convex curve except possibly in the radial directions $\pm u$ or $\pm v$, since $\boxtimes_{u,v}(K)$ is a disk.

Since $\{\pm u, \pm v\} \cap \{\pm u', \pm v'\} = \emptyset$ and $\boxtimes_{u',v'}(K)$ is also a disk, the curve ∂K must be locally convex in the radial directions $\pm u$ or $\pm v$ as well. In other words, ∂K is a convex curve, and K is a convex body such that $\boxtimes(K)$ is a disk. It follows from Proposition 3 that K is a disk. \square

2. Cyclic rearrangement inequalities for star bodies

Cyclic rearrangement will now be used to prove a family of Bonnesen-type isoperimetric inequalities for star-shaped sets.

THEOREM 1. *Let K be a star body in \mathbb{R}^2 , and let u, v be a pair of independent unit vectors in \mathbb{R}^2 . Let θ denote the angle between u and v .*

Suppose that $\rho_K(u) = \rho_K(-u) = b$ and $\rho_K(v) = \rho_K(-v) = a$. Then

$$(2) \quad P(K)^2 - 4\pi A(K) \geq 4\pi(b - a)^2 |\sin \theta|.$$

If K is a convex body, then equality holds in (2) if and only if K is a disk.

If K is a star body and if equality holds in (2) with respect to any non-orthogonal basis $\{u, v\}$, or with respect to two or more disjoint orthogonal frames $\{\pm u_i, \pm v_i\}$ such that $\rho_K(u_i) = \rho_K(-u_i)$ and $\rho_K(v_i) = \rho_K(-v_i)$, then K must be a disk.

Expressed in the terms of the radial function ρ_K , the inequality (2) becomes

$$P(K)^2 - 4\pi A(K) \geq 4\pi(\rho_K(u) - \rho_K(v))^2 |u \times v|,$$

where $|u \times v|$ is the vector cross product of the unit vectors u and v .

Proof. Since $\rho_K(u) = \rho_K(-u)$ and $\rho_K(v) = \rho_K(-v)$, we can perform a cyclic rearrangement of K with respect to the basis $\{u, v\}$, as in Figure 1. Let $K' = \boxtimes_{u,v}(K)$. Then $P(K') = P(K)$ and

$$A(K') = A(K) + (b - a)^2 |\sin \theta|.$$

Hence,

$$P(K)^2 - 4\pi A(K) = P(K')^2 - 4\pi A(K') + 4\pi(b - a)^2 |\sin \theta|.$$

By the isoperimetric inequality (1), $P(K')^2 - 4\pi A(K') \geq 0$, so that (2) now follows.

If K is convex and equality holds in (2), then $P(K')^2 - 4\pi A(K') = 0$. From the equality conditions of the isoperimetric inequality (1) it follows that $K' = \boxtimes(K)$ is a disk. It then follows from Proposition 3 that K is a disk.

If K is a star body and if equality holds in (2) with respect to any non-orthogonal basis $\{u, v\}$, then $P(K')^2 - 4\pi A(K') = 0$ and $K' = \boxtimes(K)$ is a disk once again. Since u and v are not orthogonal, K must also be a disk by Proposition 2.

If K is a star body and if equality holds in (2) with respect to two or more disjoint bases $\{\pm u, \pm v\}$, then $K'_{u,v} = \boxtimes_{u,v}(K)$ is a disk with respect to two or more disjoint bases $\{\pm u, \pm v\}$, so that K is a disk, by Proposition 4. \square

Note that if K is merely star-shaped, and not convex, it is possible for equality to hold in (2) when K is not a disk, as in Figure 3.

A star body will be called *symmetric* if K is origin-symmetric; that is, $K = -K$. This occurs if and only if the radial function ρ_K is an even function on the circle \mathbb{S}^1 .

The following theorem is an immediate consequence of Theorem 1.

THEOREM 2 (Cyclic rearrangement inequality for symmetric star bodies). *Let K be a symmetric star body in \mathbb{R}^2 . Let b, a respectively denote the radii of K in the directions of two independent unit vectors u, v in \mathbb{R}^2 . Let θ denote the angle between u and v . Then*

$$(3) \quad P(K)^2 - 4\pi A(K) \geq 4\pi(b - a)^2 |\sin \theta|.$$

If K is a convex body, then equality holds in (3) if and only if K is a disk.

If K is not a convex body, then the inequality (3) is strict whenever u and v are not orthogonal.

If K is a symmetric star body then equality holds in (3) with respect to two or more disjoint orthogonal frames $\{\pm u_i, \pm v_i\}$ if and only if K is a disk.

Theorem 2 no longer applies when the central symmetry condition is omitted. However, the following variation of Theorem 2 holds for non-symmetric compact *convex* bodies.

THEOREM 3. *Let K be a convex body in \mathbb{R}^2 , and let u, v be a pair of independent unit vectors in \mathbb{R}^2 . Let ℓ denote the line parallel to u that cuts K into two parts having equal areas.*

Let b denote the radius of K along ℓ measuring from the midpoint of $K \cap \ell$; that is, let $b = (1/2) \text{length}(K \cap \ell)$. Let a denote the radius of K in the direction parallel to v that lies on the side of ℓ where K has lesser perimeter, measuring again from the midpoint of $K \cap \ell$. Finally, let θ denote the angle between u and v . Then

$$(4) \quad P(K)^2 - 4\pi A(K) \geq 4\pi(b - a)^2 |\sin \theta|.$$

Proof. The hypotheses of the theorem assert that $K = K_1 \cup K_2$, where $K_1 \cap K_2 = K \cap \ell$ and $A(K_1) = A(K_2)$, while $P(K_1) \leq P(K_2)$. Let Z denote the rotation of K_1 by the angle π around the midpoint of $K \cap \ell$, and let $K' = K_1 \cup Z$.

Although the set K' may not be convex, it is star-shaped and symmetric with respect to the midpoint of $K \cap \ell$. Moreover, $A(K') = A(K)$ and $P(K') \leq P(K)$, while $b = \rho_{K'}(u)$ and $a = \rho_{K'}(v)$. It follows that

$$P(K)^2 - 4\pi A(K) \geq P(K')^2 - 4\pi A(K') \geq 4\pi(b - a)^2 |\sin \theta|,$$

where the second inequality follows from Theorem 2. □

Note that equality may hold in Theorem 3 in some instances where the body K is not a disk.

Theorems 1, 2, and 3 do not give the strongest results possible in this context. In [1] (see also [15], [17]) Bonnesen showed that if K is a compact convex set in the plane then

$$P(K)^2 - 4\pi A(K) \geq 4\pi d^2,$$

where d is the minimal width over all annuli containing the boundary ∂K . Using a different approach, Fuglede [3] extended this inequality to any compact planar region having a simple closed curve as its boundary.

More comparisons to Bonnesen's original inequalities are treated in the last section of this article.

3. Using cyclic rearrangement to prove the isoperimetric inequality

We assumed the isoperimetric inequality (1) in the proof of Theorems 1, 2, and 3. However, it is also possible to use cyclic rearrangement to prove the isoperimetric inequality (1) directly.

Independent Proof of Inequality (1). Let K be a symmetric star body in \mathbb{R}^2 .

If $S \subseteq \mathbb{R}^2$, denote by $c(S)$ the *convex hull* of S . Let Λ_K denote the collection of all finite iterates of the composed operator

$$\zeta_{u,v}(S) = c(\boxtimes_{u,v}(S))$$

on the symmetric star body K with respect to all bases $\{u, v\}$ of \mathbb{R}^2 . Note that all such iterates of ζ are symmetric convex sets. Moreover, $A(\zeta_{u,v}(S)) = A(S)$ for all bases $\{u, v\}$ if and only if S is a disk; that is, if and only if the radial function of S is constant.

Neither the convex hull operation nor the \boxtimes operator increase perimeter. Hence, if $L \in \Lambda_K$, then $P(L) \leq P(K)$. It follows that the diameter of L is bounded, since

$$\text{diameter}(L) \leq \frac{P(L)}{2} \leq P(K).$$

In other words, the diameter is uniformly bounded above by the value $P(K)$ over the collection Λ_K . This implies that the collection Λ_K has a compact closure $\bar{\Lambda}_K$ with respect to the Hausdorff metric on compact sets, so that the continuous function $A(L)$ must attain a maximum at some compact convex set $\tilde{K} \in \bar{\Lambda}_K$. (This topological assertion follows from the Blaschke selection theorem; see any of [2], [17], [20].)

Since area is maximized over the closure of Λ_K at \tilde{K} , it follows that $A(\zeta(\tilde{K})) = A(\tilde{K})$ with respect to any frame $\{u, v\}$, so that \tilde{K} is a disk. Moreover, since ζ does not decrease area, $A(K) \leq A(\tilde{K})$, with equality iff $K = \tilde{K}$. For if K has radii $a < b$ with respect to any pair of directions u, v differing by an acute angle $\theta > 0$, then

$$A(\tilde{K}) \geq A(\zeta(K)) = A(K) + (b - a)^2 |\sin \theta| > A(K),$$

where the symmetrization ζ is performed with respect to u and v .

Because \tilde{K} is a disk, we have $P(\tilde{K})^2 = 4\pi A(\tilde{K})$. Since the operator ζ can never increase perimeter, it follows that

$$(5) \quad P(K)^2 \geq P(\tilde{K})^2 = 4\pi A(\tilde{K}) \geq 4\pi A(K) + 4\pi(b - a)^2 |\sin \theta|.$$

It follows immediately from (5) that if K is a symmetric star body, centered at the origin, then $P(K)^2 - 4\pi A(K) \geq 0$. Moreover, if $P(K)^2 - 4\pi A(K) = 0$ then $\rho_K(u) = \rho_K(v)$ for all pairs u, v , so that K must be a disk. This verifies (1) for symmetric star bodies.

To verify (1) for more general sets, recall that the convex hull of a compact set has at once greater (or equal) area and smaller (or equal) perimeter. It is therefore sufficient to demonstrate (1) for compact *convex* sets.

If K is a compact convex set, choose a line ℓ that divides K into two parts K_1 and K_2 of equal area, and suppose $P(K_1) \leq P(K_2)$. Let x denote the midpoint of $K \cap \ell$, and let K' denote the union of K_1 with the 180° rotation of K_1 around the point x . Then K' is centrally symmetric and star-shaped

with respect to the point x , so that

$$P(K)^2 - 4\pi A(K) \geq P(K')^2 - 4\pi A(K') \geq 0$$

once again.

If $P(K)^2 - 4\pi A(K) = 0$, then $P(K')^2 - 4\pi A(K') = 0$ as well, so that K' is a disk. Therefore, the K_1 portion of K is a half-disk, for *all* choices of orientation for the line ℓ . Since K is convex, these variously oriented half-disks must be concentric with the same radius, so that K itself must be a disk.

This verifies the isoperimetric inequality (1) for compact convex sets, and thereby for all compact sets having well-defined perimeter. \square

4. Comparisons to Bonnesen's inequality

The classical inequality of Bonnesen runs as follows [6], [15], [16], [17]:

Suppose that K is a compact convex set in \mathbb{R}^2 . Let R denote the circumradius of K , and let r denote the inradius of K .

Then

$$(6) \quad P(K)^2 - 4\pi A(K) \geq \pi^2(R - r)^2.$$

This assertion was later strengthened to apply to any region K of the plane bounded by a Jordan curve [15]. Here the *inradius* r of K is defined to be the maximal radius over all disks contained inside K , while the *circumradius* R of K is the minimal radius taken over all disks containing K .

Bonnesen was also able to strengthen (6) by replacing the constant π^2 with 4π , but only for the case where K is *convex* and the inradius and circumradius are realized by concentric circles [1], [15], [17]. This result was later extended by Fuglede [3] to any planar region enclosed by a simple closed curve. In other words, if K is a planar region enclosed by a simple closed curve, then

$$(7) \quad P(K)^2 - 4\pi A(K) \geq 4\pi d^2,$$

where d is the minimal width over all annuli containing the boundary ∂K .

For non-convex star bodies, the inequalities of Theorems 1, 2, and 3 are intermediate in strength between Bonnesen's original inequality (6) and Fuglede's generalization (7).

One important instance for which Theorem 2 agrees with the strongest Bonnesen (and Fuglede) bound is the set of ellipses. If E is an ellipse, then the inradius r and circumradius R are attained in orthogonal directions, so that $\theta = \pi/2$. In this instance, Theorem 2 asserts that

$$P^2 - 4\pi A \geq 4\pi(R - r)^2,$$

in agreement with Bonnesen's inequality (7). The use of cyclic rearrangement to address the isoperimetric problem for ellipses appears in a book by Steinhaus [18, p. 87] and was a principal inspiration for this article.

We conclude with the observation that cyclic rearrangement preserves far more than perimeter. The boundary of the rearranged set $\boxtimes(K)$ is equidissectable to the original boundary ∂K by a finite set of rigid motions, and has, for example, identical local curvature except at four points. This suggests that the symmetrization technique provided by cyclic rearrangement is likely to offer insight into the behavior of many other geometric functionals besides area and perimeter, such as valuations on convex bodies [9], [11], [12], dual mixed volumes [7], [13], [14], and other valuations on star-shaped sets [4], [8].

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